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Unitary operator bases and q-deformed algebras

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Abstract

Starting from the Schwinger unitary operator bases formalism constructed out of a finite dimensional state space, the well-known q-deformed commutation relation is shown to emerge in a natural way, when the deformation parameter is a root of unity.

I. INTRODUCTION

From the studies of deformed algebras, which appeared in connection with problems in statistical mechanics and in quantum field theory (QFT), it came out that the q-deformation

parameter is, in its general form, a complex number. In many applications it assumes a real value while in other cases its imaginary part also plays a physical role. Apart from the basic quantum mechanical study of the q-deformed oscillator by Biedenharn [1] and MacFarlane [2], in which a real deformation parameter is assumed, Floratos [3], on the other hand, in his study of the q-oscillator many-body problem, also discusses the case where q is a pure complex number. Furthermore, in the particular case when q is a root of unity, it can be shown that the underlying state space, characterizing the physical system, is finite dimensional. The q-deformed algebras generate a suitable framework in this case and has been explicitly used in connection with the phase problem in optics [4]; moreover, it has also been pointed out their importance in QFT [5].

A long time ago, Schwinger [6] has pointed out that it is possible to construct an operator basis, in the operator space, once we are given a finite dimensional state space. The two fundamental unitary operators from which the basis is constructed satisfy the Weyl commutation relation and act cyclically on the corresponding state space, thus admitting as many roots of unity, as eigenvalues, as is the dimension of the space. Here we will show how the Schwinger operator basis can be used as a natural tool in order to obtain the q-deformed commutation relation in the particular case when q is a root of unity.

II. THE UNITARY OPERATOR BASES

For the complete quantum description of a physical system, a set of operators must be found in such a way as to permit the construction of all possible dynamical quantities related to that system. The elements of that set are then identified as the elements of a complete operator basis.

One particular set, consisting of unitary operators, has been studied by Schwinger [6] and will be briefly recalled here. Let us consider a N - dimensional linear, normed space of states to be understood as the quantum phase-space of the relevant system. We can define a unitary operator V through the mapping of an orthonormal system $\{|u_k\rangle\}_{k=0...N-1}$ defined constitutes a complete orthonormal operators basis, with which we can construct all possible dynamical quantities pertaining to that system [6]. In this way, an operator decomposition in this basis is written as

$$\hat{O} = \sum_{m,n=0}^{N-1} O(m,n)\,\hat{S}_1(m,n) \ , \tag{2.5}$$

where

$$O(m,n) = Tr\left[\hat{S}_1^{\dagger}(m,n)\,\hat{O}\right] .$$

A very interesting property manifested by the operator basis $\{\hat{S}_1\}$ is the factorization property

$$\hat{S}_1(m,n) = \prod_{l=1}^{h} \hat{S}_{1l}(m_l,n_l) \ ,$$

where the sub-bases

$$\hat{S}_{1l}(m_l,n_l) = \frac{U_l^{m_l} V_l^{n_l}}{\sqrt{P_l}} \ , \ m_l, n_l = 0,1,...P_l - 1 \ ,$$

obey the commutation relations

$$V_{l_1} U_{l_2} = U_{l_2} V_{l_1} \ , \ l_1 \neq l_2 \ ,$$

$$V_{l_1} U_{l_2} = \exp\left(\frac{2\pi i}{P_{l_1}}\right) U_{l_2} V_{l_1} \ , \ l_1 = l_2 \ ,$$

where h is the total number of primes factors in N including repetitions, with P_l a prime factor of N. This decomposition shows that the factorized basis is constructed from operator sub-bases, each of which associated with a prime number of states, the pair of operators U and V of each sub-basis being classified by the value of the prime integer $P_l = 2, 3, 5, ...$ It is straightforward to verify that the pair U and V associated with the canonical coordinate-momentum pair $q - p$ is obtained in the particular case $P_l = \infty$. Then, according to Schwinger, due to this factorization property and mutual orthogonality, each of these sub-bases is associated to a particular degree of freedom of the physical system.

In order to emphasize the complete symmetry between U and V, we want also to observe that we could have introduced the new form for the operator basis elements

$$\hat{S}_2(m,n) = \frac{U^m V^n}{\sqrt{N}} \exp\left(\frac{i\pi mn}{N}\right) = \frac{V^n U^m}{\sqrt{N}} \exp\left(\frac{-i\pi mn}{N}\right) ,$$

which preserves all properties already discussed under the substitutions $U \to V$ and $V \to U^{-1}$, combined with $m \to n$ and $n \to -m$.

For different degrees of freedom we must conveniently choose the range of variation of the state labels in order to correctly treat the system kinematics; for instance, it is important to emphasize again the canonical case, i.e., $P_l = \infty$, for, in such a case, the unitary operators are immediately identified with the well-known shift operators

$$V \to e^{iq\hat{P}}$$

$$U \to e^{ip\hat{Q}}$$

when one considers the symmetric interval $m, n = -\frac{N-1}{2}, ..., +\frac{N-1}{2}$, and then takes the $N \to \infty$ limit by prime numbers [6]. However, it is also possible to perform a construction of the unitary operators U and V in such a way to obtain an explicit "angle - action" pair, characterizing an Abelian two-dimensional rotation; in this case it can be shown that

$$V \to \exp\left(i\frac{2\pi}{N}\hat{J}\right) \tag{2.6}$$

$$U \to \exp\left(i\hat{\Theta}\right) . \tag{2.7}$$

Here, the interval of variation of the state labels are suitably defined to be $m = -\frac{N-1}{2}, ..., +\frac{N-1}{2}$ and $n = -\frac{N-1}{2}\pi, ..., +\frac{N-1}{2}\pi$ in such a form that, in the limit of $N \to \infty$, one recovers $m = \{-\infty, ..., +\infty\}$, running by integers, and $n = \{-\pi, \pi\}$ [11].

For the sake of completeness it is important to go back to the operator decomposition procedure, Eq.(2.5), and discuss the importance of the particular choice of the operator basis. In fact, in order to emphasize the discrete phase space character of the description, we see that a particularly suitable choice for the unknown function $g(k)$ must use an anti-symmetric function of the state label k so as to select the vacuum state. Now, the natural antisymmetric periodic function defined on the circle is the $\sin(\theta)$ function, what requires odd N's. Therefore, the proposed annihilation operator is then written as

$$a = \sum_{k=0}^{N-1} \frac{\sin\left(\frac{2\pi k}{N}\right)}{\sin\left(\frac{2\pi}{N}\right)} \mid v_{k-1}\rangle\langle v_k \mid \ .$$

According to the discussion in section 2, we can decompose these operators in the operator basis,

$$\hat{O} = \sum_{m,n=0}^{N-1} O\,(m,n)\,\hat{G}\,(m,n)$$

obtaining

$$a^{\dagger} = U$$

and

$$a = U^{-1}\frac{V - V^{-1}}{\omega - \omega^{-1}}$$

respectively.

The question that can be posed now is if there exists some relation between the bilinear products of the creation and annihilation operators, $a^{\dagger}$, a. Starting from the Weyl relation, Eq. (2.4), the definitions Eqs. (2.6) and (2.7), where instead of $\hat{J}$ we now use $\hat{N}$, the number operator and using the above expressions for a and $a^{\dagger}$, we can immediately obtain the following relation

$$aa^{\dagger} - \omega a^{\dagger}a = \omega^{-\hat{N}} \ .$$

Therefore, we have seen that, starting from the Schwinger unitary operators, the well-known q-deformed commutation relation emerges in a natural way, when the deformation parameter is a root of unity. Furthermore, it is immediate to verify that the creation and annihilation operators, a and $a^{\dagger}$ proposed here are directly related to the h and g functions proposed by Floratos in his discussion of q-deformed algebras for the bosonic case [3].

IV. REMARKS AND CONCLUSIONS

The main objective of this paper was to show that the q-deformed algebras can be put in correspondence to Schwinger's unitary operator bases formalism, when the deformation parameter is a root of unity.

Furthermore, it was shown that this formalism is the natural arena for the discussion of recent work on general finite dimensional quantum mechanics problems. Particularly, the Schwinger's formalism was used to represent any operator acting on any finite dimensional state spaces [11,12]. To be specific, it has also been used to study the Liouvillian dynamics in the general finite dimensional phase spaces [13] as well as to describe physical systems from the particular case of a spin 1/2 ($N = 2$ space) up to the canonical continuous case (as the limit $N \to \infty$). The special case of phase and number operators appearing in connection with quantum optics has also been treated within this framework [14]. This latter problem, or equivalently its Pegg-Barnett description [4], being just a particular case of the general Schwinger formalism, can therefore be also embodied in the q-deformed algebra context along the lines studied here.

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