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ABSTRACT

Conventional laser theory deals with discrete "cavity modes" and introduces artificial mechanisms to simulate radiation field losses due to beam extraction. A more realistic model of a laser with a transmitting window, previously investigated by Lang, Scully and Lamb, is reanalysed semiclassicaly. In the previous work, the continuous radiation field spectrum was treated as the limit of a discrete but very dense spectrum of modes, and the narrowness of the laser linewidth was attributed to the locking together of these modes. Working entirely within the continuous spectrum, we find that the treatment is considerably simplified, leading to an explicit solution for single-quasimode operation within the linear approximation, that exhibits the role of the excitation conditions in the buildup of the laser field. As was pointed out by Lang and Scully, the transmission loss plays the role of an effective noise source, that verifies the fluctuation-dissipation theorem. However, the continuum treatment does not support the mode locking mechanism, but rather the usual explanation of the laser linewidth in terms of gain narrowing. The nonlinear analysis agrees with the main conclusion of Lang, Scully and Lamb that the "mode amplitude" of the conventional theory should be interpreted as a collective variable, that satisfies the well-known rotating wave Van der Pol oscillator equation. The domain of validity of this result and of the laser quasimode concept depends on an approximation analogous to the Weisskopf-Wigner one, which is well satisfied in the usual situation of very long-lived quasimodes.

I. INTRODUCTION

In laser theory, the losses in the radiation field are usually represented in terms of artificial loss mechanisms. In the most thoroughly discussed case, that of a gas laser, the model for the laser cavity involves two perfectly reflecting mirrors, leading to a discrete set of "normal modes" of the electromagnetic field, the so-called "Fox-Li quasimodes"[1]. The radiation losses in a real laser (transmission through the mirrors and diffraction losses) are simulated by the "ad hoc" introduction of a homogeneous medium smeared throughout the cavity. In semi-classical treatments, this is taken as a conducting medium; in the quantum theory of the laser, different models of field loss reservoirs have been employed.

All these models, however, are quite artificial and are only remotely related with the real situation, where the damping of the field is mainly due to the extraction of the laser beam through a semitransparent mirror. The "modes" of the laser cavity should therefore be interpreted[2] as resonances in a continuous spectrum, which may be represented, e.g., in terms of complex poles of the scattering matrix (cf. Appendix A).

To justify the usual procedure, it is claimed that the results should be insensitive to the nature of the loss mechanism. This is supported by the fact that different artificial loss mechanisms apparently lead to equivalent results. However, at least some of these results, like the laser threshold

condition, depend only on energy balance or rate equations, so that their insensitivity to the nature of the loss mechanism is not surprising.

Although the usual approach simplifies the theoretical analysis, enabling one to work with a discrete spectrum of "laser modes", and even though it may lead to correct results, it leaves unanswered several questions of principle. Examples of such questions are: Why is the laser line so narrow in "single-mode" operation? Does the laser threshold correspond to a phase transition?

Within the usual treatment, there is no way to form a satisfactory conceptual picture of the nature of a "laser mode". Such questions can only be analysed within the framework of a theory that does not disguise the existence of a continuous spectrum for the radiation field.

Lang et al[3] (hereafter referred to as LSL) employed a very simple and interesting model of a laser with a transmitting window[4] to investigate the above questions. The relationship between this more realistic model for the radiation losses and the fluctuation - dissipation theorem was also discussed[5].

Although the model leads to a continuous spectrum for the radiation field, it was found expedient in these investigations to treat it as the limit of a discrete spectrum, obtained by inserting a perfectly reflecting mirror at an arbitrarily large distance outside the laser cavity. To avoid the return of the light beam after reflection by this outside mirror, it was still necessary to include a fictitious conducting medium distributed throughout space. By considering the interaction between two neighboring modes in the dense spectrum of the radiation field, the authors concluded that the narrowness of the laser linewidth in "single-mode" operation is due to the nonlinear effect of mode locking, rather than to the usually invoked gain-narrowing mechanism.

In the present work, we reexamine the LSL model, still within the context of semiclassical theory. However, we work from the outset with the continuous spectrum. This has several advantages. First, the mathematical treatment becomes much simpler. Secondly, it is no longer necessary to introduce a fictitious damping medium: the radiation loss is entirely due to the beam extraction, which is what one wants. Finally, while many aspects brought out by the work with the discretized version remain unchanged, a few points appear in a different light.

We agree with the main conclusion reached by LSL, namely, that the "mode amplitudes" employed in the usual theories of the laser should be interpreted as collective variables. However, we find no evidence to support the contention that the narrowness of the laser line is due to mode locking. The picture that emerges instead agrees with the traditional explanation in terms of gain narrowing. The role of noise also appears more clearly in the continuum treatment, leading to an improved derivation of the relationship with the fluctuation-dissipation theorem.

In Sect.II (cf. also Appendix A), we discuss the normal modes of the electromagnetic field and their connection with Fox-Li quasimodes. In Sect.III, we obtain the basic equation for the field in the presence of active atoms, following the standard semiclassical approach[1]. The solution of the basic equation in the linear approximation is investigated in Sect. IV, where we also discuss the time evolution of the field from an arbitrary initial state. The breakdown of the linear approximation above the laser threshold leads to the consideration of the nonlinear theory in Sect. V. The nonlinear evolution equation is solved in terms of a complex collective variable associated with the "mode amplitude", as was done by LSL. The relationship with the fluctuation-dissipation theorem is investigated in Sect. VI, and the line narrowing mechanism is discussed in Sect. VII. The conclusions and the outlook on further extensions of the model (including the fully quantized theory) are presented in Sect. VIII. To facilitate comparison with the LSL treatment, we adhere closely to their notation.

II. MODEL AND FIELD MODES

The laser cavity in the LSL model is a free space region bounded by two plane parallel plates, one of which is totally reflecting, whereas the other one is semitransparent. We take the origin at this semitransparent plate and the z-axis perpendicular to the plates. If ℓ is the separation between the plates, the laser cavity is the region $0 < z < \ell$, and the outside region is the left half-space, $-\infty < z < 0$ (fig.1). This system is a close relative of the Fabry-Perot etalon[6].

We take the plate coating of the semitransparent window as a dielectric film, rather than a half-silvered mirror, so as to introduce no ohmic losses. This film is modelled as a limiting case of a very thin layer with a very large dielectric constant, described by

$$\varepsilon(z) = \varepsilon_0 \left[1 + \eta\, \delta(z)\right], \qquad \text{(II.1)}$$

where $\delta(z)$ is the Dirac delta function and η is a real parameter with dimensions of length, with determines the transparency of the window.

The left half-space plays the role of field loss reservoir, the damping being entirely due to the beam extraction through the window. We consider only longitudinal field modes, so that the model becomes effectively one-dimensional, in the sense that the field depends only on the z coordinate.

The normal modes of propagation are stationary solutions of Maxwell's equations that satisfy the boundary conditions. As shown in Appendix A, they have a continuous frequency spectrum, and the eigenfunction (electric field amplitude) associated with a circular frequency $\omega_k = ck$ is given by

$$U_k(z) = L_k \sin\left[k(z-\ell)\right], \qquad 0 \le z \le \ell,$$

$$= (2/\pi)^{1/2} \sin(kz-\delta_k), \qquad -\infty < z \le 0, \qquad \text{(II.2)}$$

where the usual δ-function continuum normalization is employed,

$$\int_{-\infty}^{\ell} U_k(z)U_{k'}(z)\,\varepsilon(z)\,dz/\varepsilon_0 = \delta(k-k'), \qquad \text{(II.3)}$$

and the expressions of L_k and of the phase shift δ_k are given in Appendix A.

A plot of L_k^2 as a function of ω_k (fig.2) shows resonances centered approximately around the Fox-Li quasimode frequencies, with spacing $\simeq c\pi/\ell$. In the LSL treatment this is replaced by a dense discrete spectrum (fig.2).

From now on we assume that the transparency of the transmitting window is very small, as is usually the case. Then, the resonance width Γ_n associated with a given Fox-Li resonant frequency ω_{on} is much smaller than the spacing between resonances. We can therefore consider a neighborhood of ω_{on} having a width large as compared with Γ_n, but still small as compared with the resonance spacing. Within this neighborhood, we can approximate L_k by a Lorentzian function M_k,

$$L_k \simeq M_k \equiv (2/\pi)^{1/2}\, \Gamma_n\, \Lambda_{on} / \left[(\omega_k-\omega_{on})^2+\Gamma_n^2\right]^{1/2}, \qquad \text{(II.4)}$$

where (cf. Appendix A) Γ_n is determined by the window transparency,

$$\Gamma_n = c/(\Lambda_{on}^2 \ell), \qquad \text{(II.5)}$$

with

$$\Lambda_{on} \equiv \eta n\pi/\ell >> 1, \qquad \text{(II.6)}$$

and ω_{on} is the resonant frequency associated with the nth Fox-Li quasimode (n>>1), given by

$$\omega_{on} = ck_{on} \approx (n\pi+\Lambda_{on}^{-1})c/\ell \; . \qquad \text{(II.7)}$$

The requirement that the transparency is very small is expressed by condition (II.6).

III. POLARIZATION OF THE ACTIVE MEDIUM

The interaction between the active atoms and the radiation field will be treated within the usual self-consistent field approximation of semiclassical laser theory[1]. The off-diagonal matrix elements of the induced electric dipole moment of the active atoms due to the action of the electric field act in their turn as a source term for the field, associated with a macroscopic polarization $\vec{P}$.

Let $\vec{E} = E(z,t)\vec{i}$ and $\vec{P}=P(z,t)\vec{i}$ be the electric field and polarization, respectively (only linearly polarized longitudinal modes are considered). Assuming, as usual, that the active medium is rarefied enough, we may approximate

$$0 = \vec{\nabla}\cdot\vec{D} = \vec{\nabla}\cdot(\epsilon_0\vec{E}+\vec{P}) \approx \vec{\nabla}\cdot\vec{E}, \qquad \text{(III.1)}$$

so that Maxwell's equations reduce to the inhomogeneous wave equation

$$\partial^2E/\partial z^2 - \mu_0\epsilon_0\partial^2E/\partial t^2 = \mu_0\partial^2P/\partial t^2, \qquad \text{(III.2)}$$

where P(z,t) is nonvanishing only within the laser cavity, i.e., for $0 < z < \ell$.

The field and the polarization can be expanded throughout space in terms of the complete orthonormal set of eigenfunctions $U_k(z)$ given by (III.2):

$$E(z,t) = \int_0^\infty E_k(t)U_k(z)dk, \qquad \text{(III.3)}$$

$$P(z,t) = \int_0^\infty P_k(t)U_k(z)dk, \qquad \text{(III.4)}$$

where (cf. Appendix A)

$$E_k(t) = \int_{-\infty}^{\ell} E(z,t)U_k(z)\epsilon(z)dz/\epsilon_0, \qquad \text{(III.5)}$$

$$P_k(t) = \int_{-\infty}^{\ell} P(z,t)U_k(z)\epsilon(z)dz/\epsilon_0. \qquad \text{(III.6)}$$

Inserting (III.3) and (III.4) in (III.1), we find

$$\ddot{E}_k(t) + \omega_k^2E_k(t) = \ddot{P}_k(t)/\epsilon_0. \qquad \text{(III.7)}$$

From now on we restrict our consideration to solutions consistent with the desired condition of single-quasimode operation, centered around a given Fox-Li resonant frequency $\omega_0 \equiv \omega_{on}$, by making the "Ansatz"

$$E_k(t) = \frac{1}{2}\left[\mathcal{E}_k(t)\exp(-i\omega_0 t)+c.c.\right], \tag{III.8}$$

$$P_k(t) = \frac{1}{2}\left[\mathcal{P}_k(t)\exp(-i\omega_0 t)+c.c.\right], \tag{III.9}$$

with the usual slowly-varying amplitude and phase assumptions[1] for $\mathcal{E}_k(t)$, $\mathcal{P}_k(t)$. Neglecting small terms, under these conditions, (III.7) becomes

$$\dot{\mathcal{E}}_k(t) + i\Delta\omega_k\mathcal{E}_k(t) = (i\omega_0/2\epsilon_0)\mathcal{P}_k(t), \tag{III.10}$$

where

$$\Delta\omega_k = \omega_k - \omega_0. \tag{III.11}$$

In the usual model with only one discrete mode and the loss introduced in an artificial manner, $\mathcal{P}_k(t)$ and $\mathcal{E}_k(t)$ are related by

$$\mathcal{P}_k(t) = a\,\mathcal{E}_k(t) - b|\mathcal{E}_k(t)|^2\mathcal{E}_k(t). \tag{III.12}$$

In the present case, the usual perturbative approach to the interaction between the active atoms and the field, carried out in the corresponding lowest order of nonlinearity (cf. Appendix B),

leads to

$$\mathcal{P}_k(t) = \int_0^\infty a_{k\mu}\,\mathcal{E}_\mu(t)\,d\mu$$

$$- \int_0^\infty \int_0^\infty \int_0^\infty b_{k\mu\rho\sigma}\;\mathcal{E}_\mu(t)\mathcal{E}_\rho^*(t)\mathcal{E}_\sigma(t)\,d\mu\, d\rho\, d\sigma, \tag{III.13}$$

where

$$a_{k\mu} = -iaM_k M_\mu, \tag{III.14}$$

$$b_{k\mu\rho\sigma} = -ibM_k M_\mu M_\rho M_\sigma, \tag{III.15}$$

with M_k given by (II.4) and

$$a = \bar{N}\ell d^2/2\hbar\gamma_{ab}, \tag{III.16}$$

$$b = 3\bar{N}\ell d^4/16\hbar^3\gamma_a\gamma_b\gamma_{ab}. \tag{III.17}$$

Here, $\bar{N}$ is the space-averaged population inversion density; d is the microscopic dipole moment; γ_a, γ_b are the decay constants respectively of the upper and lower states of the two-level atoms, and $\gamma_{ab}=(\gamma_a+\gamma_b)/2$. Furthermore, we have assumed central tuning, i.e., that ω_0 is tuned to the atomic transition frequency.

The result (III.13) already takes into account that only continuum modes belonging to the same Fox-Li band, centered at ω_0, interact appreciably among themselves (cf. (III.26) below). It is for this reason that, e.g. in (III.14),

M_k and M_μ appear instead of L_k and L_μ, i.e., the approximation (II.4) can be employed. The integrals over the continuous spectrum in (III.13) thus extend effectively over a single band, but, in view of the fall-off of the Lorentzian factors, they may be extended to infinity. This corresponds to a kind of Weisskopf-Wigner approximation (see the discussion following (IV.20)).

An important difference between (III.12) and (III.13) is that, in the latter, the electric susceptibility is effectively nonlocal in momentum space. This coupling between modes also occurs in the LSL treatment, where

$$\mathcal{P}_k(t) = \sum_\mu a_{k\mu}\mathcal{E}_\mu(t) - \sum_{\mu\rho\sigma} b_{k\mu\rho\sigma}\ \mathcal{E}_\mu(t)\mathcal{E}^*_\rho(t)\mathcal{E}_\sigma(t), \qquad \text{(III.18)}$$

in terms of the discretized spectrum of modes in their analysis.

The mode coupling takes place through the agency of the active atoms, and it is affected both by their spatial distribution and by the cavity Q-factor. Indeed, we have (cf. Appendix B)

$$a_{k\mu} \propto N(k-\mu) \equiv \int_{-\infty}^{\ell} N(z)U_k(z)U_\mu(z)\varepsilon(z)dz/\varepsilon_0, \qquad \text{(III.19)}$$

where $N(z)$ is the population inversion density.

If $N(z)$ were uniform throughout space $(-\infty<z<\ell)$, one would have

$$a_{k\mu} = a_k\delta(k-\mu), \qquad \text{(III.20)}$$

in view of the orthonormality of the eigenfunctions $U_k(z)$. The mode coupling is therefore related to the nonuniformity of $N(z)$. In particular, if $N(z)$ is uniform within the laser cavity and zero outside, (III.19) becomes

$$N(k-\mu) = \overline{N}\int_0^{\ell} U_k(z)U_\mu(z)dz \quad . \qquad \text{(III.21)}$$

In the absence of the semitransparent mirror, $U_k(z)$ would be given by

$$U_k(z) = (2/\pi)^{1/2}\ \sin\left[k(z-\ell)\right] \quad , \quad -\infty<z\leq\ell \quad , \qquad \text{(III.22)}$$

so that we would have

$$a_{k\mu} \propto \overline{N}\ \sin\left[(k-\mu)\ell\right]/(k-\mu) \quad , \qquad \text{(III.23)}$$

yielding significant coupling between neighboring quasimode bands (Appendix A), within the range

$$|\omega_k - \omega_\mu| \sim c\pi/\ell \quad . \qquad \text{(III.24)}$$

In the actual situation, corresponding to the presence of a highly reflecting semitransparent mirror at $z = 0$, the eigenfunctions are given by (II.2) and we get, instead of (III.23),

$$a_{k\mu} \propto \bar{N} M_k M_\mu \sin\left[(k-\mu)\ell\right]/(k-\mu) \quad . \qquad \text{(III.25)}$$

The factors M_k and M_μ overlap significantly only if (cf. (II.4) to (II.6))

$$|\omega_k - \omega_\mu| \leq \Gamma_n = c\pi/\Lambda_{on}^{\;2}\,\ell << c\pi/\ell \quad , \qquad \text{(III.26)}$$

so that we may set $k = \mu$ in the last factor of (III.25), leading to (III.14). Thus, the cavity Q-factor restricts significant mode interaction to a single Fox-Li band. In the limit as $\Lambda_{on} \to \infty$, the mirror becomes perfectly reflecting, the spectrum becomes discrete and the modes decouple, so that we recover the result (III.12) of the usual model.

Finally, inserting (III.13) in (III.10), we obtain the basic equation of semiclassical laser theory that will be employed in our treatment:

$$\dot{\mathcal{E}}_k(t) + i\,\Delta\omega_k\,\mathcal{E}_k(t) = \int_0^\infty \alpha_{k\mu}\,\mathcal{E}_\mu(t)\,d\mu$$

$$- \int_0^\infty\!\int_0^\infty\!\int_0^\infty \beta_{k\mu\rho\sigma}\,\mathcal{E}_\mu(t)\,\mathcal{E}^*_\rho(t)\,\mathcal{E}_\sigma(t)\,d\mu\,d\rho\,d\sigma \quad , \qquad \text{(III.27)}$$

where (cf. (III.14) to (III.17))

$$\alpha_{k\mu} = \alpha\, M_k M_\mu \quad , \qquad \beta_{k\mu\rho\sigma} = \beta\, M_k M_\mu M_\rho M_\sigma \quad , \qquad \text{(III.28)}$$

$$\alpha = \omega_o\,\bar{N}\,\ell\, d^2/4\hbar\,\varepsilon_o\,\gamma_{ab} \quad , \qquad \text{(III.29)}$$

$$\beta = 3\omega_o\,\bar{N}\,\ell\, d^4/32\hbar^3\,\varepsilon_o\,\gamma_a\,\gamma_b\,\gamma_{ab} \quad . \qquad \text{(III.30)}$$

The corresponding equation of LSL differs from (III.27) in essentially two respects: (a) The spectrum is rendered discrete (though very dense) by introducing an additional perfectly reflecting mirror at $z = -L\,(L >> \ell)$; (b) To avoid the return of outgoing radiation from this additional mirror, a homogeneous fictitious damping medium is introduced throughout space, associated with an additional damping term $\gamma\,\dot{\mathcal{E}}_k$ on the left-hand side.

In the present treatment, no such term needs to be inserted: the transmission loss is already contained within (III.27). This will be explicitly demonstrated in the next Section.

IV. LINEAR THEORY

In this Section, we solve (III.27) in the linear approximation, obtained by neglecting nonlinear terms:

$$\dot{\mathcal{E}}_k(t) + i\,\Delta\omega_k\,\mathcal{E}_k(t) = \alpha\int_0^\infty M_k\, M_\mu\,\mathcal{E}_\mu(t)\,d\mu \quad . \qquad \text{(IV.1)}$$

We deal with this integro-differential equation, which must be solved subject to given initial conditions $\mathcal{E}_k(0)$, by applying the Laplace transform. We set

$$F_k(p) = \mathcal{L}[\mathcal{E}_k(t)] = \int_0^\infty \mathcal{E}_k(t) \exp(-pt)\,dt \quad , \tag{IV.2}$$

valid for Re $p > \lambda$, where λ is known as the abscissa of convergence. This means that the solution must be bounded by $\exp(\lambda t)$ as $t \to \infty$.

The transformed equation is

$$p\, F_k(p) - \mathcal{E}_k(0) + i\, \Delta\omega_k\, F_k(p) = \alpha \int_0^\infty M_k\, M_\mu\, F_\mu(p)\, d\mu \quad , \tag{IV.3}$$

which is a Fredholm-type integral equation, instead of the original integro-differential equation. Since (IV.3) has a separable kernel, it can be solved exactly.

For this purpose, we first solve (IV.3) with respect to $F_k(p)$, multiply both sides by M_k and integrate over k , yielding

$$\left[1 - \alpha \int_0^\infty \frac{M_k^2}{(p + i\Delta\omega_k)}\, dk\right] \int_0^\infty M_\mu\, F_\mu(p)\, d\mu = \int_0^\infty \frac{M_k\, \mathcal{E}_k(0)}{(p+i\Delta\omega_k)}\, dk \quad . \tag{IV.4}$$

Substituting M_k by (II.4) in the integral within square brackets and taking into account that $\omega_o = \omega_{on} \gg \Gamma_n$, we see

that this integral can be evaluated by extending the range of integration to $-\infty$. Dropping the band index for simplicity, so that

$$\Gamma_n \to \Gamma \quad , \qquad \Lambda_{on} \to \Lambda \quad , \tag{IV.5}$$

the net result is

$$\Delta(p) \int_0^\infty M_\mu\, F_\mu(p)\, d\mu = \int_0^\infty \frac{M_k\, \mathcal{E}_k(0)}{(p+i\Delta\omega_k)}\, dk \quad , \tag{IV.6}$$

where

$$\Delta(p) = (p - p_o)/(p + \Gamma) \quad , \tag{IV.7}$$

$$p_o = \alpha\, M^2 - \Gamma \quad , \tag{IV.8}$$

$$M^2 = \int_{-\infty}^\infty M_k^2\, dk = \pi \Gamma \Lambda^2 / c \quad . \tag{IV.9}$$

Inserting back (IV.6) into (IV.3), we get

$$\Delta(p)\, F_k(p) = \Delta(p)\, \frac{\mathcal{E}_k(0)}{(p+i\Delta\omega_k)} + \frac{\alpha\, M_k}{(p+i\Delta\omega_k)} \int_0^\infty \frac{M_\mu\, \mathcal{E}_\mu(0)}{(p+i\Delta\omega_\mu)}\, d\mu \quad . \tag{IV.10}$$

The factor $\Delta(p)$ is nothing but the Fredholm determinant.

Indeed, according to (IV.4) and (IV.6),

$$\Delta(p) = 1 - \int \mathcal{K}(\mu,\mu)\,d\mu \quad , \tag{IV.11}$$

where $\mathcal{K}(\mu,\mu)$ is the diagonal part of the kernel

$$\mathcal{K}(k,\mu) = \alpha M_k M_\mu/(p + i\Delta\omega_k) \tag{IV.12}$$

of the integral equation (IV.3).

According to the Fredholm alternative, there are now only two possibilities, depending on whether $\Delta(p) = 0$ (i.e., by (IV.7), $p = p_o$) or $\Delta(p) \neq 0$ (i.e., $p \neq p_o$).

(a) $p = p_o$. In this case, by (IV.10), either the integral on the right-hand side vanishes for $p = p_o$ or $F_k(p_o) \to \infty$. This possibility is related with the problem of division by zero in distribution theory. The distribution

$$F_k(p) = c_k\,\delta(p - p_o) \tag{IV.13}$$

is a nontrivial solution of the homogeneous integral equation, with $\mathcal{E}_k(0) = 0$. However, it is not an admissible Laplace transform, so that, within the class of solutions allowable on physical grounds, we can ignore this possibility from now on.

(b) $p \neq p_o$. In this case, $\Delta(p) \neq 0$ and the associated homogeneous equation (with $\mathcal{E}_k(0) = 0$) has only the

trivial solution $F_k(p) = 0$, i.e., $\mathcal{E}_k(t) = 0$. The inhomogeneous equation has a unique solution, obtained by dividing both sides of (IV.10) by $\Delta(p)$.

By partial fraction decomposition, this solution can be rewritten as follows:

$$F_k(p) = \frac{1}{(p + i\Delta\omega_k)} \left[\mathcal{E}_k(0) + \frac{\alpha M_k (\Delta\omega_k + i\Gamma)}{(p_o + i\Delta\omega_k)} \right.$$

$$\left. \times \fint_0^\infty \frac{M_\mu \mathcal{E}_\mu(0)}{\omega_\mu - \omega_k} d\mu \right] - \alpha M_k \fint_0^\infty \frac{(\Delta\omega_\mu + i\Gamma) M_\mu \mathcal{E}_\mu(0)\,d\mu}{(\omega_\mu - \omega_k)(p_o + i\Delta\omega_\mu)(p + i\Delta\omega_\mu)}$$

$$+ \frac{\alpha^2 M_k M^2}{(p - p_o)(p_o + i\Delta\omega_k)} \int_0^\infty \frac{M_\mu \mathcal{E}_\mu(0)}{(p_o + i\Delta\omega_\mu)} d\mu \quad , \tag{IV.14}$$

where $\fint$ denotes the Cauchy principal value. Note that the sum of the two principal value integrals is regular at $\omega_\mu = \omega_k$.

Finally, performing the Laplace inversion, we obtain the solution of the integro-differential equation (IV.1):

$$\mathcal{E}_k(t) = \mathcal{L}^{-1}\left[F_k(p)\right] = \mathcal{E}_k(0)\,\exp(-i\Delta\omega_k t)$$

$$+ \frac{\alpha M_k (\Delta\omega_k + i\Gamma)}{(p_o + i\Delta\omega_k)} \fint_0^\infty \frac{M_\mu \mathcal{E}_\mu(0)}{\omega_\mu - \omega_k} d\mu\,\exp(-i\Delta\omega_k t)$$

$$- \alpha M_k \int_0^{\infty} \frac{(\Delta\omega_\mu + i\Gamma) M_\mu \mathcal{E}_\mu(0) \exp(-i\Delta\omega_\mu t)}{(\omega_\mu - \omega_k)(p_o + i\Delta\omega_\mu)} d\mu$$

$$+ \frac{\alpha^2 M_k M^2}{(p_o + i\Delta\omega_k)} \int_0^{\infty} \frac{M_\mu \mathcal{E}_\mu(0)}{(p_o + i\Delta\omega_\mu)} d\mu \exp(p_o t) \quad . \qquad \text{(IV.15)}$$

It can readily be checked that the initial condition is satisfied. The only approximation in going over from (IV.4) to (IV.6) is the extension of the range of integration to $-\infty$ to evaluate $\Delta(p)$, which is an excellent approximation, since ω_o lies in the optical domain.

The first term of (IV.15), the only one that survives for $\alpha = 0$, represents free uncoupled oscillation at frequency ω_k (cf. (III.8), (III.11)). The other terms result from the coupling among the modes through the agency of the active atoms ($\alpha \neq 0$), yielding oscillation at frequency ω_k as well as at other frequencies.

The most significant term is the last one, which contains the factor $\exp(p_o t)$. If $p_o > 0$, this term diverges as $t \to \infty$, so that the linear theory eventually breaks down. The critical point $p_o = 0$ defines the laser threshold of operation. According to (IV.8), we have $p_o = \alpha M^2 - \Gamma$, so that, above threshold, $\alpha M^2 > \Gamma$. We can identify αM^2 and Γ respectively with the gain and loss parameters. The loss parameter Γ is just the halfwidth of the resonant Fox-Li

quasimode, arising exclusively from the transmission loss, as expected.

It is instructive to consider particular cases of the general solution (IV.15), associated with specific choices of the initial wave packet $\mathcal{E}_k(0)$.

(i) - Let us take

$$\mathcal{E}_k(0) = c_o \delta(\omega_k - \omega_o), \qquad \text{(IV.16)}$$

which corresponds to the excitation of a single continuum mode, with the resonant frequency $\omega_k = \omega_o$, at t=0. Inserting (IV.16) in (IV.15), we get

$$\mathcal{E}_k(t) = c_o \delta(\omega_k - \omega_o) + \alpha c_o \Lambda M_k \left[\frac{\alpha M^2 \exp(p_o t)}{p_o(p_o + i\Delta\omega_k)} - \right.$$

$$\left. - \frac{(\Delta\omega_k + i\Gamma)}{\Delta\omega_k (p_o + i\Delta\omega_k)} \exp(-i\Delta\omega_k t) + \frac{i\Gamma}{p_o \Delta\omega_k} \right] , \qquad \text{(IV.17)}$$

which allows one to obtain the field $E(z,t)$ from (III.3) and (III.8).

In particular, we find for the internal field $E^{(i)}(z,t)$ $(0 < z < \ell)$

$$E^{(i)}(z,t) = c_o \Lambda \left[1 + \mathcal{O}(\alpha)\right] \sin\left[k_o(z-\ell)\right] \exp(-i\omega_o t) +$$

$$+ \alpha c_o M^2 (\Lambda/p_o) \sin\left[k_o(z-\ell)\right] \exp(-i\omega_o t + p_o t) + \mathcal{O}(1) , \qquad \text{(IV.18)}$$

where $\mathcal{O}(\alpha)$ denotes a correction term of order α and $\mathcal{O}(1)$ denotes terms that vanish as $t\to\infty$.

The free-field solution ($\alpha=0$), given by the first term of (IV.18), is undamped; this is due to the special choice of the initial condition (IV.16) as a stationary free-field eigenfunction. In the presence of active atoms, the free-field solution is still dominant below threshold ($p_o<0$) as $t\to\infty$. However, above threshold, the second term of (IV.18), which is exponentially increasing, becomes dominant within the range of validity of the linear theory.

(ii) - Let us now take

$$\mathcal{E}_o(k) = c_o M_k, \tag{IV.19}$$

corresponding to an initial wave packet whose spectral profile is resonant with a single Fox-Li band, centered at ω_o. Substituting (IV.19) back in (IV.15) to get $\mathcal{E}_k(t)$, and substituting the result in (III.3) and (III.8), we find for the internal field in this case

$$E^{(i)}(z,t) = (2c_o/\ell)\sin\left[k_o(z-\ell)\right]\exp(i\omega_o t)\,.$$
$$\times\left[\exp(-\Gamma t) + \alpha\exp(p_o t)\right] + \mathcal{O}(\alpha)\,. \tag{IV.20}$$

The term in $\exp(-\Gamma t)$ again represents the free-field solution ($\alpha=0$). Now, however, in contrast with (IV.18), it is exponentially damped, with lifetime $\tau=1/\Gamma$, where Γ is the halfwidth of the laser quasimode. This confirms our contention

that the effect of transmission damping is already contained in (III.27).

In contrast with case (i), now, in the presence of active atoms, the free-field solution is no longer dominant for $p_o<0$, because $p_o=\alpha M^2-\Gamma>-\Gamma$. The dominant term, for large t, is the term in $\exp(p_o t)$, both above and below threshold.

The different behavior of the free-field solution in cases (i) and (ii) illustrates the dependence of the decay due to transmission on the excitation; note that the initial wave packet in both cases is centered on the same resonant band of the laser cavity.

It should also be pointed out that the exponential decay found in case (ii) is only an approximation, albeit a very good one. Indeed, in the evaluation of (IV.15) for the initial conditions given, as in (IV.4), we have several times replaced the true spectral profile L_k of the cavity modes by the Lorentzian approximation M_k (cf. (II.4)), and we have extended over an infinite range integrations that should actually extend only over the resonant band. These approximations are similar to the Weisskopf-Wigner one; if they were not made, there would appear (small) deviations from exponential decay at large times.

V. NONLINEAR THEORY

For a laser operating above threshold, as was mentioned in Sect.IV, the validity of the linear theory breaks down as soon as the intensity of the radiation field builds up to a large enough value for the nonlinear terms in (III.27) to become important.

In order to solve the equation under these conditions, we introduce a collective variable

$$A(t) = \int_0^\infty M_k \mathcal{E}_k(t)\,dk, \qquad (V.1)$$

which is generally complex. The discrete analogue of A(t) was introduced by LSL; the definition is suggested by (III.3), (II.2) and (II.4), which indicate that A(t) plays the role of amplitude associated with a single Fox-Li quasimode. In (V.1), again, the integral is effectively extended only over the resonant band, centered on ω_o.

In terms of the new variable, (III.27) becomes

$$\dot{\mathcal{E}}_k(t) + i\Delta\omega_k \mathcal{E}_k(t) = M_k G[A(t)], \qquad (V.2)$$

where

$$G[A(t)] = \alpha A(t) - \beta |A(t)|^2 A(t). \qquad (V.3)$$

Multiplying by M_k and integrating over k, we get (cf.IV.9))

$$\dot{A}(t) + i\int_0^\infty M_k \Delta\omega_k \mathcal{E}_k(t)\,dk = M^2 G[A(t)]. \qquad (V.4)$$

The second term on the left-hand side may be expressed in terms of the initial data by a procedure similar

to that of LSL. For this purpose, one writes down the integrated form of (V.2), giving $\mathcal{E}_k(t)$ in terms of $\mathcal{E}_k(0)$ and of the right-hand side, and one substitutes the result back into (V.4). After suitable transformations (again involving Weisskopf-Wigner-like approximations), one finally gets

$$\dot{A}(t) = p_o A(t) - \beta M^2 |A(t)|^2 A(t) + F(t), \qquad (V.5)$$

where

$$F(t) = \int_0^\infty (\Gamma - i\Delta\omega_k) M_k \mathcal{E}_k(0) \exp(-i\Delta\omega_k t)\,dk \qquad (V.6)$$

depends on the initial conditions.

This expression for F(t) differs from the corresponding one in LSL not only by the difference between our integral and their sum (discretization), but also by the presence of an extra damping factor $\exp(-\gamma t)$ in their work. This arises from their introduction of a fictitious damping medium throughout space in order to simulate the transmission loss. Because of this extra factor, their term corresponding to F(t) vanishes as $t\to\infty$, so that they neglect it. They do point out, however, that it acts like a noise source associated with the mirror transparency, and the role of this term is discussed in a paper by Lang and Scully[5].

In our model, no artificial damping is introduced, so that the extra damping factor $\exp(-\gamma t)$ is not present, and the effects due to F(t) will have to be discussed. We begin, however, by assuming the initial conditions to be such that $F(t)$ may be neglected in (V.5), and leave for later discussion the effects of a nonvanishing $F(t)$.

In the linear regime ($\beta=0$), with $F(t)=0$, the solution of (V.5) is

$$A(t) = A(0)\exp(p_o t), \qquad (V.7)$$

which corresponds to the contribution of the last term in (IV.15).

Let us now discuss the general solution of (V.5) for $F(t)=0$. Setting

$$A(t) = a(t)\exp[-i\phi(t)], \qquad (V.8)$$

we find that, under these conditions, (V.5) yields

$$\dot{a}(t) = p_o a(t) - \beta M^2 a^3(t), \qquad (V.9)$$

$$\dot{\phi}(t) = 0, \qquad (V.10)$$

so that $\phi(t)=\phi_o$ (initial value). Multiplying by $a(t)$ both sides of (V.9) and setting

$$I(t) = a^2(t) = |A(t)|^2, \qquad (V.11)$$

the differential equation for $I(t)$ can be immediately integrated, with the following result:

$$I(t)/I_\infty = \{1+\left[(I_\infty/I_o)-1\right]\exp(-2p_o t)\}^{-1}, \qquad (V.12)$$

where $I_o=a^2(0)$ is the initial value, and

$$I_\infty = p_o/\beta M^2 = (\alpha M^2-\Gamma)/\beta M^2 \qquad (V.13)$$

is the asymptotic value of $I(t)$ as $t\to\infty$ above threshold ($p_o>0$). Below threshold, $I(t)\to 0$ as $t\to\infty$, according to (V.12).

The expression (V.13), which also follows by setting $\dot{a}=0$ in (V.9), corresponds to the stable amplitude of oscillation of the laser above threshold. Note that this well-known result here refers to the collective variable $A(t)$, rather than to a single-mode amplitude.

Above threshold, with $I_\infty/I_o>>1$, we see from (V.12) that $I(t)$ grows exponentially at first, in agreement with (V.7), but then it bends over and approches the stable value (V.13). The rise time for stabilization is of the order of

$$\tau_o = \frac{1}{2}\, p_o^{-1}\ln(I_\infty/I_o). \qquad (V.14)$$

Unless the laser is operating far above threshold, we have $p_o/\Gamma = (\alpha M^2-\Gamma)/\Gamma \lesssim 1$, so that, typically, τ_o is $>>\tau$, where $\tau=1/\Gamma$ is the lifetime associated with the free oscillation of the Fox-Li quasimode (cf.(IV.20)). Since τ is usually much larger than the atomic transition lifetime, the adiabatic hypothesis employed in laser theory is justified.

At threshold (p_o=0), the solution of (V.9) becomes

$$I(t)/I_o = (1+2\beta M^2 I_o t)^{-1}, \tag{V.15}$$

which corresponds to a very slow decay, instead of the exponential decay found below threshod.

It follows from the above discussion that the general solution of (V.5) when $F(t)=0$ is

$$A(t) = \sqrt{I(t)}\exp(-i\phi_o), \tag{V.16}$$

where I(t) is given by (V.12). Thus, above threshold, with $F(t)=0$, the complex amplitude A(t) evolves towards a value that loses memory of its original modulus (cf.(V.13)), but it retains the initial phase.

What happens if $F(t)\neq 0$? Let

$$F(t)=f(t)\exp[-i\theta(t)]. \tag{V.17}$$

Then, instead of (V.9), (V.10), we get from (V.5)

$$\dot{a}(t) = p_o a(t)-\beta M^2 a^3(t)+f(t)\cos[\psi(t)], \tag{V.18}$$

$$\dot{\phi}(t) = [f(t)/a(t)]\sin[\psi(t)], \tag{V.19}$$

where $\psi(t)=\theta(t)-\phi(t)$.

According to (V.18), (V.19), the effect of F(t) is to produce fluctuations in both the amplitude a(t) and the phase $\phi(t)$. However, since $a(t)/f(t)$ usually becomes very large for asymptotic times, due to the effect of laser amplification on a(t), one may disregard fluctuations in the amplitude, but this is not so for the phase. Fluctuation effects are further discussed in the following Section.

VI. FLUCTUATION-DISSIPATION THEOREM

The initial conditions by which F(t) is determined (cf. (V.6)) are usually stochastic, rather than deterministic, so that F(t) plays the role of a noise source term and (V.5) becomes a Langevin-type equation. It corresponds, in fact, to the rotating wave Van der Pol oscillator equation that has been extensively discussed by Lax[7].

As was pointed out by Lang and Scully[5], it is reasonable to identify $\mathcal{E}_k(0)$ in (V.6) with the thermal radiation field that is initially present, for which (angular brackets denote ensemble averages)

$$\langle \mathcal{E}_k(0)\rangle = 0 , \tag{VI.1}$$

$$\langle \mathcal{E}_k^*(0)\,\mathcal{E}_{k'}(0)\rangle = G_o(k)\,\delta(k-k') , \tag{VI.2}$$

where $G_o(k)$, which corresponds to the spectral density of thermal radiation, is proportional to $\bar{n}(k)+\frac{1}{2}$, where $\bar{n}(k)$ is the thermal photon occupation number.

Since the width Γ of the resonant Fox-Li band is usually much smaller than that of the thermal distribution, $G_o(k)$ will overlap with a large number of cavity bands. If, in spite of this, we confine our attention only to the single resonant band, it follows immediately from (V.6) and (VI.1) that

$$\langle F(t)\rangle = 0 \quad , \tag{VI.3}$$

whereas (VI.2) yields, with the help of (II.5),

$$\langle F^*(t)F(t')\rangle = (2\Gamma/\pi\ell)\int_0^\infty d\omega_k\, G_o(k)\, . \times \exp[i\omega_k(t-t')] \quad . \tag{VI.4}$$

If we assume that: (a) The only significant contribution to (VI.4) comes from the resonant band, so that, in view of the slow variation of $G_o(k)$, we may replace it by $G_o(k_o)$; (b) In spite of this, the range of integration in (VI.4) may be extended to $-\infty$, we find that these assumptions lead to

$$\langle F^*(t)F(t')\rangle = (4\Gamma/\ell)G_o(k_o)\,\delta(t-t') \quad . \tag{VI.5}$$

This is equivalent to the result found by Lang and Scully[5],

which they associate with the fluctuation-dissipation theorem. It would also characterize $F(t)$ as a Markoffian noise source.

It should be pointed out, however, that (VI.5) is obtained through the use of rather drastic approximations. While assumption (a) is consistent with our use of resonance approximations and restriction to a single band at several stages of the treatment that led to (V.6), it is not consistent with assumption (b) and with neglecting the overlap between $G_o(k)$ and nonresonant bands.

If we extend the integration in (VI.4) only over the resonant band, we find, instead of (VI.5), a correlation time $\tau = 1/\Gamma$, consistent with the loss of memory due to transmission through the laser window. The Markoffian approximation is then valid only over time intervals much larger than τ.

On the other hand, in view of the slow variation with frequency of the quasimode width (cf. (II.5)), we can interpret (VI.4) directly as a version of the fluctuation-dissipation theorem[8], without neglecting the overlap with other bands. Thus, the approximation (VI.5) is unnecessary.

VII. LINE NARROWING MECHANISM

According to LSL[3], the usual explanation of the narrowness of the laser line in terms of "gain narrowing" is inadequate, in view of the nonlinearity of the problem. They propose a different explanation in terms of mode locking.

Their argument is based on the following points:

(a) The dominant term in the general solution (IV.15) above threshold (within the linear approximation) is of the form

$$\mathcal{E}_k(t) = c_o\, M_k \exp(p_o t)/(p_o + i\Delta\omega_k) \quad , \qquad \text{(VII.1)}$$

where c_o is a constant determined by the initial conditions. In terms of the mode amplitudes (III.8),

$$E_k(t) = \mathrm{Re}\left[\mathcal{E}_k(t)\, \exp(-i\omega_o t)\right] \quad , \qquad \text{(VII.2)}$$

substitution of the result (VII.1) implies that all modes oscillate at the same frequency ω_o, and that their phases ϕ_k, relative to the phase ϕ_o of the central mode ($\Delta\omega_k = 0$), are given by

$$\phi_k - \phi_o = -\tan^{-1}(\Delta\omega_k/p_o) \quad , \qquad \text{(VII.3)}$$

which is interpreted as signifying that a "phase locking" has occurred.

(b) To justify this interpretation, the discrete version of (III.27) in the linear approximation is considered:

$$\dot{\mathcal{E}}_k(t) + (i\Delta\omega_k + \gamma)\,\mathcal{E}_k(t) = \alpha\, M_k \sum_\mu M_\mu\, \mathcal{E}_\mu(t) \quad , \qquad \text{(VII.4)}$$

and it is applied to the (admittedly unphysical) hypothetical

situation in which there are only two modes coupled by (VII.4). It is further assumed, for simplicity, that the corresponding mode frequencies ω_1 and ω_2 are symmetrically placed with respect to the resonance frequency ω_o, so that

$$\omega_1 - \omega_o = \omega_o - \omega_2 = \Delta\omega/2 \quad ; \quad M_1 = M_2 = \mathcal{M} \quad . \qquad \text{(VII.5)}$$

Setting

$$\mathcal{E}_j(t) = a_j(t)\, \exp[-i\phi_j(t)] \quad , \quad j = 1,2 \quad , \qquad \text{(VII.6)}$$

with $a_1 = a_2$ (by symmetry), there results a phase locking equation for $\Delta\phi = \phi_1 - \phi_2$,

$$\frac{d}{dt}(\Delta\phi) = \Delta\omega - 2\alpha\mathcal{M}^2 \sin(\Delta\phi) \quad , \qquad \text{(VII.7)}$$

with $\Delta\omega$ as detuning parameter and $2\alpha\mathcal{M}^2$ as locking coefficient. The interaction between the modes that leads to this phase locking results from the presence of active atoms ($\alpha \neq 0$).

Let us examine first what happens with point (b) in the present treatment. The linearized version of (III.27) is given by (IV.1). Setting

$$\mathcal{E}_k(t) = a_k(t)\, \exp[-i\phi_k(t)] \quad , \qquad \text{(VII.8)}$$

$$A(t) = \int_0^\infty M_\mu \mathcal{E}_\mu(t)\,dt = a(t)\,\exp[-i\phi(t)] \quad , \qquad \text{(VII.9)}$$

we find that (IV.1) yields a pair of coupled equations for a_k and ϕ_k ,

$$\dot{a}_k = \alpha M_k\, a(t)\, \cos[\phi_k(t) - \phi(t)] \quad , \qquad \text{(VII.10)}$$

$$\dot{\phi}_k = \Delta\omega_k - \alpha\, M_k\, [a(t)/a_k(t)]\, \sin[\phi_k(t) - \phi(t)] \quad . \qquad \text{(VII.11)}$$

Rewriting the last equation for another mode k' and subtracting, we find

$$\frac{d}{dt}(\phi_k - \phi_{k'}) = (\omega_k - \omega_{k'}) - \alpha\, a(t)\left[\frac{M_k}{a_k}\sin(\phi_k - \phi) - \frac{M_{k'}}{a_{k'}}\sin(\phi_{k'} - \phi)\right] \quad . \qquad \text{(VII.12)}$$

Except for the special case considered by LSL , where it reduces to (VII.7), this is no longer a phase-locking equation. In constrast with (VII.7), the last term in (VII.12) does not depend only on the difference $\phi_k - \phi_{k'}$, and there is no analogue of the constant locking coefficient. If one tries to save the mode-locking mechanism by going over to the non-linear theory, this is of no avail: clearcut mode-locking effects are not present.

Actually, although the usual explanation in terms of gain narrowing is criticized by LSL for not taking non-linearity into account, their discussion remains within the framework of the linear approximation. In a different sense, this approximation involves nonlinearity, associated with the coupling among modes mediated by the active atoms, through the collective variable $A(t)$.

Direct inspection of the general solution of (IV.1), given by (IV.15), indicates that phase locking is generally not apparent. Although (VII.1) does represent the dominant term above threshold for sufficiently large times, the linear approximation breaks down under these conditions, and one cannot properly associate a spectral line shape with an indefinitely growing exponential term. As has already been mentioned, inclusion of the nonlinear stabilizing terms does not improve the situation: evidence for mode locking remains absent.

One can try to understand the mechanism of line narrowing by examining what happens as the laser threshold is approached from below. Under these circumstances, the linear approximation (IV.15) is well justified, and its last term, given by (VII.1), will usually be dominant, as we have seen in Sect. IV (e.g., for resonant initial conditions such as (IV.19)).

Since $p_o < 0$, we can properly Fourier analyse the electric field associated with (VII.1), (VII.2), and we find that the spectrum is proportional to

$$f_k = \left[(\Delta\omega_k)^2 + p_o^2\right]^{-1} = \left[(\Delta\omega_k)^2 + (\Gamma - \alpha M^2)^2\right]^{-1} \quad . \qquad \text{(VII.13)}$$

Thus, the effective linewidth below threshold is given by $-p_o = \Gamma - \alpha M^2$, i.e., by the quasimode linewidth minus the gain. This approaches zero as the threshold is approached from below ($p_o \uparrow 0$). This result is consistent with the usual interpretation in terms of gain narrowing.

VIII. CONCLUSION

(a) Summary of Results

The LSL model is probably the simplest one where the laser transmission loss can be treated properly, i.e., taking full acount of the continuous nature of the spectrum of field modes. Here we have remained within the framework of semiclassical theory; some comments about the fully quantized treatment will be presented below.

In the linear approximation, our treatment in the continuum is considerably simpler than the corresponding discretized version given by LSL. It is no longer necessary to introduce a fictitious damping medium distributed throughout space to simulate the transmission loss. Furthermore, the discretized treatment leads to a set of linear equations; to discuss its solution requires finding the roots of a complicated transcendental equation[3], and it is difficult to find the general solution for arbitrary initial conditions.

In contrast, the transmission loss is already built in within the continuum treatment. The basic equation in the linear approximation is a separable integral equation that can be solved exactly. The Laplace transform method leads naturally to the general solution for arbitrary initial conditions, allowing one to discuss the effect of the excitation on the field build-up, as we have seen in Sect. IV.

In the nonlinear theory, the crucial observation is due to LSL: the "laser mode" of the usual treatment should be interpreted as a collective concept; correspondingly, the "mode amplitude" is actually a collective variable. When suitable approximations, akin to the Weisskopf-Wigner approach, are made, this collective variable is found to satisfy the usual rotating wave Van der Pol oscillator equation. This is a very reasonable result, since it is only within the range of applicability of the Weisskopf-Wigner-like approximation that the collective variable concept preserves its validity.

Fluctuations are introduced in the semiclassical theory through the stochastic character of the initial conditions, leading to a noise source term in the Van der Pol equation. The continuum treatment leads naturally to this term and also allows us to give an improved discussion of its connection with the fluctuation-dissipation theorem.

The mode-locking mechanism invoked by LSL to explain the narrowness of the laser linewidth is not apparent in the present treatment. If one follows the build-up of the laser oscillation as the threshold is approached from below, the results support instead the usual explanation in terms of a gain-narrowing mechanism.

(b) Fully Quantized Theory

A fully quantized version of the LSL model has been developed[9]. The field quantization is carried out in terms of modes of the continuous spectrum, which are defined throughout space (cf. (II.2)). On the other hand, it is necessary to perform a projection onto the subspace associated with the laser cavity. Indeed, the coupling with the active atoms is confined to this internal region, and we want to obtain the density operator associated with the internal field.

For this purpose, one can introduce collective field operators, analogous to the collective variables employed in the semiclassical theory. These operators are suitably weigthed and normalized superpositions of the usual creation and annihilation operators for modes of the continuous spectrum, extended over each single Fox-Li band. They may be interpreted as creation and annihilation operators of photons associated with each collective quasimode.

An indirect procedure, similar to that employed by Bonifacio and Lugiato[10] in the theory of superradiance, is

then adopted. One wants to show that the density operator associated with the internal field, in the absence of active atoms, obeys an effective evolution equation equivalent to the usual one in the quantum theory of damping, where the dissipation operator arises entirely from the transmission losses. For this purpose, one applies the reconstruction theorem[2], by showing that this equation yields the correct set of coherence functions of all orders. This is only true, however, within the domain of validity of the exponential decay law for these temporal coherence functions.

The interaction with the active atoms is then introduced, and one recovers the results of the usual treatment, with, however, a basic difference in the interpretation: they apply to collective variables, rather than to the discrete modes of a lossless cavity.

Both in the semiclassical and in the fully quantized theory, therefore, the range of validity of the quasi-mode concept is limited to the domain of applicability of the Weisskopf-Wigner exponential decay approximation. For the situations in which one is ordinarily interested, corresponding to very long-lived quasimodes, this is not a serious restriction. For short-lived quasimodes, however, the situation would have to be reexamined.

(c) Outlook

Within the framework of the LSL model, some of the questions posed in the Introduction have been answered.

However, one must not forget the limitations that are inherent to the model. It is a one-dimensional model, in the sense that the field is taken to depend only on the longitudinal coordinate. A more realistic three-dimensional model of transmission loss would be far more difficult to treat.

Other restrictions arise from the approximations employed both in the formulation of the problem and in its solution. In particular, the self-consistent field method corresponds to a mean field approximation. The removal of some or all of these restrictions seems to be a necessary prerequisite for a meaningful discussion of the laser threshold region and of the phase transition analogy.

APPENDIX A - NORMAL MODE SPECTRUM

In this Appendix we obtain the spectrum of normal modes of propagation of our system, defined by fig.1 and (II.1). We consider only longitudinal modes, with $\vec{E}=E(z,t)\vec{i}$, $\vec{H}=H(z,t)\vec{j}$. For monochromatic waves of circular frequency ω_k,

$$E(z,t) = U_k(z)\exp(-i\omega_k t), \qquad \text{(A-1)}$$

we find that Maxwell's equations yield

$$U''_k(z) + k^2U_k(z)=0, \qquad k=\omega_k/c, \qquad \text{(A-2)}$$

subject to the boundary conditions $U_k(\ell)=0$ and

$$U_k(0+) = U_k(0-) = U_k(0), \qquad \text{(A-3)}$$

$$U'_k(0+) - U'_k(0-) = -\eta k^2U_k(0), \qquad \text{(A-4)}$$

where the discontinuity in (A-4) arises from the delta function term in (II.1).

The solutions, subject to the usual continuous spectrum normalization (II.3), are of the form (II.2). Substitution into the boundary conditions (A-3) and (A-4) yields

$$\sin\delta_k = (\pi/2)^{1/2} L_k \sin(k\ell), \qquad \text{(A-5)}$$

$$(\pi/2)L_k{}^2 = (1+\Lambda^2\sin^2 k\ell - \Lambda\sin 2k\ell)^{-1}, \tag{A-6}$$

where

$$\Lambda = \Lambda(k) = \eta k. \tag{A-7}$$

Setting

$$t = \tan(k\ell), \tag{A-8}$$

we find

$$(\pi/2)L_k^2 = (1+t^2)/\left[t^2+(\Lambda t-1)^2\right]. \tag{A-9}$$

The function L_k^2 has peaks at the points

$$t_n = \tan(k_n\ell) = \Lambda^{-1}(k_n) = (\eta k_n)^{-1} \equiv \Lambda_n^{-1} \ll 1, \tag{A-10}$$

where we assume, from now on, that the window transparency is very small (cf. (II.6)). The peak values of L_k^2 are given by

$$(\pi/2)\left[L_k(t_n)\right]^2 = 1 + \Lambda_n^2 \approx \Lambda_n^2, \tag{A-11}$$

and the peak halfwidths are

$$\Delta t_n = 1/\Lambda_n^2 \approx \ell\Delta k_n. \tag{A-12}$$

The resonance frequencies k_n are the roots of the transcendental equation (A-10). In the optical range we have

$$k_n\ell = n\pi + \theta_n, \quad n \gg 1, \quad |\theta_n| \ll 1. \tag{A-13}$$

Accordingly, (A-10) may be solved by iteration, with the following result:

$$k_n \approx k_{on} = (n\pi + \Lambda_{on}^{-1})/\ell = \omega_{on}/c, \tag{A-14}$$

where

$$\Lambda_{on} = \eta n\pi/\ell. \tag{A-15}$$

Note that the quasimode resonance frequencies ω_{on} are shifted, with respect to the modes of a totally reflecting cavity, by $\Delta\omega_{on} = c/\Lambda_{on}\ell$.

Substituting the above approximations in (A-9) and setting (cf. (A-12))

$$\Gamma_n = c/\Lambda_{on}^2\ell, \tag{A-16}$$

we find, in the neighborhood of $\omega_k = \omega_{on}$,

$$L_k^2 \approx M_k^2 = (2/\pi)\Gamma_n^2\,\Lambda_{on}^2/\left[(\omega_k-\omega_{on})^2 + \Gamma_n^2\right], \tag{A-17}$$

which shows a Lorentzian line shape with linewidth Γ_n given by (A-16). The dependence of Γ_n on n follows from (A-15):

$$\Gamma_n = c\ell/(\pi\eta n)^2, \tag{A-18}$$

which is a slowly-varying function of the resonance frequency within the optical domain.

The result (A-16) for the quasimode linewidth has a simple physical interpretation. The transmissivity T of the laser window (defined as the ratio of transmitted to incident intensity for plane wave incidence) can readily be computed from the boundary conditions (A-3) and (A-4), with the following result:

$$T = (1 + \Lambda^2/4)^{-1} \approx 4/\Lambda^2 \;\; (<<1), \tag{A-19}$$

since we assume $\Lambda >> 1$.

After N double traversals of the laser cavity, which take a time $t=2N\ell/c$, the intensity damping factor is

$$(1-T)^N \approx \exp(-NT) = \exp(-cTt/2\ell).$$

By (IV.20), this should equal $\exp(-2\Gamma t)$, so that

$$\Gamma = cT/4\ell = c/\Lambda^2\ell,$$

in agreement with (A-16).

From (A-5), (A-8) and (A-9), we also get

$$\sin\delta_k = t/\left[t^2+(\Lambda t-1)^2\right]^{1/2}, \tag{A-20}$$

$$\cos\delta_k = (\Lambda t-1)/\left[t^2+(\Lambda t-1)^2\right]^{1/2}. \tag{A-21}$$

With the same approximations employed in (A-17), i.e., within the resonance band centered at $\omega_k=\omega_{on}$, these results become

$$\sin\delta_k \approx \Gamma_n/\left[(\omega_k-\omega_{on})^2 + \Gamma_n^2\right]^{1/2}, \tag{A-22}$$

$$\cos\delta_k \approx (\omega_k-\omega_{on})/\left[(\omega_k-\omega_{on})^2 + \Gamma_n^2\right]^{1/2}. \tag{A-23}$$

The element S(k) of the S-matrix associated with (II.2) is given by

$$S(k) = \exp(2i\delta_k) = \frac{\Lambda t-1-it}{\Lambda t-1+it}. \tag{A-24}$$

The poles k_n of the S-matrix are the roots of

$$\Lambda t - 1 + it = 0, \tag{A-25}$$

where t is replaced by (A-8).

By making the same approximations already employed above, we find that

$$\omega_n = ck_n \approx \omega_{on} - i\Gamma_n, \tag{A-26}$$

where

$$\omega_{on} = (c/\ell)(n\pi+\Lambda_{on}^{-1}), \quad (A-27)$$

$$\Gamma_n = c/\Lambda_{on}^2\ell, \quad (A-28)$$

which coincide with (A-14) and (A-16), respectively. Thus, the resonance frequencies and the widths of the laser quasimodes are respectively given by the real and imaginary parts of the poles of the S-matrix, as expected[2]. This agrees with the physical interpretation of the quasimodes in terms of "decaying states"[11].

APPENDIX B - THE COEFFICIENTS $a_{k\mu}, b_{k\mu\rho\sigma}$

The main difference between the calculation of these coefficients, defined by (III.13), and the corresponding calculation for a discretized spectrum[12,1], lies in the substitution of sums by integrals; e.g., (III.3) replaces the discrete expansion $E(z,t) = \sum_k A_k(t)U_k(z)$, with suitable changes in normalization.

The following simplifying assumptions are made: (i) Atomic motion is neglected; (ii) The population inversion density N(z) is assumed to be time independent; (iii) The rotating wave approximation is employed; (iv) The Fox-Li resonance frequency ω_o coincides with the atomic transition frequency (central tuning).

Under these conditions, we find

$$a_{k\mu} = -id^2N(k-\mu)/\hbar\gamma_{ab}, \quad (B-1)$$

where $N(k-\mu)$ is defined by (III.19) and γ_{ab} and d are defined following (III.17). Since N vanishes outside the laser cavity and we may assume that $|k-\mu|\ell<<1$ (cf. (III.25), (III.26)), we get

$$N(k-\mu) \simeq \frac{1}{2}\ell\bar{N}\, M_k M_\mu, \quad (B-2)$$

where $\bar{N}$ is the space average of N(z).

Substituting (B-2) in (B-1), we are led to (III.14). Similar considerations lead to (III.15).

FOOTNOTES AND REFERENCES

* Supported by CAPES.

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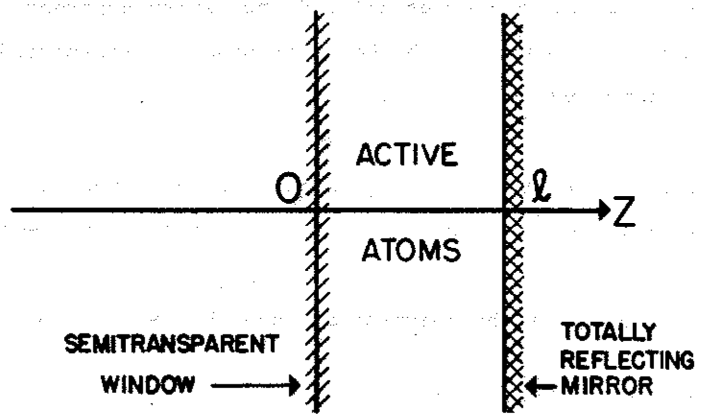
FIGURE CAPTIONS

FIG.1 - The laser cavity is the region $0 < z < \ell$ between the totally reflecting mirror at $z=\ell$ and the semitransparent window at $z=0$, through which the laser beam is transmitted to the outside region $-\infty < z < 0$.

FIG.2 - Plot of L_k^2 as a function of ω_k, showing Fox-Li quasimode resonances ω_{on} with halfwidths $\Gamma_n <<$ spacing $c\pi/\ell$ between resonances. The thin vertical lines represent the discretized version of the spectrum employed by LSL.


ACTIVE
ATOMS
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WINDOW
TOTALLY
REFLECTING
MIRROR

Fig. 1

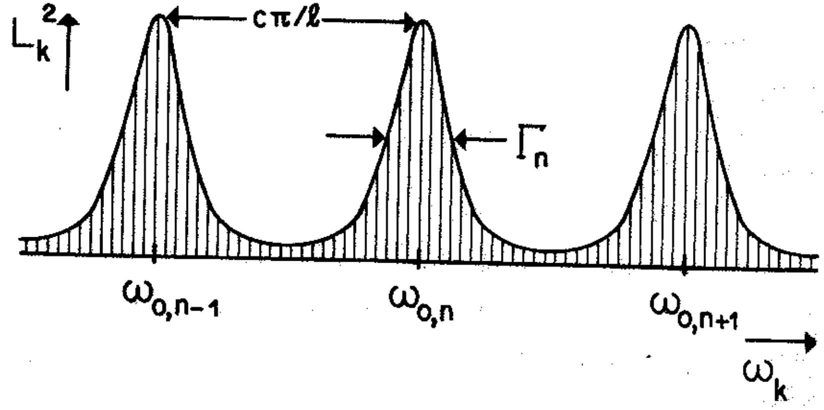
L_k^2
$c\pi/\ell$
Γ_n
$\omega_{o,n-1}$
$\omega_{o,n}$
$\omega_{o,n+1}$
ω_k

Fig. 2