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Abstract: We study periodic solutions of the equation $\frac{1}{\tau}\dot{X}(t) + X(t) = f(X(t-1))$, with $f(X)$ given by $f_1(X) = AX(1-X)$ or $f_2(X) = \pi\mu(1-sinX)$, grouped in some sets characterized by different dominant frequencies. Numerical results with $f(X) = f_1(X)$ are given. Varying the parameter τ we find a period-doubling cascade (possibly complete) in one of these sets, with reasonable agreement with the contraction rate of Feigenbaum. We show that, at least partially, many other period-doubling cascades occur in other sets, at different values of τ, and these cascades "tend" to satisfy the same contraction rate of Feigenbaum. Based on results of Ikeda, Kondo & Akimoto (1982), Ikeda & Matsumoto (1987) and Mallet-Paret (1988) we obtain a lower bound for the parameter value for the existence of more complex dynamics. We conjecture that this fact is related to the violation of the so called "negative-feedback condition".

1) Introduction

Most of the physical systems that have been studied have their evolution determined only by their present state. This is the case of classical mechanics. However, a great deal of dynamical phenomena has its evolution determined not only by its present but also by its past. This kind of system appears, for example, in particle dynamics with eletromagnetic interaction (Bel (1982), Driver (1963)), in many biological systems (Mackey & Glass (1977), Hadeler (1979)) in optical systems (Ikeda, Kondo & Akimoto (1982)), and in many other fields.

Quite often, the dependence on the past that appears in realistic models is quite complicated. In certain cases it is possible to restrict the dependence on the past to a single time delay, which is one of the simplest cases of past dependence. In this paper we shall study a particular kind of scalar delayed differential equation:

$$\frac{1}{\tau}\dot{X}(t) + X(t) = f(X(t-1)), \tag{1}$$

where f(x) is unimodal and τ is the time delay. Among the practical applications of equation (1) we shall be concerned with two particular ones. First, a model for population dynamics of an isolated species of fly with limited food supply (Perez, Malta & Coutinho (1978)). In particular, we are going to use an expression proposed by Maynard Smith (1968), where:

$$f(X) = f_1(X) = AX(1-X). \tag{2}$$

Second, a model for the dynamics in a nonlinear optical cavity (Ikeda, Kondo & Akimoto (1982)). In this case we have:

$$f(X) = f_2(X) = \pi\mu[1 - sin(X)]. \tag{3}$$

Although (1) is a first order differential equation, the initial information which defines a solution of (1) is a function $\phi(t)$ defined in some unitary interval. Given $\phi(t), t \in [-1,0]$,one way to solve this equation is to integrate it step by step in the intervals $[t+n, t+n+1]$, imposing continuity at the boundary of each interval. This procedure leads to a function map defined by (1) as:

$$X_{n+1}(\theta) = X_n(1)e^{-\tau\theta} + \tau\int_0^\theta e^{-\tau(\theta-s)}f(X_n(s))ds, \tag{4}$$
$$0 \le \theta \le 1, \qquad n = 0,1,2,...,$$

where: $X_n(\theta)$, is a continuous function defined on $[0,1]$, $X_0(\theta) = \phi(\theta+1)$.

The solution of (1) with $X(t) = \phi(t)$ for $t \in [-1,0]$, is given by $X(t) = X(\theta+n-1) = X_n(\theta)$, for $t = \theta+n-1$, $t > 0$, $n = 0,1,2,\ldots$, $\theta \in [0,1]$.

The above procedure makes clear that we are dealing with a system of infinite dimensional "phase space" (Hale (1977)). The state of the system at each instant t is given by the function $X(t-\theta), \theta \in [0,1]$. Equation (1), with f(X) given by (2) or (3), may have a very complex behaviour with chaotic solutions and so on, and due to its reasonably simple formal solution (4) it is a good paradigm for studying non-trivial dynamics in systems with infinite degrees of freedom (see Ikeda & Matsumoto (1987)).

This paper is organized as follows. Next section begins with a presentation of the very interesting results of Mallet-Paret (1988), that impose severe restrictions on the behaviour of the asymptotic solutions of (1) for a certain class of f(X). We then present a method for determining boundaries of solutions of (1) and make applications using (2) and (3). We establish the ranges of A and μ (appearing in (2) and (3)) for which the Mallet-Paret results are valid.

In section 3 we briefly discuss the phenomenology of the solutions of (1) with $f(X)$ given by (2) and (3) (see Ikeda, Kondo & Akimoto (1982); Ikeda & Matsumoto (1987)), stressing that this equation has many simultaneously stable solutions. It is possible to group a great number of these solutions in some sets called "branches" (Ikeda, Kondo & Akimoto (1982), Ikeda & Matsumoto (1987)). When the Mallet-Paret (1988) results hold for the solutions of (1), these "branches" plus the connections between them exhaust the "global solution phase space" (or the set of solutions that are defined for $t \in (-\infty,+\infty)$) and that attracts the solution of all initial value problem). We then present a numerical study of the bifurcation structure of some stable periodic solutions of branch 1 (or fundamental branch, following Ikeda, Kondo & Akimoto (1982)) for equation (1) with $f(X)$ given by (2). There are evidences on behalf of the conjecture that this equation has a full period-doubling cascade as we vary τ or A separately, satisfying the Feigenbaum contraction rate (Feigenbaum (1980)) on both τ and A.

In section 4 we show that the period-doubling bifurcations that we find in branch 1 imply many other period-doubling bifurcations in other branches for different values of τ. We also show that these other period-doubling bifurcations "tend" to satisfy the contraction rate of Feigenbaum. This result is interesting because it establishes some relation between the bifurcation structure within branches for different values of τ.

We conclude with section 5, stressing that the dynamics of (1), with f(x) given by (2) or (3), can be understood in two levels. First the dynamics within the branches, as we have done in section 3. And second the dynamics "between branches", as, for example, stated by the theorem of Mallet-Paret (1986) (see also Ikeda & Matsumoto (1987)). From our estimates of the validity of the theorem of Mallet-Paret for (1) with f(X) given by (3), and the numerical results of Ikeda, Kondo & Akimoto (1982), Ikeda & Matsumoto (1987), we conjecture that the "merging phenomenon" (the fusion of solutions of different branches) observed in the second work is related to the violation of the conditions of the theorem of Mallet-Paret.

2)"Negative-feedback" condition, Morse Decomposition, and boundary of oscillatory solutions.

We shall begin by presenting some results of Mallet-Paret (1988) which are formulated for the scalar equation,

$$\dot{X}(t) = g(X(t), X(t-1)), \tag{5}$$

with the requirement that the function $g(\xi,\eta)$ satisfies the negative-feedback condition, that means:

$$\begin{cases} \eta g(0,\eta) > 0, & \text{for all} \quad \eta \neq 0, \\ \dfrac{\partial g(\xi,\eta)}{\partial \eta}|_{(0,0)} > 0. \end{cases} \tag{6}$$

Before stating the results of Mallet-Paret it is necessary to introduce some definitions. First, we say that the phase space Ξ of a given dynamical system admits a Morse Decomposition (Conley (1978), Mallet-Paret (1988)), if it is possible to define a finite and ordered collection $S_1 < S_2 < S_3 < \ldots S_M$ of compact invariant subsets (called Morse Sets) of Ξ , such that if $\alpha(x)$ and $\omega(x)$ are the alpha and the omega limit sets, respectively, (see Guckenheimer & Holmes (1983)) of the orbit through x, $x \in \Xi$, then there exists $N \geq K$ such that $\alpha(x) \subseteq S_N$ and $\omega \subseteq S_K$; furthermore, if $N = K$ then the orbit through x is contained in S_N. The Morse Sets S_N, together with the connecting orbit sets $C_K^N = \{x \in \Xi | \, \alpha(x) \subseteq S_N \quad \text{and} \quad \omega(x) \subseteq S_K\}$ for $N > K$, exhaust the phase space Ξ.

Now it is necessary to define the functional $V[X(t)]$ that , to each global solution of (5) and $t \in (-\infty, +\infty)$, associates an integer number greater than zero in the following way:

- let σ be the first zero of $X(\theta)$ for $\theta \in [t, \infty)$;

- then $V[X(t)]$ is equal to the number of zeros (counting multiplicity) of $X(\theta)$ for $\theta \in (\sigma-1, \sigma]$.
- if σ does not exist then $V[X(t)] = 1$;

Mallet-Paret (1988) proved that for global solutions of (5), with $g(\xi, \eta)$ satisfying the negative-feedback condition (6), if $t_1 \leq t_2$ then $V[X(t_1)] \geq V[X(t_2)]$. This means that the number of zeros of a global solution of (5), per unit interval, that contains a zero on its right boundary, is non increasing. He also showed that $V[X(t)]$ never assumes even integer values and that it is a bounded function.

Let us state the main result of Mallet-Paret (1988) in another way. Let us call Ψ the set of global solutions of (5), and let us define the sets:

$$S_N = \{X \in \Psi - \{0\} \mid V[X(t)] = N \quad \text{for all} \quad t \in \mathcal{R}, \quad \text{and} \quad 0 \notin \alpha(X) \cup \omega(X)\} \qquad N = 0, 1, 2, \ldots$$

If the origin is hyperbolic we also define the set $S_{N^*} = \{0\}$ where N^* is the dimension of the unstable manifold of the origin. Using these definitions Mallet-Paret (1988) proved that the space of global solutions of (5) has a Morse decomposition, where the sets S_N are the Morse sets. It is important to point out that the set Ψ attracts all the initial value problem of (5) (Mallet-Paret (1988), Hale (1977)) and, therefore, it contains all the relevant information for asymptotic studies.

Now, we want to apply the Mallet-Paret results to equation (1) with f(X) given by (2). First of all we have to make a trivial translation on X such that the point X_{eq}, defined by the nontrivial root of $X = f_1(X)$, will be placed at the origin (see figure 1).

From fig. 1b we notice that the function $g(\xi, \eta) = -\xi + (-A+2)\eta - A\eta^2$ does not satisfy the condition (6). Nevertheless, if we show that the asymptotic solutions $X_{as}(t)$ of (1), with $f(X)$ given by (2), for a certain class of initial functions, satisfy $X_{as}(t) > 1 - X_{eq}$ or, equivalently, $Y_{as} > Y_c$(see figure 1), then we can trivially redefine $f_1(X)$ as:

$$f_1^*(X) = \begin{cases} f_1(X) & if \quad X \geq 1 - X_{eq} + \epsilon, \\ f_1(1 - X_{eq} + \epsilon) & if \quad X < 1 - X_{eq} + \epsilon, \end{cases} \tag{7}$$

which then satisfies the negative-feedback condition (6).

In order to follow the program above, let us restrict our attention to equation (1), with $f(X)$ given by (2), with initial functions $\phi(t)$ satisfying $0 < \phi(t) < 1, t \in [-1, 0]$ (we will denote by $X(t, \phi)$ the solution X(t) with initial condition $\phi(t)$). This condition ensures that the solutions remain in the interval I = (0,1) for $0 < A < 4$. In fact,

i) let $t', t' > 0$, be the first instant of time at which $X(t', \phi) = 0$ or $X(t', \phi) = 1$;
ii) if $X(t', \phi) = 0$, then from (1), with $f(X)$ given by (2), $\dot{X}(t') = f_1(X(t'-1)) > 0$, so that $X(t', \phi)$ is increasing when it goes through zero, which is impossible by hypothesis;

iii) if $X(t', \phi) = 1$, then from (1) with $f(X)$ given by (2), $\dot{X}(t') = -1 + f_1(X(t' - 1))$, but as $f_1(X) \subset I$, for $0 < X < 1$, and $0 < A < 4$, then $\dot{X}(t') < 0$, which is again impossible.

As the solutions remain bounded they can be divided in two types. Either they tend monotonically to one fixed point, in this case $X = X_{eq}$ or $X = 0$; or they oscillate. The first case is not generic if the fixed points are unstable. Let us analyse the second case.

It is easy to see from (4) that the solution $X(t, \phi)$, $t > 0$, becomes smoother at each integration step (Hale (1977), Bellman & Cooke (1963)). Therefore we can say that if $t > 1$, then at each extremum of $X(t, \phi)$ we have $\dot{X}(t) = 0$. Now let $X(t_1)$ be the first maximum of $X(t, \phi)$ for $t > 1$. At this point, from (1) with $f(X)$ given by (2), we have:

$$X(t_1) = f_1(X(t_1 - 1)) < f_{1\,max},$$

where $f_{1\,max}$ is the maximum of f_1, i.e., $A/4$. All other maxima of $X(t, \phi)$ for $t < t_1$ will satisfy the same property. Now let $X(t_2)$ be the first maximum of $X(t, \phi)$ for $t > t_1 + 1$, then we have:

$$X(t_2) = f_1(X(t_2 - 1)) > f_1(f_{1\,max}),$$

see figure 1. Hence it follows that

$$f_1(f_{1\,max}) < X(t, \phi) < f_{1\,max}, \qquad for\, t > 1.$$

Therefore, if:

$$f_1(f_{1\,max}) = f_1(A/4) > 1 - X_{eq} \qquad \text{is true,} \tag{8}$$

then it is possible to redefine $f_1(X)$ as in (7), with f_1^* satisfying condition (6) (see figure 1a).

From the inequality (8) we get the condition:

$$A < 3.67857\ldots,$$

so that the set Ψ, of the global solutions of (1) with $f(X)$ given by(2), admits a nontrivial Morse Decomposition as stated in the introduction.

The procedure above can be easily generalized for (1) with other functions $f(X)$ (unimodal or not). For example for the equation (1) with $f(X)$ given by (3), we have that if condition $f_2(0) < \pi - X_{eq}$ is satisfied (see figure 2), then it is possible to apply Mallet-Paret results. This condition implies that

$$\mu < 0.760\ldots.$$

Notice that if (1) satisfies the negative-feedback condition, the result of Mallet-Paret gives a good description of the general features of the global dynamics of (1). When this condition is not satisfied the behaviour of the system can be more complicated. An example of a phenomenon that cannot occur if the system satisfies the negative-feedback condition is the connection between the fundamental solution and its first odd harmonic, (Ikeda, Kondo & Akimoto (1982)) and it can occur when the system does not satisfy this condition, as found by Ikeda , Kondo & Akimoto (1982) on their study of equation (1) with $f(X)$ given by (3). As a matter of fact our estimate $\mu = 0.760\ldots$ constitutes a lower bound for the existence of this connection. Ikeda, Kondo & Akimoto (1982) found it numerically for the parameter $\mu = 0.775\ldots$, a value not far from the lower bound obtained by us.

3)Phenomenology of Periodic Solutions and Bifurcations

For some values of the parameters A and μ it is not difficult to find numerical periodic solutions of (1) with $f(X)$ given by (2) or (3), respectively (rigorous proof of existence of such periodic solutions may be found in the mathematical literature; see for example Mallet-Paret (1988)). If equation (1) has a periodic solution $X_0(t)$, with period p_0, for some value $\tau = \tau_0$ then we can write:

$$\frac{1}{\tau_0}\dot{X}_0(t) + X_0(t) = f(X_0(t-1-np_0)). \tag{9}$$

Making the transformations $t = t'(1+np_0)$ and $X_n = X_0(t(1+np_0))$ we get:

$$\frac{1}{\tau_0(1+np_0)}\dot{X}_n(t) + X_n(t) = f(X_n(t-1)). \tag{10}$$

Therefore, if $X_0(t)$ is a solution of (1) for $\tau = \tau_0$ with period $p = p_0$, then $X_n(t)$ is also solution of (1) for $\tau = \tau_n = \tau_0(1+np_0)$ with period $p = p_n = \frac{p}{1+np}$. This relation does not say anything about the stability of the solutions $X_0(t)$ and $X_n(t)$. Probably $X_0(t)$ is stable and $X_n(t)$ unstable (numerical evidence). Though, for small values of τ, there may be situations in which both $X_0(t)$ and $X_n(t)$ are stable (numerical evidence).

When the function $f(X)$ in (1) satisfies the negative-feedback condition (6), the solutions for different values of τ, $X_n(t)$ and $X_m(t)$, are in different Morse Sets.

The above procedure is independent of the negative-feedback condition (6), hence we are going to group the solutions, obtained by re-scaling some specific periodic solutions, in sets called "branches" (following Ikeda, Kondo & Akimoto (1982)).

We know that equation (1), with f(X) given by (2) or (3), for some values of τ, A (μ), has periodic solutions, with period greater than two, originated from the Hopf bifurcation of the nontrivial fixed point. If we apply the above scaling procedure to these solutions, $X_0(t) \longrightarrow X_{n-1}(t)$, we can define many other periodic solutions for different values of τ. The branch n is constituted by the solution $X_{n-1}(t)$ plus all other solutions that have "approximately" the same dominant frequency of X_{n-1} for the same values of parameters (see Ikeda, Kondo & Akimoto). It is important to note that when the negative-feedback condition (6) is satisfied the branches coincide with the Morse Sets S_n, and they are better characterized in this way. When the negative-feedback condition is not satisfied we can decide if a solution belongs or not to some branch analysing its power spectrum. Notice that in the latter case solutions can exist that cannot belong to any branch or that belong to two different branches (they have characteristic frequencies from both branches) as it occurs in the "merging phenomenon" (fusion of two different branches) mentioned by Ikeda & Matsumoto (1987).

Given this "definition" of branch we are going to present some results about the bifurcation structure of the branch 1 of equation (1) with $f(X)$ given by (2).

First of all, using a numerical procedure based on (4), we integrate (1) with $f(X)$ given by (2), for $3 < A < 4$ and $\tau_e < \tau < \tau_\infty$, where τ_e is the value of τ where the fixed point X_{eq} becomes unstable and τ_∞ is the estimated Feigenbaum's accumulation point (see below).

What we observe is that period-doubling is an important kind of bifurcation that occurs in periodic solutions, and that for a wide range of τ (A), fixed A (τ), we find a period-doubling cascade. As a matter of fact we could detect, in some cases, up to five period doublings for the same parameter, τ or A. Our results are summarized in figure 3 and table 1.

Our results agree with the conjecture that equation (1) with $f(X)$ given by (2) has a full period-doubling cascade in both directions τ ($A = const.$) and A ($\tau = const.$) for certain ranges of τ and A (see Chow & Green (1985); Derstine et al. (1983); Gao et al. (1983); Gao, Yuan & Narducci (1983); Gao et al. (1984); Ikeda, Kondo & Akimoto (1982); de Oliveira & Malta (1987)). In other words, one of the possible "routes to chaos" in this system may be the Feigenbaum's route (Feigenbaum (1980)). It is interesting to point out that there is an argument of Mallet-Paret & Nussbaum (1986) on behalf of a truncation of the period-doubling cascade, at least for large τ, before the point of period-doubling accumulation. The support for this argument is that the solutions of (1) may have some singularities for large τ (see figure 4), and that these singularities may work as a noise. This noise would be responsible for the premature truncation of the period-doubling cascade (see Crutchfield & Huberman (1980)). If this truncation existed our numerical precision would not allow for its detection. It should be remarked that the phenomenology of the stable periodic solutions belonging to other branches are much more complicated than that of branch 1. Branches 2, 3,... also exhibit period-doubling bifurcations giving rise to solutions called "isomers" by Ikeda & Matsumoto (1987). There can exist many simultaneously stable isomers and the bifurcation structure that leads to irregular solutions is not well understood. In the next section we present some implications to other branches of our results for the branch 1.

4) Relations between the bifurcation structure in different branches.

We can apply the same transformations used in the beginning of section 3 (see (9), (10)) to study the implications to other branches of the period-doubling bifurcations of solutions of (1) belonging to a certain branch. First let us verify the implications of the first period-doubling bifurcation in branch 1, that happens at $\tau = \tau^*_{1,2}$ for fixed A. For simplicity we shall assume that for $\tau = \tau^*_{1,2}$ there exist two solutions, $X_{1,1}(t)$ and $X_{1,2}(t)$ (the first index labels the branch and the second the order of the solution in the period-doubling cascade), the first with period $p_{1,1} = p$ and the second with period $p_{1,2} = 2p$. Making the same kind of transformation that we made to get equation (10) from (9), we define new solutions (indicated by the prime):

$$\begin{cases} X'_{n,1}(t) = X_{1,1}((1+(n-1)p_{1,1})t), \\ \tau'^*_{n,1} = \tau^*_{1,2}(1+(n-1)p_{1,1}), \\ p'_{n,1} = \dfrac{p_{1,1}}{1+(n-1)p_{1,1}} \qquad for\ n = 1,2,3,\ldots \end{cases} \tag{11}$$

$$\begin{cases} X'_{n,2}(t) = X_{1,2}((1+(n-1)p_{1,2})t), \\ \tau'^*_{n,2} = \tau^*_{1,2}(1+(n-1)p_{1,2}), \\ p'_{n,2} = \dfrac{p_{1,2}}{1+(n-1)p_{1,2}} \qquad for\ n = 1,2,3,\ldots \end{cases} \tag{12}$$

The solutions $X'_{n,1}$ defined by (11) are obviously in the branch n due to our definition of branch. But we do not know to which branch belong the solutions $X'_{n,2}$. As $X_{1,1}(t)$ and $X_{1,2}(t)$ are exactly the same function at $\tau = \tau^*_{1,2}$, we can do a simple analysis of the relations between $X'_{n,2}(t)$ and $X'_{n,1}(t)$. Remembering that $p_{1,2} = 2p$ and $p_{1,1} = p$ we have:

$$X'_{n,2}(t) = X_{1,2}((1+(n-1)2p)t) = X_{1,1}(1+(2n-2)p)t)$$
$$X'_{m,1}(t) = X_{1,1}((1+(m-1)p)t).$$

Now, if $m-1 = 2n-2$, so that $m = 2n-1$, then $X'_{m,1}(t) = X'_{n-1}(t)$. This means that each function $X'_{n,2}(t)$ is related to a function $X'_{2n-1,1}(t)$, $n = 1,2,3,\ldots$, for $\tau = \tau^*_{1,2}(1+2(n-1)p_{1,1})$, in the same way as $X_{2,1}(t)$ is related to $X_{1,1}(t)$. In other words, the first period-doubling bifurcation in branch 1 is directly related to the "first" period-doubling bifurcation in the odd numbered branches.

At this point it is convenient to generalize our notation. We shall assume that solutions of equation (1) have a period-doubling cascade in branch 1 as determined in the last section for $f(X)$ given by (2). A solution of (1) belonging to branch n and corresponding to the $j^{\underline{th}}$ period-doubling in this branch will be denoted by $X_{n,j}(t)$. Notice that we are not claiming that all these $X_{n,j}(t)$ exist. We only claim that at least some of these functions exist. The period of $X_{n,j}$ and the values of the parameter τ at which the solution exists will be denoted by $p_{n,j}$ and $\tau_{n,j}$ respectively. The particular value of the parameter $\tau, \tau = \tau^*_{n,j}$, corresponds to the point at which the $j\underline{th}$ period-doubling bifurcation occurs.

With this notation, and using the result that the first period-doubling bifurcation of branch 1 implies the first period-doubling bifurcation in the odd numbered branches, we can rewrite expressions (11) and (12) as:

$$\begin{cases} X_{n,1}(t) = X_{1,1}((1+(n-1)p_{1,1})t), \\ \tau^*_{n,1} = \tau^*_{1,1}(1+(n-1)p_{1,1}), \\ p_{n,1} = \dfrac{p_{1,1}}{1+(n-1)p_{1,1}} \qquad for\ n = 1,2,3,\ldots \end{cases}$$

$$\begin{cases} X_{2n-1,2}(t) = X_{1,2}((1+(n-1)p_{1,2})t), \\ \tau^*_{2n-1,2} = \tau^*_{1,2}(1+(n-1)p_{1,2}), \\ p_{2n-1,2} = \dfrac{p_{1,2}}{1+(n-1)p_{1,2}} \qquad for\ n = 1,2,3,\ldots \end{cases}$$

Our procedure can be generalized for the second, third,..., period-doubling bifurcations. In the case above the period-doubling bifurcation in branch 1 does not have correspondence in branch 2 because, even though the solution $X_{1,1}(t)$ has a correspondence in branch 2, the solution $X_{1,2}(t)$ does not have a direct correspondence in branch 2. At the second period-doubling bifurcation in branch 1 we have the same phenomenon. Even though the solution $X_{1,2}(t)$ has a correspondence in the branches 3, 5, 7, 9,..., the solution $X_{1,3}(t)$ has correspondence only in the branches 5, 9,..., and so on. This is illustrated in figure 5.

The relations displayed in figure 5 are given by (after some index manipulations):

$$\begin{cases} X_{1+2(-(n-1)),j}(t) = X_{1,j}((1+(n-1)p_{1,j})t), \\ \tau^*_{1+2(-(n-1)),j} = \tau^*_{1,j}(1+(n-1)p_{1,j}), \\ p_{1+2(-(n-1)),j} = \dfrac{p_{1,j}}{1+(n-1)p_{1,j}} \qquad for\ n=1,2,3,\ldots \end{cases} \tag{13}$$

The relations (13) are appropriate for checking the contraction rate of the parameter intervals of period-doubling in a branch L, $L = 1+2^{(M-1)}$, where L is a large number. In other words, we want to answer the question:

$$\begin{aligned} &\text{If} \quad \frac{\tau^*_{1,j}-\tau^*_{1,j-1}}{\tau^*_{1,j+1}-\tau^*_{1,j}} \approx \delta = 4.6692\ldots \quad \text{for large } j, \\ &\text{is it true that} \quad \frac{\tau^*_{L,j}-\tau^*_{L,j-1}}{\tau^*_{L,j+1}-\tau^*_{L,j}} \approx \delta \quad \text{for large } j? \end{aligned} \tag{14}$$

From (13) we have the relations:

$$\begin{cases} \tau^*_{1+2(-2(-,j-1} = \tau^*_{1,j-1}(1+2^{(M-j+1)}p_{1,j-1}), \\ \tau^*_{1+2(-2(-,j} = \tau^*_{1,j}(1+2^{(M-j)}p_{1,j}), \\ \tau^*_{1+2(-2(-,j+1} = \tau^*_{1,j+1}(1+2^{(M-j-1)}p_{1,j+1}), \end{cases} \tag{15}$$

As $\tau^*_{1,j-1}$ is very close to $\tau^*_{1,j+1}$ (j is large) it is reasonable to disregard the continuous variation of the period due to the variation of τ and to assume that:

$$p_{1,j-1} \approx 2^{-1}p_{1,j} \quad and \quad p_{1,j+1} \approx 2p_{1,j}. \tag{16}$$

Substituting (16) in (15) it is easy to check that (14) is true. Therefore, if branch 1 exhibts a full period-doubling cacade satisfying the Feigenbaum's contraction rate, then branch $1+2^{(M-1)}$, M large, will exhibit at least $M-1$ period-doubling bifurcations, and for large n, $n < (M-1)$, the relation $\delta_n \approx 4.6692\ldots$ will be satisfied.

It should be remarked that if there exists a period-doubling cascade in branch 2 (we have numerical evidences that this is true), the same procedure can be applied to it, and analogous correspondence will be found between bifurcations in branch 2 and bifurcations in the even numbered branches. For instance, the first period-doubling bifurcation of branch 2 will correspond to the first period-doubling bifurcation in branches 4, 6, 8,...; the second period-doubling will have correspondence in branches 6, 10, 14,..., and so on. As a matter of fact, the above

procedure can be applied to any member of the period-doubling cascade of any branch.

One of the difficulties of studying the stable solution of higher order branches is the small domain of stability of these solutions. Moreover period-doubling cascade of unstable solutions may exist and in this case we need a numerical scheme different from that used in section 3 to follow it. We have made numerical tests of the stability of the solutions determined by (13) in higher order branches. We concluded that sometimes there is a positive correlation between the stability of the solutions of branch 1 and the corresponding solutions of higher order branches, and sometimes there is a negative correlation. Therefore it is quite possible that period-doubling cascades of unstable solutions happen in higher order branches.

We close this section by poiting out that the solutions that are originated by period-doubling bifurcation, from the same re-scaled solution, in different branches, may be quite different. This happens, for instance, with the first period-doubling bifurcation in branches 1 and 2. To understand this phenomenon it is sufficient to draw the square wave solutions of periods 2 and 2/3, respectively, of the function map $X(t) = f(X(t-1))$ associated with (1) for large τ, and analyse the period-doubling bifurcation of each solution. It is easily seen that the two resulting solutions are different. An analogous phenomenon occurs for solutions of (1), providing another explanation for the fact that the first period-doubling bifurcation of branch 1 is not directly related to the first period-doubling bifurcation of branch 2.

5) Conclusion

Introducing a convenient "definition" of branch the study of the dynamics of equation (1) is separated in two levels: the bifurcations of solutions belonging to a particular branch and the connection between different branches (i.e. how are the orbits that join these sets (see Mallet-Paret (1988)), or how they mix (see Ikeda, Kondo & Akimoto (1982), Ikeda & Matsumoto (1987)).

From the results of Mallet-Paret (1988), Ikeda, Kondo & Akimoto (1982) and Ikeda & Matsumoto (1987), we notice that, possibly, both, the satisfaction of the condition of negative-feedback and the violation of this condition, play important roles on the dynamics of (1). The "satisfaction" leads to Mallet-Paret results and the "violation" is related to numerical results of Ikeda, Kondo & Akimoto and Ikeda & Matsumoto that show that near the point where this condition is not satisfied important changes may occur in the system, such as the connection between inferior and superior branches [Ikeda, Kondo & Akimoto], and the "final branching merging". This last phenomenon is described by Ikeda & Matsumoto (1987), and is related to the fusion of chaotic solutions of different branches giving rise to a solution presenting what they call "developped chaos". Again the point at which it occurs corresponds to $\mu = 0.775$ (see both Ikeda, Kondo & Akimoto (1982) and Ikeda &Matsumoto (1987) to get this information), for large τ, which is close to our lower-bound value, 0.76, for the violation of the negative-feedback condition. The violation of the negative-feedback condition seems to imply the existence of "connections", and "merging" phenomena, but this is a conjecture that must be further investigated.

We would like to stress that the relation between the solutions of equation (1) belonging to different branches is a very interesting feature. It is possible for a pattern that happened at some value of τ to be repeated in another scale at some other value of τ. Moreover analogous sequence of bifurcations may be repeated at different values of τ in different scales, and this is a curious symmetry of the systems described by equation (1).

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6) References

L. Bel, "Spontaneous predictivisation", *Lect. Notes in Phys.* **162** (1982), 21-49.

R. Bellman & K. L. Cooke, "Differential-Difference Equations", Acad. Press-USA (1963).

S. N. Chow & D. Green Jr., "Some results on singular delay-differential equations", *Chaos, Fractals and Dynamics*, ed by P. Fischer & W. R. Smith, Marcel Dekker (1985), 161-182.

C. Conley, *Am. Math. Soc.-Regional Conferences* **39** (1978).

J. P. Crutchfield & B. A. Huberman, "Fluctuations and the onset of chaos",*Phys. Lett.* **77A** (1980), 407-410.

M. W. Derstine, H. M. Gibbs, F. A. Hopf & D. L. Kaplan, "Alternate paths to chaos in optical bistability", *Phys. Rev.* **A27** (1983), 3200-3208.

R. D. Driver, "A functional differential system of neutral type arising in a two body problem of classical electrodynamics",*Nonlinear Differemtial Equations and Nonlinear Mechanics*, Acad. Press (1963), 474-484.

M. Feigenbaum, "Universal behaviour in nonlinear systems",*Los Alamos Science*, Summer (1980).

J. Y. Gao , L. M. Narducci, L. S. Schulman, M. Squicciarini & J. M. Yuan, "Route to chaos in a hybrid bistable system with delay", *Phys. Rev* **A28** (1983), 2910-2914.

J. Y. Gao , J. M. Yuan & L. M. Narducci, "Instabilities and chaotic behavior in a hybrid bistable system with short delay", *Opt. Comm.* **44** (1983), 201-206.

J. Y. Gao, L. M. Narducci, H. Sadiky, M. Squicciarini & J. Yuan, "Higher-order bifurcations in a bistable system with delay", *Phys. Rev.* **A30** (1984), 901-904.

J. Guckenheimer & P. Holmes, "Nonlinear Oscillations, Dynamical Systems, and Bifurcations", Springer-Verlag USA, (1983).

K. P. Hadeler, "Delay equations in biology", *Lec. Notes in Math.* **730** (1979), 136-156.

J. K. Hale, "Theory of Functional Differential Equations", Springer-Verlag USA, (1977).

K. Ikeda , K. Kondo & O. Akimoto, "Successive Higher harmonic bifurcations in systems with delayed feedback", *Phys. Rev. Lett.* **49** (1982), 1467-1470.

K. Ikeda & K. Matsumoto, "High dimensional chaotic behavioor in systems with time delay feedback", *Physica* **29D** (1987), 223-235.

M. C. Mackey & L. Glass, "Oscillation and chaos in physiological control systems", *Science* **197** (1977), 287-289.

J. Mallet-Paret, "Morse decompositions for delay differential equations", *Jour. Diff. Eq.* **72** (1988), 270-315.

J. Mallet-Paret & R. D. Nussbaum , "A bifurcation gap for a sigularly perturbed delay equation",*Chaotic Dynamics and Fractals*, ed. by M. F. Barnsley & G. S. Denko, Academic Press-USA (1986), 263-286.

C. R. de Oliveira & C. P. Malta, "Bifurcations in a class of time delay equations", *Phys. Rev.* **A36** (1987), 3997-4001.

J. F. Perez, C. P. Malta & F. A. B. Coutinho, "Qualitative analysis of oscillations in isolated populations of flies", *Jour. Theor. Biol.* **71** (1978), 505-514.

J. M. Smith, "Mathematical Ideas in Biology", Cambridge Press (1968).

Table caption

Table 1 - calculation of some contraction rates $\delta_n = \frac{\lambda_{n+1}-\lambda n}{\lambda_{n+2}-\lambda_{n+1}}$, where λ_n is the value of the varying parameter (A or τ) where the stable periodic solution of branch 1 suffered the n^{th}period-doubling bifurcation. This contraction is based on data presented in figure 3 (dashed lines). The errors presented are due to numerical resolution problems.

Figure captions

Figure 1: (a) the graph of function (2), and (b) the graph of (2) translated of $\frac{-A-1}{A}$.

Figure 2: Graph of function (3) showing the fixed point X_{eq} of (1) with $f(X)$ given by (3).

Figure 3: - full line: (from below to above) curves of first, second and third period-doubling, and curve of accumulation of period-doubling; - dotted line: the corresponding period-doubling and accumulation points of the quadratic map; -dashed lines: bifurcation parameters used in the estimation of Feigenbaum's contraction rate (see table 1), vertical lines, $\tau = 4.013$, $\tau = 7.438$, horizontal lines, $A = 3.65$, $A = 3.85$.

figure 4: A typical periodic solution of (1) with $f(X)$ given by (2), with period near four, showing its singularities ($\tau = 50.0, A = 3.5$).

Figure 5: The point $\tau^*_{1,1}$ represents the appearence of a periodic solution in branch 1 with period greater than 2 (as it happens for (1) with $f(X)$ given by (2), see section 3). Notice that the period-doubling bifurcations in branch 1 are mapped into other branches bifurcations at different values of τ.

Fixed parameter	δ_1	δ_2
$A = 3.65$	$2.135 \pm 3 \times 10^{-3}$	$4.06 \pm 2 \times 10^{-2}$
$A = 3.85$	$4.46 \pm 8 \times 10^{-2}$	$4.5 \pm 2 \times 10^{-1}$
$\tau = 7.438$	$4.53 \pm 7 \times 10^{-2}$	$4.6 \pm 2 \times 10^{-1}$
$\tau = 4.013$	$4.62 \pm 9 \times 10^{-2}$	$4.8 \pm 2 \times 10^{-1}$

Table 1

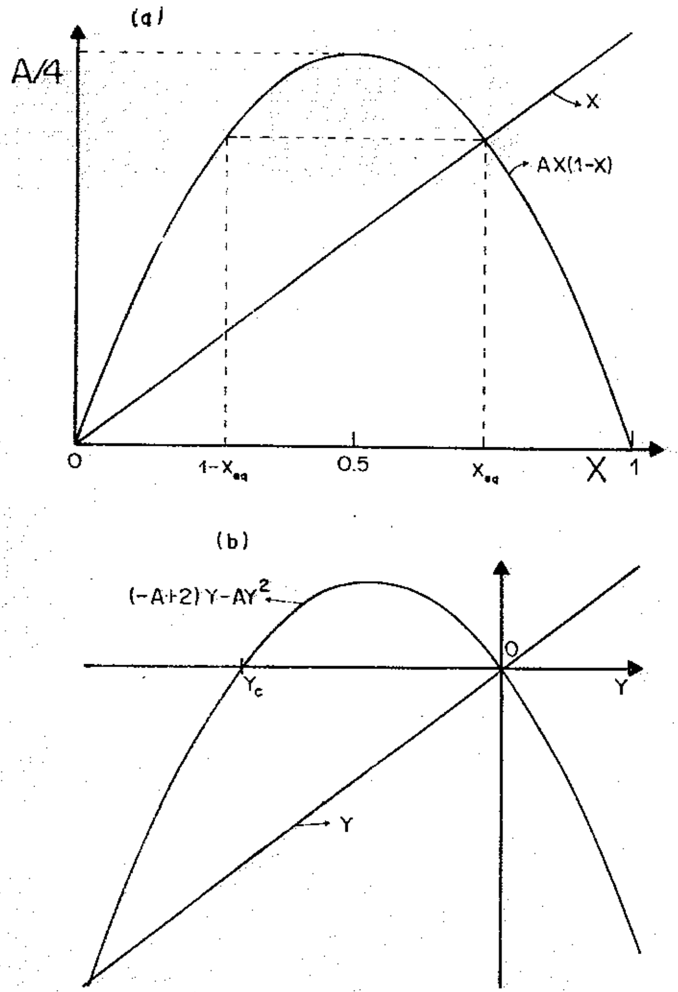

FIGURE 1

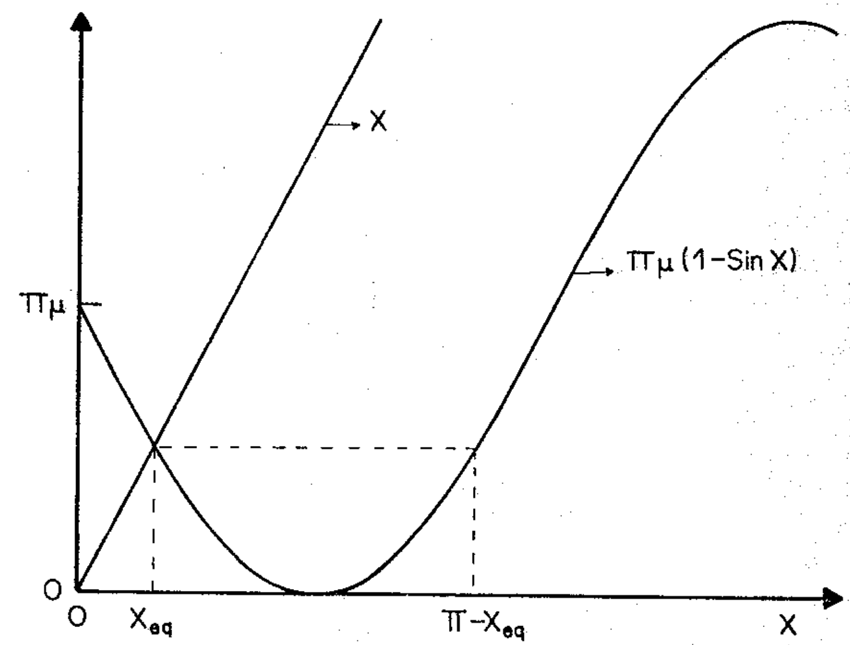

FIGURE 2

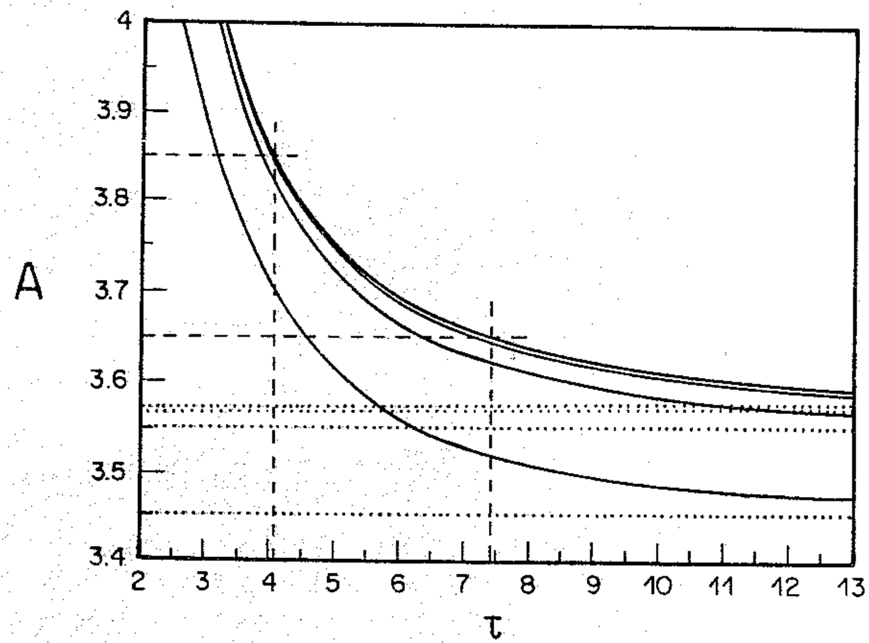

FIGURE 3

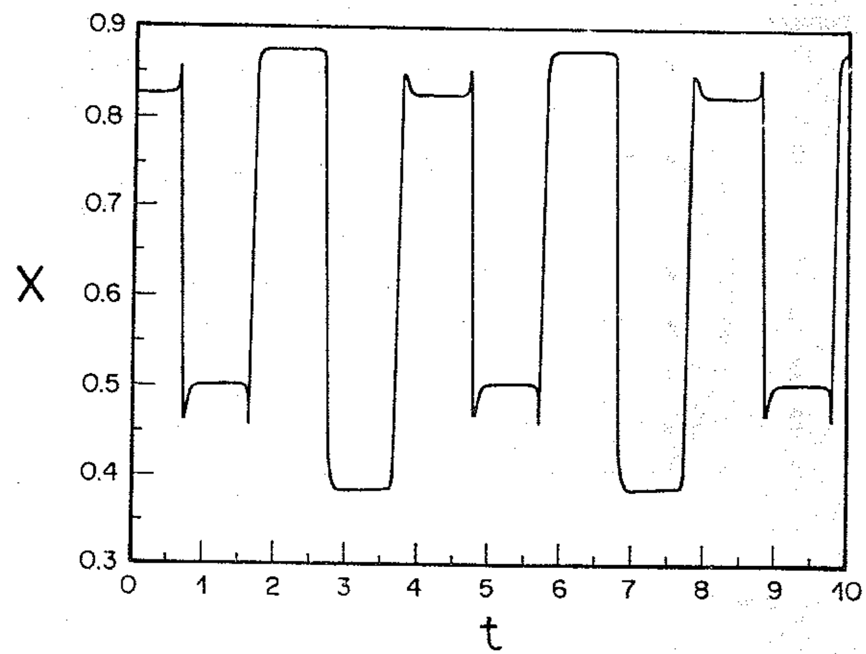

FIGURE 4

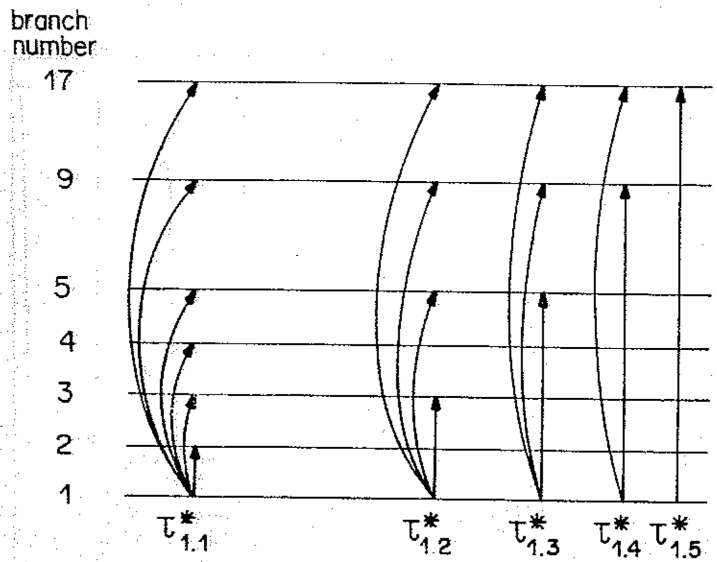

FIGURE 5