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THE 3-GRAVITON VERTEX FUNCTION IN THERMAL QUANTUM GRAVITY

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The high temperature limit of the 3-graviton vertex function is studied in thermal quantum gravity, to one loop order. The leading (T^4) contributions arising from internal gravitons are calculated and shown to be twice the ones associated with internal scalar particles, in correspondence with the two helicity states of the graviton. The gauge invariance of this result follows in consequence of the Ward and Weyl identities obeyed by the thermal loops, which are verified explicitly.

I. INTRODUCTION

There has been much work on quantum gravity at finite temperatures, which are high compared with typical frequencies of the gravitational field. If the temperature is well below the Planck scale, perturbation theory can be used to calculate the n-graviton functions, with internal lines which correspond to matter in thermal equilibrium. The functions for $n = 1$ and $n = 2$ have been studied previously [1,2] and show a leading T^4 behavior. Subsequently, the work has been extended to the $n = 3$ case, with a single loop of internal scalar particles. Furthermore, it has been shown on general grounds, based on the Ward and Weyl identities, that the partition function in a gravitational field is determined uniquely in terms of the $O(\kappa)$ contributions [3]. Consequently, the contributions from internal scalars, gluons and gravitons should be the same, up to simple numerical factors which just count degrees of freedom.

The main purpose of this paper is to calculate the high-temperature limit of the 3-graviton vertex function with a single loop of internal gravitons. Besides verifying the general arguments presented in reference [3], this study of thermal quantum gravity might offer new insights into the general structure of the metric dependence of the partition function at high temperature. The calculation of the 3-graviton function is considerably more complicated than that of the 3-gluon one [4]. The method we use is an extension of that in reference [5], in which the thermal Yang-Mills n-point functions were related to the forward-scattering amplitudes for the thermal Yang-Mills particles. This method simplifies considerably the calculations in the present case, were we consider the forward scattering amplitudes for the thermal gravitons in a gravitational field.

In order to illustrate the method and to derive several results which will be important later on, we first consider in Section II the graviton self-energy function. We evaluate the leading temperature corrections and check the Ward and Weyl identities which relate the self-energy function to the energy-momentum tensor. In Section III, we derive the T^4

terms in the 3-graviton function, for the self-interacting thermal gravitons. We verify the Ward and Weyl identities connecting the 3- and 2-point graviton functions. The leading contributions have the same form as the one associated with internal scalar particles, differing only by a factor of 2 in correspondence with the two helicities of a physical graviton. Some mathematical details which arise during these calculations are given in the Appendices.

II. THE GRAVITON SELF-ENERGY

We consider here the leading high-temperature corrections to the graviton self-energy in a space-time which is asymptotically flat. Hence, we expand the metric tensor $g^{\mu\nu}$ in terms of the deviation from the Minkowski metric $\eta^{\mu\nu}$ as follows

$$\sqrt{-g}g^{\mu\nu} \equiv \tilde{g}^{\mu\nu} = \eta^{\mu\nu} + \kappa\phi^{\mu\nu}, \tag{2.1}$$

where $\kappa = \sqrt{32\pi G}$ and $\phi^{\mu\nu}$ denotes the graviton field. This enables us to evaluate perturbatively the thermal Green functions by expanding the Einstein's Lagrangian written in the form [6]:

$$\mathcal{L}_{grav} = \frac{1}{2\kappa^2}(\tilde{g}^{\rho\sigma}\tilde{g}_{\lambda\mu}\tilde{g}_{\kappa\nu} - \frac{1}{2}\tilde{g}^{\rho\sigma}\tilde{g}_{\mu\kappa}\tilde{g}_{\lambda\nu} - 2\delta^{\sigma}_{\kappa}\delta^{\rho}_{\lambda}\tilde{g}_{\mu\nu})\tilde{g}^{\mu\kappa}_{,\rho}\tilde{g}^{\lambda\nu}_{,\sigma}. \tag{2.2}$$

It is convenient to fix the gauge by choosing:

$$\mathcal{L}_{fix} = -\frac{1}{\kappa^2}(\partial_\mu\tilde{g}^{\mu\nu})^2, \tag{2.3}$$

which yields a contribution of the gravitational ghosts given by:

$$\mathcal{L}_{ghost} = \bar{\xi}_\nu[\eta_{\nu\lambda}\partial^2 - \kappa(\phi_{\mu\nu,\lambda\mu} - \phi_{\mu\rho}\eta_{\nu\lambda}\partial_\rho\partial_\mu - \phi_{\mu\rho,\mu}\eta_{\nu\lambda}\partial_\rho + \phi_{\mu\nu,\mu}\partial_\lambda)]\xi_\lambda. \tag{2.4}$$

The relevant Feynman rules following from the above Lagrangian density are summarized in Appendix A.

The Feynman diagrams which contribute to the graviton self-energy function are shown in Fig. 1. These graphs represent [5] the forward scattering amplitude of a thermal graviton with on-shell momenta $q_\alpha = (q, \vec{q})$ as indicated in Fig. 2. This amplitude must be multiplied by the corresponding Bose distribution function of the thermal graviton and integrated over its 3-momentum $\vec{q}$. In this way, we can express the thermal self-energy graviton function as:

$$\Gamma^2_{(\alpha\beta)(\mu\nu)}(k) = \frac{1}{8\pi^2}\int_0^\infty \frac{qdq}{e^{q/T}-1}\int\frac{d\Omega}{4\pi}\Gamma^2_{(\alpha\beta)(\mu\nu)}(k,q). \tag{2.5}$$

The high-temperature limit of the forward scattering amplitude $\Gamma^2_{(\alpha\beta)(\mu\nu)}(k,q)$ is governed by those parts of the Feynman integrals which are superficially most divergent. To obtain these, one needs to expand the Feynman denominator:

$$\frac{1}{k^2+2q\cdot k} = \frac{1}{2q\cdot k} - \frac{k^2}{(2q\cdot k)^2} + \cdots. \tag{2.6}$$

In this case there is a "super-leading" term of the form:

$$\frac{q_\alpha q_\beta q_\mu q_\nu}{q\cdot k}, \tag{2.7}$$

which cancels between the graphs of Fig. 2 and the corresponding crossed diagrams. We are then left with the leading contributions which are functions of degree two in q. Rescaling the null vector q_α by $q_\alpha = qQ_\alpha$, where $Q_\alpha = (1,\hat{Q})$ with $Q^2 = 0$, these can be expressed in terms of the graviton energy density

$$\rho_g = \frac{1}{\pi^2}\int_0^\infty \frac{q^3dq}{e^{q/T}-1} = \frac{\pi^2T^4}{15}. \tag{2.8}$$

From the Feynman rules developed in Appendix A, we find that the contributions to the Forward scattering amplitude $\hat{\Gamma}^2_{(\alpha\beta)(\mu\nu)}(k,Q)$ associated with the graphs in Fig. 2. are given respectively by:

$$\begin{aligned}\frac{1}{\kappa^2}\hat{\Gamma}^{2a}_{(\alpha\beta)(\mu\nu)}(k,Q) = &\frac{10\,k_\nu\,Q_\alpha\,Q_\beta\,Q_\mu}{k\cdot Q} + \frac{10\,k_\mu\,Q_\alpha\,Q_\beta\,Q_\nu}{k\cdot Q} + \frac{10\,k_\beta\,Q_\alpha\,Q_\mu\,Q_\nu}{k\cdot Q}\\ &+ \frac{10\,k_\alpha\,Q_\beta\,Q_\mu\,Q_\nu}{k\cdot Q} - \frac{10\,k^2\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu}{(k\cdot Q)^2}\\ &- 10\,Q_\mu\,Q_\nu\,\eta_{\alpha\beta} - 10\,Q_\alpha\,Q_\beta\,\eta_{\mu\nu},\end{aligned} \tag{2.9}$$

$$\frac{1}{\kappa^2}\hat{\Gamma}^{2b}_{(\alpha\beta)(\mu\nu)}(k,Q) = -\frac{8\,k_\nu\,Q_\alpha\,Q_\beta\,Q_\mu}{k\cdot Q} - \frac{8\,k_\mu\,Q_\alpha\,Q_\beta\,Q_\nu}{k\cdot Q} - \frac{8\,k_\beta\,Q_\alpha\,Q_\mu\,Q_\nu}{k\cdot Q} - \frac{8\,k_\alpha\,Q_\beta\,Q_\mu\,Q_\nu}{k\cdot Q} + \frac{8\,k^2\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu}{(k\cdot Q)^2}, \tag{2.10}$$

$$\frac{1}{\kappa^2}\hat{\Gamma}^{2c}_{(\alpha\beta)(\mu\nu)}(k,Q) = 10(\eta_{\alpha\beta}Q_\mu Q_\nu + \eta_{\mu\nu}Q_\alpha Q_\beta). \tag{2.11}$$

Note that equations (2.9) and (2.11) contain terms involving the Minkowski metric tensor. However, such terms cancel out in the total sum which gives twice the contribution one gets from a single loop of internal scalar particles:

$$\frac{1}{\kappa^2}\hat{\Gamma}^{2}_{(\alpha\beta)(\mu\nu)}(k,Q) = 2\left(\frac{k_\nu\,Q_\alpha\,Q_\beta\,Q_\mu}{k\cdot Q} + \frac{k_\mu\,Q_\alpha\,Q_\beta\,Q_\nu}{k\cdot Q} + \frac{k_\beta\,Q_\alpha\,Q_\mu\,Q_\nu}{k\cdot Q} + \frac{k_\alpha\,Q_\beta\,Q_\mu\,Q_\nu}{k\cdot Q} - \frac{k^2\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu}{(k\cdot Q)^2}\right). \tag{2.12}$$

This result is expected from the Ward identity relating the self-energy function to the energy-momentum tensor:

$$(2\eta^{\alpha\lambda}k_1{}^{\beta} - \eta^{\alpha\beta}k_1{}^{\lambda})\Gamma^2_{(\alpha\beta)(\mu\nu)}(k_1) = \eta^{\beta\lambda}(\delta^\alpha_\mu k_{1\,\nu} + \delta^\alpha_\nu k_{1\,\mu})\frac{T_{\alpha\beta}}{2}, \tag{2.13}$$

and from the invariance under Weyl transformations which requires:

$$\eta^{\alpha\beta}\Gamma^2_{(\alpha\beta)(\mu\nu)}(k) = \frac{T_{\mu\nu}}{2}. \tag{2.14}$$

As shown in reference [3], these relations fix uniquely the self-energy function in terms of the energy-momentum tensor. Since the contributions to $T_{\mu\nu}$ from internal scalars and gravitons are all the same [2] apart from simple numerical factors counting the degrees of freedom, the result expressed by eq. (2.12) should be expected.

III. THE 3-GRAVITON VERTEX FUNCTION

We now turn to the leading temperature corrections of the 3-point graviton function. The thermal loop diagrams which are relevant to our discussion are shown in Fig. 3. and the corresponding forward scattering amplitudes are represented in Fig. 4. In the high temperature limit we require large values of momenta $q_\alpha = (q,\vec{q})$ associated with the thermal graviton. Since the thermal particle is on shell, each Feynman denominator in the diagrams in Fig. 4 has the form:

$$(2q\cdot k + k^2)^{-1} \quad \text{with} \quad k = \sum k_i, \tag{3.1}$$

the sum being over some set of indices i. We may expand each denominator in powers of $(k^2/2q\cdot k)$, and also the numerators in powers of $k_{i\mu}/q$. The first term has a denominator which is quadratic in $(q\cdot k)^{-1}$, and a numerator with the single tensor structure $q_\alpha q_\beta q_\mu q_\nu q_\rho q_\sigma$. However, these terms cancel when all graphs are added, as a consequence of the eikonal identity:

$$(q\cdot k_1)^{-1}(q\cdot k_2)^{-1} + (q\cdot k_1)^{-1}(q\cdot k_3)^{-1} + (q\cdot k_2)^{-1}(q\cdot k_3)^{-1} = 0, \tag{3.2}$$

since $k_1 + k_2 + k_3 = 0$. The next terms are down by a power of k_i/q, being individually "super-leading". Also these turn out to cancel out by a combination of the eikonal identity and the requirement of Bose symmetry.

With the super-leading terms out of the way, we now consider the leading contributions. To this end we take the integrand to one further term in powers of the external momenta. Then the leading terms will become homogeneous functions of q of degree 2. We proceed as in the previous section, rescaling q_α by a factor of q: $q_\alpha = qQ_\alpha$, and integrating over q. Then the leading behavior of the 3-graviton vertex can be expressed as:

$$\Gamma^3_{(\alpha\beta)(\mu\nu)(\rho\sigma)}(k_1,k_2,k_3) = \frac{\rho_g}{8}\int\frac{d\Omega}{4\pi}\hat{\Gamma}^3_{(\alpha\beta)(\mu\nu)(\rho\sigma)}(k_1,k_2,k_3,Q), \tag{3.3}$$

where the graviton energy density ρ_g (equation (2.8)) is proportional to T^4. Using the Feynman rules given in Appendix A, we perform the necessary algebra in order to find the leading contributions to the forward scattering amplitude $\hat{\Gamma}^3_{(\alpha\beta)(\mu\nu)(\rho\sigma)}(k_1,k_2,k_3,Q)$. The calculation is considerably involved, requiring the use of vast algebraic manipulations. The corresponding corrections, associated with the graphs shown in Fig. 4., can be written in the form given in Appendix B. Here we discuss only the main features of these complex algebraic structures.

The contributions from diagram (4a), involving only three-graviton vertices has the general from (see eq. B1):

$$\begin{aligned}\hat{\Gamma}^{3a}_{(\alpha\beta)(\mu\nu)(\rho\sigma)}(k_1,k_2,k_3,Q) = {} & AQ_\alpha Q_\beta Q_\mu Q_\nu Q_\rho Q_\sigma + B^1_\alpha Q_\beta Q_\mu Q_\nu Q_\rho Q_\sigma + (\alpha \leftrightarrow \beta) \\ & + B^2_\mu Q_\alpha Q_\beta Q_\nu Q_\rho Q_\sigma + \cdots \\ & + C^1_{\alpha\beta} Q_\mu Q_\nu Q_\rho Q_\sigma + \cdots + D^{12}_{\alpha\mu} Q_\beta Q_\nu Q_\rho Q_\sigma + \cdots \\ & + E^{12}_\alpha \eta_{\mu\nu} Q_\beta Q_\rho Q_\sigma + \cdots + F^{12}_\alpha \eta_{\beta\mu} Q_\nu Q_\rho Q_\sigma + \cdots \\ & + G^1_\alpha \eta_{\mu\sigma} Q_\beta Q_\nu Q_\rho + \cdots + H^1 \eta_{\alpha\beta} Q_\mu Q_\nu Q_\rho Q_\sigma + \cdots \\ & + I^{12} \eta_{\alpha\mu} Q_\beta Q_\nu Q_\rho Q_\sigma + \cdots \\ & + J\,(\eta_{\alpha\beta}\eta_{\mu\nu} Q_\rho Q_\sigma + \cdots) + J'\,(\eta_{\alpha\beta}\eta_{\mu\rho} Q_\nu Q_\sigma + \cdots) \\ & + J''\,(\eta_{\alpha\mu}\eta_{\beta\nu} Q_\rho Q_\sigma + \cdots) + J'''\,(\eta_{\alpha\mu}\eta_{\beta\rho} Q_\nu Q_\sigma + \cdots)\,.\end{aligned} \tag{3.4}$$

Here J, J', J'', J''' are constants, A, H^i, I^{ij} are scalars functions of Q and the external momenta, B^i_α, E^{ij}_α, F^{ij}_α, G^i_α are vector functions of Q and the external momenta not proportional to Q_α, and $C^i_{\alpha\beta}$, $D^{ij}_{\alpha\beta}$ are tensor functions containing neither Q_α nor Q_β. The ellipses denote the addition of as many terms as are necessary to symmetrize under $(\alpha \leftrightarrow \beta)$, $(\mu \leftrightarrow \nu)$, $(\rho \leftrightarrow \sigma)$ and under the permutations of $(k_1,\ \alpha,\ \beta)$, $(k_2,\ \mu,\ \nu)$, $(k_3,\ \rho,\ \sigma)$. Note that each term is a function of degree two in Q and of zero degree in the external momenta.

The contribution from the ghost particles in Fig. (4b) is less complicated algebraically than the previous one, since the terms proportional to the Minkowski metric η are absent

(see eq. B2). On the other hand, the corrections arising from graph (4c) give, as shown in eq. (B3), only terms which contain explicitly the Minkowski metric η. Finally, the contributions associated with diagram (4d), which involves the 5-graviton coupling, have a structure proportional to $\eta \otimes \eta$, as can be seen from equation (B4).

However, the terms depending explicitly on the Minkowski metric cancel when all graphs are added. The final result for the forward scattering amplitude is rather simple and can be written in the form (cf. eq. (B5)):

$$\begin{aligned}\hat{\Gamma}^3_{(\alpha\beta)(\mu\nu)(\rho\sigma)}(k_1,k_2,k_3,Q) = {} & \hat{A} Q_\alpha Q_\beta Q_\mu Q_\nu Q_\rho Q_\sigma + \hat{B}^1_\alpha Q_\beta Q_\mu Q_\nu Q_\rho Q_\sigma + (\alpha \leftrightarrow \beta) \\ & + \hat{B}^2_\mu Q_\alpha Q_\beta Q_\nu Q_\rho Q_\sigma + \cdots + \hat{C}^1_{\alpha\beta} Q_\mu Q_\nu Q_\rho Q_\sigma + \cdots \\ & + \hat{D}^{12}_{\alpha\mu} Q_\beta Q_\nu Q_\rho Q_\sigma + \cdots,\end{aligned} \tag{3.5}$$

where $Q_\alpha = (1,\hat{Q})$, and

$$\hat{A} = \frac{k_1^2 k_2\cdot k_3}{(k_1\cdot Q)^2 k_2\cdot Q k_3\cdot Q} + (\textit{cyclic permutations}), \tag{3.6}$$

$$\hat{B}^1_\alpha = -\frac{k_2\cdot k_3 k_{1\alpha}}{k_1\cdot Q k_2\cdot Q k_3\cdot Q} - \frac{k_3^2 k_{2\alpha}}{k_2\cdot Q(k_3\cdot Q)^2} - \frac{k_2^2 k_{3\alpha}}{(k_2\cdot Q)^2 k_3\cdot Q}, \tag{3.7}$$

$$\hat{C}^1_{\alpha\beta} = \frac{k_{2\alpha}k_{3\beta} + k_{2\beta}k_{3\alpha}}{k_2\cdot Q k_3\cdot Q}, \tag{3.8}$$

$$\hat{D}^{12}_{\alpha\mu} = \frac{k_{1\alpha}k_{3\mu}}{k_1\cdot Q k_3\cdot Q} + \frac{k_{2\mu}k_{3\alpha}}{k_2\cdot Q k_3\cdot Q}. \tag{3.9}$$

The other terms in (3.5) containing the coefficients $\hat{B}^i$, $\hat{C}^i$ and $\hat{D}^{ij}$ can be obtained respectively from (3.7), (3.8) and (3.9) by symmetry.

The important point about (3.5) is that, under Lorentz transformations in the asymptotic Minkowski space, it is a covariant function of the null vector Q_α. We remark that equation (3.5) is a homogeneous expression of Q of degree 2 and of zero degree in the external momenta. It gives precisely twice the contribution arising from a single loop of internal scalar particles.

In order to understand this result, we now consider the Ward identity which follows as a consequence of the invariance under general coordinate transformations. This relation,

which connects the 3-graviton vertex (eqs. (3.3) and (3.5)) to the self energy function (eqs. (2.5), (2.8) and (2.12)), is verified explicitly by:

$$(2\eta^{\alpha\lambda}k_1^{\beta} - \eta^{\alpha\beta}k_1^{\lambda})\Gamma^3_{(\alpha\beta)(\mu\nu)(\rho\sigma)}(k_1,k_2,k_3) = [\eta^{\beta\lambda}(\delta^{\alpha}_{\mu}k_{1\nu} + \delta^{\alpha}_{\nu}k_{1\mu}) + \delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}k_3{}^{\lambda}]\Gamma^2_{(\alpha\beta)(\rho\sigma)}(k_3) + (k_2,\ \mu,\ \nu) \leftrightarrow (k_3,\ \rho,\ \sigma). \tag{3.10}$$

Since the thermal corrections from the ghost-ghost-graviton vertex functions are subleading [3], these terms do not contribute to Eq. (3.10). Consequently, the Ward identity has the same form as in a physical gauge, indicating that the leading T^4 contributions are gauge independent. This property can be ascertained by considering also the Weyl invariance of the theory, which reflects its invariance under scale transformations. From this, we verify the trace identity:

$$\eta^{\alpha\beta}\,\Gamma^3_{(\alpha\beta)(\mu\nu)(\rho\sigma)}(k_1,k_2,k_3) = \Gamma^2_{(\mu\nu)(\rho\sigma)}(k_2) + \Gamma^2_{(\rho\sigma)(\mu\nu)}(k_3). \tag{3.11}$$

It has been argued in [3] that the use of the Ward and Weyl identities, together with the structure indicated by equation (3.5), are sufficient to fix uniquely the 3-point vertex in terms of the self-energy function. As we have seen, the same identities determine the 2-point function in terms of the energy-momentum tensor $T^{\mu\nu}$. Hence, these relations are sufficient to determine uniquely the 3-point graviton vertex in terms of $T^{\mu\nu}$.

It is well known [7] that in gauge theories, the thermal corrections to the energy-momentum tensor should be gauge invariant quantities. The contributions to $T^{\mu\nu}$ from internal scalars and gravitons are the same, up to a factor of 2 which counts the graviton degrees of freedom. From the above arguments, it then follows that the leading T^4 contributions to the 3-graviton vertex function, from a single loop of internal gravitons, must be gauge invariant. Furthermore, these corrections should be twice the ones arising from an internal loop of scalar particles, as expected from the two helicity states of the graviton.

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APPENDIX A

In this appendix we derive the Feynman rules which are relevant for the perturbative study of quantum gravity. To this end, we write the total Lagrangian density obtained from the sum of (2.2), (2.3) and (2.4) as:

$$\mathcal{L} = \sum_{j=2}^{\infty} \kappa^{j-2}\mathcal{L}_{(j)} + \mathcal{L}_{ghost}, \tag{A.1}$$

where the terms in the series are given by increasing powers of the graviton field defined in eq. (2.1). Noticing that:

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} - \kappa\phi_{\mu\nu} + \kappa^2\phi_{\mu\alpha}\phi_{\alpha\nu} - \kappa^3\phi_{\mu\alpha}\phi_{\alpha\beta}\phi_{\beta\nu} + O(\kappa^4), \tag{A.2}$$

and inverting the tensor which multiplies $\phi_{\alpha\beta}\phi_{\mu\nu}$, we find that the graviton propagator in momentum space is given by

$$D^{grav.}_{(\alpha\beta)(\mu\nu)}(k) = \frac{1}{2k^2}(\eta_{\alpha\mu}\eta_{\beta\nu} + \eta_{\alpha\nu}\eta_{\beta\mu} - \eta_{\alpha\beta}\eta_{\mu\nu}). \tag{A.3}$$

In the same way, the quadratic part in the ghost field leads to the ghost propagator

$$D^{ghost}_{\alpha\beta}(k) = \frac{\eta_{\alpha\beta}}{k^2}. \tag{A.4}$$

The ghost-ghost-graviton vertex can be easily derived from eq. (2.4). We obtain in momentum space that:

$$\Gamma^{ghost}_{(\alpha\beta)(\mu)(\nu)}(k_1,k_2,k_3) = \frac{\kappa}{2}[\eta_{\mu\nu}(k_{2\alpha}k_{3\beta} + k_{3\alpha}k_{2\beta}) - (\eta_{\alpha\mu}k_{1\beta} + \eta_{\beta\mu}k_{1\alpha})k_{2\nu}]. \tag{A.5}$$

The graviton-graviton couplings in momentum space represent an algebraic problem for which we have used the computer program Mathematica. We find for these couplings the following expressions:

The three graviton coupling

$$
\begin{aligned}
&\frac{4}{\kappa}\times\Gamma^{3}_{(\alpha\beta)(\mu\nu)(\rho\sigma)}(k_1,k_2,k_3) = \\
&\quad [\ -4\,k_{2\rho}\,k_{3\nu}\,\eta_{\alpha\mu}\,\eta_{\beta\sigma} - k_2\cdot k_3\,\eta_{\alpha\rho}\,\eta_{\beta\sigma}\,\eta_{\mu\nu} \\
&\qquad + 2\,k_2\cdot k_3\,\eta_{\alpha\nu}\,\eta_{\beta\sigma}\,\eta_{\mu\rho} + 2\,k_2\cdot k_3\,\eta_{\alpha\mu}\,\eta_{\beta\rho}\,\eta_{\nu\sigma} \\
&\qquad - 2\,k_{2\alpha}\,k_{3\beta}\,\eta_{\mu\rho}\,\eta_{\nu\sigma} - k_2\cdot k_3\,\eta_{\alpha\mu}\,\eta_{\beta\nu}\,\eta_{\rho\sigma} \\
&\qquad + k_{2\alpha}\,k_{3\beta}\,\eta_{\mu\nu}\,\eta_{\rho\sigma} \\
&\qquad + (\textit{symmetrization under } (\alpha\leftrightarrow\beta),\ (\mu\leftrightarrow\nu),\ (\rho\leftrightarrow\sigma))\ ] \\
&\qquad + (\textit{permutations of } (k_1,\alpha,\beta),\ (k_2,\mu,\nu),\ (k_3,\rho,\sigma))\,.
\end{aligned}
\tag{A.6}
$$

The four graviton coupling

$$
\begin{aligned}
&\frac{4}{\kappa^2}\times\Gamma^{4}_{(\alpha\beta)(\mu\nu)(\rho\sigma)(\lambda\tau)}(k_1,k_2,k_3,k_4) = \\
&\quad [\ -2\,k_{3\mu}\,k_{4\nu}\,\eta_{\alpha\sigma}\,\eta_{\beta\tau}\,\eta_{\lambda\rho} + k_{3\mu}\,k_{4\nu}\,\eta_{\alpha\rho}\,\eta_{\beta\sigma}\,\eta_{\lambda\tau} \\
&\qquad - k_3\cdot k_4\,\eta_{\alpha\rho}\,\eta_{\beta\nu}\,\eta_{\lambda\tau}\,\eta_{\mu\sigma} - 4\,k_{3\lambda}\,k_{4\sigma}\,\eta_{\alpha\rho}\,\eta_{\beta\nu}\,\eta_{\mu\tau} \\
&\qquad + 2\,k_3\cdot k_4\,\eta_{\alpha\sigma}\,\eta_{\beta\nu}\,\eta_{\lambda\rho}\,\eta_{\mu\tau} - k_3\cdot k_4\,\eta_{\alpha\rho}\,\eta_{\beta\sigma}\,\eta_{\lambda\mu}\,\eta_{\nu\tau} \\
&\qquad + 2\,k_3\cdot k_4\,\eta_{\alpha\rho}\,\eta_{\beta\lambda}\,\eta_{\mu\sigma}\,\eta_{\nu\tau} + k_{3\mu}\,k_{4\nu}\,\eta_{\alpha\lambda}\,\eta_{\beta\tau}\,\eta_{\rho\sigma} \\
&\qquad - k_3\cdot k_4\,\eta_{\alpha\lambda}\,\eta_{\beta\nu}\,\eta_{\mu\tau}\,\eta_{\rho\sigma} - 2\,k_{3\mu}\,k_{4\nu}\,\eta_{\alpha\rho}\,\eta_{\beta\lambda}\,\eta_{\sigma\tau} \\
&\qquad + 2\,k_3\cdot k_4\,\eta_{\alpha\rho}\,\eta_{\beta\nu}\,\eta_{\lambda\mu}\,\eta_{\sigma\tau} \\
&\qquad + (\textit{symmetrization under } (\alpha\leftrightarrow\beta),\ (\mu\leftrightarrow\nu),\ (\rho\leftrightarrow\sigma),\ (\lambda\leftrightarrow\tau))\ ] \\
&\qquad + (\textit{permutations of } (k_1,\alpha,\beta),\ (k_2,\mu,\nu),\ (k_3,\rho,\sigma),\ (k_4,\lambda,\tau))\,.
\end{aligned}
\tag{A.7}
$$

The five graviton coupling

$$
\begin{aligned}
&\frac{4}{\kappa^3}\times\Gamma^{5}_{(\alpha\beta)(\mu\nu)(\rho\sigma)(\lambda\tau)(\gamma\delta)}(k_1,k_2,k_3,k_4,k_5) = \\
&\quad [\ -2\,k_{4\rho}\,k_{5\sigma}\,\eta_{\alpha\mu}\,\eta_{\beta\tau}\,\eta_{\delta\nu}\,\eta_{\gamma\lambda} + k_{4\rho}\,k_{5\sigma}\,\eta_{\alpha\tau}\,\eta_{\beta\lambda}\,\eta_{\delta\nu}\,\eta_{\gamma\mu} \\
&\qquad + k_{4\rho}\,k_{5\sigma}\,\eta_{\alpha\mu}\,\eta_{\beta\tau}\,\eta_{\delta\gamma}\,\eta_{\lambda\nu} - 4\,k_{4\gamma}\,k_{5\tau}\,\eta_{\alpha\mu}\,\eta_{\beta\rho}\,\eta_{\delta\sigma}\,\eta_{\lambda\nu} \\
&\qquad - 2\,k_{4\rho}\,k_{5\sigma}\,\eta_{\alpha\mu}\,\eta_{\beta\gamma}\,\eta_{\delta\tau}\,\eta_{\lambda\nu} - 2\,k_{4\rho}\,k_{5\sigma}\,\eta_{\alpha\tau}\,\eta_{\beta\delta}\,\eta_{\gamma\mu}\,\eta_{\lambda\nu} \\
&\qquad - k_4\cdot k_5\,\eta_{\alpha\mu}\,\eta_{\beta\tau}\,\eta_{\delta\sigma}\,\eta_{\gamma\rho}\,\eta_{\lambda\nu} + 2\,k_4\cdot k_5\,\eta_{\alpha\mu}\,\eta_{\beta\rho}\,\eta_{\delta\tau}\,\eta_{\gamma\nu}\,\eta_{\lambda\sigma} \\
&\qquad + 2\,k_4\cdot k_5\,\eta_{\alpha\mu}\,\eta_{\beta\tau}\,\eta_{\delta\nu}\,\eta_{\gamma\rho}\,\eta_{\lambda\sigma} + k_{4\rho}\,k_{5\sigma}\,\eta_{\alpha\mu}\,\eta_{\beta\gamma}\,\eta_{\delta\nu}\,\eta_{\lambda\tau} \\
&\qquad - k_4\cdot k_5\,\eta_{\alpha\mu}\,\eta_{\beta\rho}\,\eta_{\delta\sigma}\,\eta_{\gamma\nu}\,\eta_{\lambda\tau} + 2\,k_4\cdot k_5\,\eta_{\alpha\mu}\,\eta_{\beta\rho}\,\eta_{\delta\sigma}\,\eta_{\gamma\lambda}\,\eta_{\nu\tau} \\
&\qquad + 2\,k_4\cdot k_5\,\eta_{\alpha\mu}\,\eta_{\beta\gamma}\,\eta_{\delta\sigma}\,\eta_{\lambda\nu}\,\eta_{\rho\tau} - k_4\cdot k_5\,\eta_{\alpha\mu}\,\eta_{\beta\gamma}\,\eta_{\delta\nu}\,\eta_{\lambda\sigma}\,\eta_{\rho\tau} \\
&\qquad - k_4\cdot k_5\,\eta_{\alpha\mu}\,\eta_{\beta\rho}\,\eta_{\delta\gamma}\,\eta_{\lambda\nu}\,\eta_{\sigma\tau} \\
&\qquad + (\textit{symmetrization under } (\alpha\leftrightarrow\beta),\ (\mu\leftrightarrow\nu),\ (\rho\leftrightarrow\sigma),\ (\lambda\leftrightarrow\tau),\ (\gamma\leftrightarrow\delta))\ ] \\
&\qquad + (\textit{permutations of } (k_1,\alpha,\beta),\ (k_2,\mu,\nu),\ (k_3,\rho,\sigma),\ (k_4,\lambda,\tau),\ (k_5,\gamma,\delta))\,.
\end{aligned}
\tag{A.8}
$$

As usual, we have energy-momentum conservation at the vertices, where all momenta are defined to be inwards.

APPENDIX B

Here we present the complete expressions for the contributions to the forward scattering amplitude which are associated with the diagrams in Fig. 4. From the Feynman rules developed in Appendix A we obtain, after a vast algebraic manipulation, the following results:

$$\frac{1}{\kappa^3}\tilde{\Gamma}^{3a}_{(\alpha\beta)(\mu\nu)(\rho\sigma)}(k_1,k_2,k_3,Q) =$$

$$\Big[- \frac{7\,k_{1\rho}\,k_{1\sigma}\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu}{k_1\cdot Q\,k_2\cdot Q} - \frac{4\,k_{1\rho}\,k_{2\sigma}\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu}{k_1\cdot Q\,k_2\cdot Q} - \frac{7\,k_{2\rho}\,k_{2\sigma}\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu}{k_1\cdot Q\,k_2\cdot Q}$$
$$+ \frac{12\,k_{1\rho}\,k_{2\mu}\,Q_\alpha\,Q_\beta\,Q_\nu\,Q_\sigma}{k_1\cdot Q\,k_2\cdot Q} + \frac{8\,k_{1\mu}\,k_{2\rho}\,Q_\alpha\,Q_\beta\,Q_\nu\,Q_\sigma}{k_1\cdot Q\,k_2\cdot Q} + \frac{8\,k_{1\rho}\,k_{2\alpha}\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\sigma}{k_1\cdot Q\,k_2\cdot Q}$$
$$+ \frac{12\,k_{1\alpha}\,k_{2\rho}\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\sigma}{k_1\cdot Q\,k_2\cdot Q} - \frac{5\,k_{1\rho}\,k_2^2\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\sigma}{k_1\cdot Q\,(k_2\cdot Q)^2} + \frac{5\,k_1^2\,k_{1\rho}\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\sigma}{(k_1\cdot Q)^2\,k_2\cdot Q}$$
$$+ \frac{5\,k_2^2\,k_{2\rho}\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\sigma}{k_1\cdot Q\,(k_2\cdot Q)^2} - \frac{5\,k_1^2\,k_{2\rho}\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\sigma}{(k_1\cdot Q)^2\,k_2\cdot Q} - \frac{7\,k_{2\mu}\,k_{2\nu}\,Q_\alpha\,Q_\beta\,Q_\rho\,Q_\sigma}{k_1\cdot Q\,k_2\cdot Q}$$
$$- \frac{8\,k_{1\mu}\,k_{2\alpha}\,Q_\beta\,Q_\nu\,Q_\rho\,Q_\sigma}{k_1\cdot Q\,k_2\cdot Q} + \frac{8\,k_{1\alpha}\,k_{2\mu}\,Q_\beta\,Q_\nu\,Q_\rho\,Q_\sigma}{k_1\cdot Q\,k_2\cdot Q} + \frac{5\,k_2^2\,k_{2\mu}\,Q_\alpha\,Q_\beta\,Q_\nu\,Q_\rho\,Q_\sigma}{k_1\cdot Q\,(k_2\cdot Q)^2}$$
$$- \frac{5\,k_1^2\,k_{2\mu}\,Q_\alpha\,Q_\beta\,Q_\nu\,Q_\rho\,Q_\sigma}{(k_1\cdot Q)^2\,k_2\cdot Q} - \frac{7\,k_{1\alpha}\,k_{1\beta}\,Q_\mu\,Q_\nu\,Q_\rho\,Q_\sigma}{k_1\cdot Q\,k_2\cdot Q} - \frac{5\,k_{1\alpha}\,k_2^2\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\rho\,Q_\sigma}{k_1\cdot Q\,(k_2\cdot Q)^2}$$
$$+ \frac{5\,k_1^2\,k_{1\alpha}\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\rho\,Q_\sigma}{(k_1\cdot Q)^2\,k_2\cdot Q} - \frac{5\,k_2^4\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\rho\,Q_\sigma}{2\,k_1\cdot Q\,(k_2\cdot Q)^3} + \frac{5\,k_1^2\,k_2^2\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\rho\,Q_\sigma}{2\,(k_1\cdot Q)^2\,(k_2\cdot Q)^2}$$
$$- \frac{5\,k_1^4\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\rho\,Q_\sigma}{2\,(k_1\cdot Q)^3\,k_2\cdot Q}$$
$$- \frac{5\,k_{1\rho}\,Q_\mu\,Q_\nu\,Q_\sigma\,\eta_{\alpha\beta}}{k_1\cdot Q} + \frac{2\,k_{1\rho}\,Q_\mu\,Q_\nu\,Q_\sigma\,\eta_{\alpha\beta}}{k_2\cdot Q} + \frac{2\,k_{2\rho}\,Q_\mu\,Q_\nu\,Q_\sigma\,\eta_{\alpha\beta}}{k_1\cdot Q}$$
$$- \frac{5\,k_{2\rho}\,Q_\mu\,Q_\nu\,Q_\sigma\,\eta_{\alpha\beta}}{k_2\cdot Q} + \frac{3\,k_{1\mu}\,Q_\nu\,Q_\rho\,Q_\sigma\,\eta_{\alpha\beta}}{k_1\cdot Q} + \frac{2\,k_{2\mu}\,Q_\nu\,Q_\rho\,Q_\sigma\,\eta_{\alpha\beta}}{k_1\cdot Q}$$
$$- \frac{5\,k_{2\mu}\,Q_\nu\,Q_\rho\,Q_\sigma\,\eta_{\alpha\beta}}{k_2\cdot Q} + \frac{k_1^2\,Q_\mu\,Q_\nu\,Q_\rho\,Q_\sigma\,\eta_{\alpha\beta}}{(k_1\cdot Q)^2} + \frac{5\,k_2^2\,Q_\mu\,Q_\nu\,Q_\rho\,Q_\sigma\,\eta_{\alpha\beta}}{2\,(k_2\cdot Q)^2}$$
$$- \frac{k_{1\nu}\,Q_\beta\,Q_\rho\,Q_\sigma\,\eta_{\alpha\mu}}{k_1\cdot Q} - \frac{k_{2\nu}\,Q_\beta\,Q_\rho\,Q_\sigma\,\eta_{\alpha\mu}}{k_1\cdot Q} + \frac{k_{2\nu}\,Q_\beta\,Q_\rho\,Q_\sigma\,\eta_{\alpha\mu}}{k_2\cdot Q}$$
$$+ \frac{k_{1\beta}\,Q_\nu\,Q_\rho\,Q_\sigma\,\eta_{\alpha\mu}}{k_1\cdot Q} - \frac{k_{1\beta}\,Q_\nu\,Q_\rho\,Q_\sigma\,\eta_{\alpha\mu}}{k_2\cdot Q} - \frac{k_{2\beta}\,Q_\nu\,Q_\rho\,Q_\sigma\,\eta_{\alpha\mu}}{k_2\cdot Q}$$
$$+ \frac{4\,k_1\cdot k_2\,Q_\beta\,Q_\nu\,Q_\rho\,Q_\sigma\,\eta_{\alpha\mu}}{k_1\cdot Q\,k_2\cdot Q} - \frac{k_{1\mu}\,Q_\beta\,Q_\rho\,Q_\sigma\,\eta_{\alpha\nu}}{k_1\cdot Q} - \frac{k_{2\mu}\,Q_\beta\,Q_\rho\,Q_\sigma\,\eta_{\alpha\nu}}{k_1\cdot Q}$$
$$+ \frac{k_{2\mu}\,Q_\beta\,Q_\rho\,Q_\sigma\,\eta_{\alpha\nu}}{k_2\cdot Q} + \frac{k_{1\beta}\,Q_\mu\,Q_\rho\,Q_\sigma\,\eta_{\alpha\nu}}{k_1\cdot Q} - \frac{k_{1\beta}\,Q_\mu\,Q_\rho\,Q_\sigma\,\eta_{\alpha\nu}}{k_2\cdot Q}$$
$$- \frac{k_{2\beta}\,Q_\mu\,Q_\rho\,Q_\sigma\,\eta_{\alpha\nu}}{k_2\cdot Q} + \frac{4\,k_1\cdot k_2\,Q_\beta\,Q_\mu\,Q_\rho\,Q_\sigma\,\eta_{\alpha\nu}}{k_1\cdot Q\,k_2\cdot Q} - \frac{k_{2\sigma}\,Q_\beta\,Q_\mu\,Q_\nu\,\eta_{\alpha\rho}}{k_1\cdot Q}$$
$$- \frac{k_{2\sigma}\,Q_\beta\,Q_\mu\,Q_\nu\,\eta_{\alpha\rho}}{k_2\cdot Q} - \frac{k_{1\beta}\,Q_\mu\,Q_\nu\,Q_\sigma\,\eta_{\alpha\rho}}{k_1\cdot Q} + \frac{k_{2\beta}\,Q_\mu\,Q_\nu\,Q_\sigma\,\eta_{\alpha\rho}}{k_2\cdot Q}$$
$$- \frac{4\,k_1^2\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\sigma\,\eta_{\alpha\rho}}{k_1\cdot Q\,k_2\cdot Q} - \frac{4\,k_1\cdot k_2\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\sigma\,\eta_{\alpha\rho}}{k_1\cdot Q\,k_2\cdot Q} - \frac{k_{2\rho}\,Q_\beta\,Q_\mu\,Q_\nu\,\eta_{\alpha\sigma}}{k_1\cdot Q}$$

$$- \frac{k_{2\rho}\,Q_\beta\,Q_\mu\,Q_\nu\,\eta_{\alpha\sigma}}{k_2\cdot Q} - \frac{k_{1\beta}\,Q_\mu\,Q_\nu\,Q_\rho\,\eta_{\alpha\sigma}}{k_1\cdot Q} + \frac{k_{2\beta}\,Q_\mu\,Q_\nu\,Q_\rho\,\eta_{\alpha\sigma}}{k_2\cdot Q}$$
$$- \frac{4\,k_1^2\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\rho\,\eta_{\alpha\sigma}}{k_1\cdot Q\,k_2\cdot Q} - \frac{4\,k_1\cdot k_2\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\rho\,\eta_{\alpha\sigma}}{k_1\cdot Q\,k_2\cdot Q} + 2\,Q_\rho\,Q_\sigma\,\eta_{\alpha\mu}\,\eta_{\beta\nu}$$
$$+ \frac{k_1\cdot Q\,Q_\mu\,Q_\sigma\,\eta_{\alpha\rho}\,\eta_{\beta\nu}}{k_2\cdot Q} + \frac{k_1\cdot Q\,Q_\nu\,Q_\sigma\,\eta_{\alpha\mu}\,\eta_{\beta\rho}}{k_2\cdot Q} + \frac{k_1\cdot Q\,Q_\nu\,Q_\rho\,\eta_{\alpha\mu}\,\eta_{\beta\sigma}}{k_2\cdot Q}$$
$$+ \frac{k_1\cdot Q\,Q_\mu\,Q_\rho\,\eta_{\alpha\nu}\,\eta_{\beta\sigma}}{k_2\cdot Q} + 6\,Q_\mu\,Q_\nu\,\eta_{\alpha\rho}\,\eta_{\beta\sigma} - \frac{4\,k_1\cdot Q\,Q_\mu\,Q_\nu\,\eta_{\alpha\rho}\,\eta_{\beta\sigma}}{k_2\cdot Q}$$
$$- \frac{5\,k_{1\rho}\,Q_\alpha\,Q_\beta\,Q_\sigma\,\eta_{\mu\nu}}{k_1\cdot Q} + \frac{2\,k_{1\rho}\,Q_\alpha\,Q_\beta\,Q_\sigma\,\eta_{\mu\nu}}{k_2\cdot Q} + \frac{2\,k_{2\rho}\,Q_\alpha\,Q_\beta\,Q_\sigma\,\eta_{\mu\nu}}{k_1\cdot Q}$$
$$- \frac{5\,k_{2\rho}\,Q_\alpha\,Q_\beta\,Q_\sigma\,\eta_{\mu\nu}}{k_2\cdot Q} - \frac{5\,k_{1\alpha}\,Q_\beta\,Q_\rho\,Q_\sigma\,\eta_{\mu\nu}}{k_1\cdot Q} + \frac{2\,k_{1\alpha}\,Q_\beta\,Q_\rho\,Q_\sigma\,\eta_{\mu\nu}}{k_2\cdot Q}$$
$$+ \frac{3\,k_{2\alpha}\,Q_\beta\,Q_\rho\,Q_\sigma\,\eta_{\mu\nu}}{k_2\cdot Q} + \frac{5\,k_1^2\,Q_\alpha\,Q_\beta\,Q_\rho\,Q_\sigma\,\eta_{\mu\nu}}{2\,(k_1\cdot Q)^2} + \frac{k_2^2\,Q_\alpha\,Q_\beta\,Q_\rho\,Q_\sigma\,\eta_{\mu\nu}}{(k_2\cdot Q)^2}$$
$$+ Q_\rho\,Q_\sigma\,\eta_{\alpha\beta}\,\eta_{\mu\nu} - \frac{2\,k_1\cdot Q\,Q_\beta\,Q_\sigma\,\eta_{\alpha\rho}\,\eta_{\mu\nu}}{k_2\cdot Q} - \frac{k_2\cdot Q\,Q_\beta\,Q_\sigma\,\eta_{\alpha\rho}\,\eta_{\mu\nu}}{k_1\cdot Q}$$
$$- \frac{2\,k_1\cdot Q\,Q_\beta\,Q_\rho\,\eta_{\alpha\sigma}\,\eta_{\mu\nu}}{k_2\cdot Q} - \frac{k_2\cdot Q\,Q_\beta\,Q_\rho\,\eta_{\alpha\sigma}\,\eta_{\mu\nu}}{k_1\cdot Q} - \frac{k_{1\sigma}\,Q_\alpha\,Q_\beta\,Q_\nu\,\eta_{\mu\rho}}{k_1\cdot Q}$$
$$- \frac{k_{1\sigma}\,Q_\alpha\,Q_\beta\,Q_\nu\,\eta_{\mu\rho}}{k_2\cdot Q} + \frac{k_{1\nu}\,Q_\alpha\,Q_\beta\,Q_\sigma\,\eta_{\mu\rho}}{k_1\cdot Q} - \frac{k_{2\nu}\,Q_\alpha\,Q_\beta\,Q_\sigma\,\eta_{\mu\rho}}{k_2\cdot Q}$$
$$- \frac{4\,k_1\cdot k_2\,Q_\alpha\,Q_\beta\,Q_\nu\,Q_\sigma\,\eta_{\mu\rho}}{k_1\cdot Q\,k_2\cdot Q} - \frac{4\,k_2^2\,Q_\alpha\,Q_\beta\,Q_\nu\,Q_\sigma\,\eta_{\mu\rho}}{k_1\cdot Q\,k_2\cdot Q} - \frac{k_1\cdot Q\,Q_\nu\,Q_\sigma\,\eta_{\alpha\beta}\,\eta_{\mu\rho}}{k_2\cdot Q}$$
$$- \frac{2\,k_2\cdot Q\,Q_\nu\,Q_\sigma\,\eta_{\alpha\beta}\,\eta_{\mu\rho}}{k_1\cdot Q} + \frac{2\,k_2\cdot Q\,Q_\beta\,Q_\sigma\,\eta_{\alpha\nu}\,\eta_{\mu\rho}}{k_1\cdot Q} + \frac{2\,k_1\cdot Q\,Q_\beta\,Q_\nu\,\eta_{\alpha\sigma}\,\eta_{\mu\rho}}{k_2\cdot Q}$$
$$+ \frac{2\,k_2\cdot Q\,Q_\beta\,Q_\nu\,\eta_{\alpha\sigma}\,\eta_{\mu\rho}}{k_1\cdot Q} - \frac{k_{1\rho}\,Q_\alpha\,Q_\beta\,Q_\nu\,\eta_{\mu\sigma}}{k_1\cdot Q} - \frac{k_{1\rho}\,Q_\alpha\,Q_\beta\,Q_\nu\,\eta_{\mu\sigma}}{k_2\cdot Q}$$
$$+ \frac{k_{1\nu}\,Q_\alpha\,Q_\beta\,Q_\rho\,\eta_{\mu\sigma}}{k_1\cdot Q} - \frac{k_{2\nu}\,Q_\alpha\,Q_\beta\,Q_\rho\,\eta_{\mu\sigma}}{k_2\cdot Q} - \frac{4\,k_1\cdot k_2\,Q_\alpha\,Q_\beta\,Q_\nu\,Q_\rho\,\eta_{\mu\sigma}}{k_1\cdot Q\,k_2\cdot Q}$$
$$- \frac{4\,k_2^2\,Q_\alpha\,Q_\beta\,Q_\nu\,Q_\rho\,\eta_{\mu\sigma}}{k_1\cdot Q\,k_2\cdot Q} - \frac{k_1\cdot Q\,Q_\nu\,Q_\rho\,\eta_{\alpha\beta}\,\eta_{\mu\sigma}}{k_2\cdot Q} - \frac{2\,k_2\cdot Q\,Q_\nu\,Q_\rho\,\eta_{\alpha\beta}\,\eta_{\mu\sigma}}{k_1\cdot Q}$$
$$+ \frac{2\,k_1\cdot Q\,Q_\beta\,Q_\nu\,\eta_{\alpha\rho}\,\eta_{\mu\sigma}}{k_2\cdot Q} + \frac{2\,k_2\cdot Q\,Q_\beta\,Q_\nu\,\eta_{\alpha\rho}\,\eta_{\mu\sigma}}{k_1\cdot Q} + \frac{2\,k_2\cdot Q\,Q_\beta\,Q_\sigma\,\eta_{\alpha\mu}\,\eta_{\nu\rho}}{k_1\cdot Q}$$
$$+ 6\,Q_\alpha\,Q_\beta\,\eta_{\mu\rho}\,\eta_{\nu\sigma} - \frac{4\,k_2\cdot Q\,Q_\alpha\,Q_\beta\,\eta_{\mu\rho}\,\eta_{\nu\sigma}}{k_1\cdot Q} - \frac{3\,k_{1\mu}\,Q_\alpha\,Q_\beta\,Q_\nu\,\eta_{\rho\sigma}}{k_1\cdot Q}$$
$$+ \frac{2\,k_{2\mu}\,Q_\alpha\,Q_\beta\,Q_\nu\,\eta_{\rho\sigma}}{k_1\cdot Q} - \frac{5\,k_{2\mu}\,Q_\alpha\,Q_\beta\,Q_\nu\,\eta_{\rho\sigma}}{k_2\cdot Q} - \frac{5\,k_{1\alpha}\,Q_\beta\,Q_\mu\,Q_\nu\,\eta_{\rho\sigma}}{k_1\cdot Q}$$
$$+ \frac{2\,k_{1\alpha}\,Q_\beta\,Q_\mu\,Q_\nu\,\eta_{\rho\sigma}}{k_2\cdot Q} - \frac{3\,k_{2\alpha}\,Q_\beta\,Q_\mu\,Q_\nu\,\eta_{\rho\sigma}}{k_2\cdot Q} + \frac{5\,k_1^2\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu\,\eta_{\rho\sigma}}{2\,(k_1\cdot Q)^2}$$

$$
\begin{aligned}
&+\frac{5\,k_2^2\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu\,\eta_{\rho\sigma}}{2\,(k_2\cdot Q)^2}+2\,Q_\mu\,Q_\nu\,\eta_{\alpha\beta}\,\eta_{\rho\sigma}-\frac{k_1\cdot Q\,Q_\beta\,Q_\nu\,\eta_{\alpha\mu}\,\eta_{\rho\sigma}}{k_2\cdot Q}\\
&-\frac{k_2\cdot Q\,Q_\beta\,Q_\nu\,\eta_{\alpha\mu}\,\eta_{\rho\sigma}}{k_1\cdot Q}-\frac{k_1\cdot Q\,Q_\beta\,Q_\mu\,\eta_{\alpha\nu}\,\eta_{\rho\sigma}}{k_2\cdot Q}-\frac{k_2\cdot Q\,Q_\beta\,Q_\mu\,\eta_{\alpha\nu}\,\eta_{\rho\sigma}}{k_1\cdot Q}\\
&+2\,Q_\alpha\,Q_\beta\,\eta_{\mu\nu}\,\eta_{\rho\sigma}\\
&+(\textit{symmetrization under }(\alpha\leftrightarrow\beta),\ (\mu\leftrightarrow\nu),\ (\rho\leftrightarrow\sigma))\,]\\
&+[(k_1,\ \alpha,\ \beta)\leftrightarrow(k_3,\ \rho,\ \sigma)]+[(k_2,\ \mu,\ \nu)\leftrightarrow(k_3,\ \rho,\ \sigma)]\,,
\end{aligned}
\tag{B.1}
$$

$$
\begin{aligned}
&\frac{1}{\kappa^3}\hat{\Gamma}^{3b}_{(\alpha\beta)(\mu\nu)(\rho\sigma)}(k_1,k_2,k_3,Q)=\\
&[-\frac{k_{1\rho}\,k_{1\sigma}\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu}{k_1\cdot Q\,k_2\cdot Q}-\frac{10\,k_{1\rho}\,k_{2\sigma}\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu}{k_1\cdot Q\,k_2\cdot Q}-\frac{k_{2\rho}\,k_{2\sigma}\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu}{k_1\cdot Q\,k_2\cdot Q}\\
&-\frac{10\,k_{1\rho}\,k_{2\mu}\,Q_\alpha\,Q_\beta\,Q_\nu\,Q_\sigma}{k_1\cdot Q\,k_2\cdot Q}+\frac{6\,k_{2\mu}\,k_{2\rho}\,Q_\alpha\,Q_\beta\,Q_\nu\,Q_\sigma}{k_1\cdot Q\,k_2\cdot Q}+\frac{6\,k_{1\alpha}\,k_{1\rho}\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\sigma}{k_1\cdot Q\,k_2\cdot Q}\\
&-\frac{10\,k_{1\alpha}\,k_{2\rho}\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\sigma}{k_1\cdot Q\,k_2\cdot Q}+\frac{4\,k_{1\rho}\,k_2^2\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\sigma}{k_1\cdot Q\,(k_2\cdot Q)^2}-\frac{4\,k_1^2\,k_{1\rho}\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\sigma}{(k_1\cdot Q)^2\,k_2\cdot Q}\\
&-\frac{4\,k_2^2\,k_{2\rho}\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\sigma}{k_1\cdot Q\,(k_2\cdot Q)^2}+\frac{4\,k_1^2\,k_{2\rho}\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\sigma}{(k_1\cdot Q)^2\,k_2\cdot Q}-\frac{k_{2\mu}\,k_{2\nu}\,Q_\alpha\,Q_\beta\,Q_\rho\,Q_\sigma}{k_1\cdot Q\,k_2\cdot Q}\\
&-\frac{6\,k_{1\alpha}\,k_{2\mu}\,Q_\beta\,Q_\nu\,Q_\rho\,Q_\sigma}{k_1\cdot Q\,k_2\cdot Q}-\frac{4\,k_2^2\,k_{2\mu}\,Q_\alpha\,Q_\beta\,Q_\nu\,Q_\rho\,Q_\sigma}{k_1\cdot Q\,(k_2\cdot Q)^2}+\frac{4\,k_1^2\,k_{2\mu}\,Q_\alpha\,Q_\beta\,Q_\nu\,Q_\rho\,Q_\sigma}{(k_1\cdot Q)^2\,k_2\cdot Q}\\
&-\frac{k_{1\alpha}\,k_{1\beta}\,Q_\mu\,Q_\nu\,Q_\rho\,Q_\sigma}{k_1\cdot Q\,k_2\cdot Q}+\frac{4\,k_{1\alpha}\,k_2^2\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\rho\,Q_\sigma}{k_1\cdot Q\,(k_2\cdot Q)^2}-\frac{4\,k_1^2\,k_{1\alpha}\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\rho\,Q_\sigma}{k_1\cdot Q^2\,k_2\cdot Q}\\
&+\frac{2\,k_2^4\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\rho\,Q_\sigma}{k_1\cdot Q\,(k_2\cdot Q)^3}-\frac{2\,k_1^2\,k_2^2\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\rho\,Q_\sigma}{(k_1\cdot Q)^2\,(k_2\cdot Q)^2}+\frac{2\,k_1^4\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\rho\,Q_\sigma}{(k_1\cdot Q)^3\,k_2\cdot Q}\\
&+(\textit{symmetrization under }(\alpha\leftrightarrow\beta),\ (\mu\leftrightarrow\nu),\ (\rho\leftrightarrow\sigma))\,]\\
&+[(k_1,\ \alpha,\ \beta)\leftrightarrow(k_3,\ \rho,\ \sigma)]+[(k_2,\ \mu,\ \nu)\leftrightarrow(k_3,\ \rho,\ \sigma)]\,,
\end{aligned}
\tag{B.2}
$$

$$
\begin{aligned}
&\frac{1}{\kappa^3}\hat{\Gamma}^{3c}_{(\alpha\beta)(\mu\nu)(\rho\sigma)}(k_1,k_2,k_3,Q)=\\
&+[\frac{4\,k_{1\rho}\,Q_\mu\,Q_\nu\,Q_\sigma\,\eta_{\alpha\beta}}{k_1\cdot Q}+\frac{4\,k_{1\mu}\,Q_\nu\,Q_\rho\,Q_\sigma\,\eta_{\alpha\beta}}{k_1\cdot Q}-\frac{2\,k_1^2\,Q_\mu\,Q_\nu\,Q_\rho\,Q_\sigma\,\eta_{\alpha\beta}}{(k_1\cdot Q)^2}\\
&-8\,Q_\rho\,Q_\sigma\,\eta_{\alpha\mu}\,\eta_{\beta\nu}-8\,Q_\mu\,Q_\nu\,\eta_{\alpha\rho}\,\eta_{\beta\sigma}+\frac{10\,k_{1\rho}\,Q_\alpha\,Q_\beta\,Q_\sigma\,\eta_{\mu\nu}}{k_1\cdot Q}\\
&+\frac{10\,k_{1\alpha}\,Q_\beta\,Q_\rho\,Q_\sigma\,\eta_{\mu\nu}}{k_1\cdot Q}-\frac{5\,k_1^2\,Q_\alpha\,Q_\beta\,Q_\rho\,Q_\sigma\,\eta_{\mu\nu}}{(k_1\cdot Q)^2}-3\,Q_\rho\,Q_\sigma\,\eta_{\alpha\beta}\,\eta_{\mu\nu}\\
&-Q_\beta\,Q_\sigma\,\eta_{\alpha\rho}\,\eta_{\mu\nu}-Q_\beta\,Q_\rho\,\eta_{\alpha\sigma}\,\eta_{\mu\nu}-2\,Q_\nu\,Q_\sigma\,\eta_{\alpha\beta}\,\eta_{\mu\rho}\\
&+2\,Q_\beta\,Q_\sigma\,\eta_{\alpha\nu}\,\eta_{\mu\rho}+2\,Q_\beta\,Q_\nu\,\eta_{\alpha\sigma}\,\eta_{\mu\rho}-2\,Q_\nu\,Q_\rho\,\eta_{\alpha\beta}\,\eta_{\mu\sigma}\\
&+2\,Q_\beta\,Q_\nu\,\eta_{\alpha\rho}\,\eta_{\mu\sigma}+2\,Q_\beta\,Q_\sigma\,\eta_{\alpha\mu}\,\eta_{\nu\rho}-20\,Q_\alpha\,Q_\beta\,\eta_{\mu\rho}\,\eta_{\nu\sigma}\\
&+\frac{10\,k_{1\mu}\,Q_\alpha\,Q_\beta\,Q_\nu\,\eta_{\rho\sigma}}{k_1\cdot Q}+\frac{10\,k_{1\alpha}\,Q_\beta\,Q_\mu\,Q_\nu\,\eta_{\rho\sigma}}{k_1\cdot Q}-\frac{5\,k_1^2\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu\,\eta_{\rho\sigma}}{(k_1\cdot Q)^2}\\
&-3\,Q_\mu\,Q_\nu\,\eta_{\alpha\beta}\,\eta_{\rho\sigma}-Q_\beta\,Q_\nu\,\eta_{\alpha\mu}\,\eta_{\rho\sigma}-Q_\beta\,Q_\mu\,\eta_{\alpha\nu}\,\eta_{\rho\sigma}\\
&-2\,Q_\alpha\,Q_\beta\,\eta_{\mu\nu}\,\eta_{\rho\sigma}\\
&+(\textit{symmetrization under }(\alpha\leftrightarrow\beta),\ (\mu\leftrightarrow\nu),\ (\rho\leftrightarrow\sigma))\,]\\
&+[\textit{ciclic permutations of }(k_1,\ \alpha,\ \beta)\ (k_2,\ \mu,\ \nu)\textit{ and }(k_3,\ \rho,\ \sigma)]\,,
\end{aligned}
\tag{B.3}
$$

$$
\begin{aligned}
&\frac{1}{\kappa^3}\hat{\Gamma}^{3d}_{(\alpha\beta)(\mu\nu)(\rho\sigma)}(k_1,k_2,k_3,Q)=\\
&+9\,Q_\rho\,Q_\sigma\,\eta_{\alpha\nu}\,\eta_{\beta\mu}+9\,Q_\rho\,Q_\sigma\,\eta_{\alpha\mu}\,\eta_{\beta\nu}+9\,Q_\mu\,Q_\nu\,\eta_{\alpha\sigma}\,\eta_{\beta\rho}\\
&+9\,Q_\mu\,Q_\nu\,\eta_{\alpha\rho}\,\eta_{\beta\sigma}+3\,Q_\rho\,Q_\sigma\,\eta_{\alpha\beta}\,\eta_{\mu\nu}+9\,Q_\alpha\,Q_\beta\,\eta_{\mu\sigma}\,\eta_{\nu\rho}\\
&+9\,Q_\alpha\,Q_\beta\,\eta_{\mu\rho}\,\eta_{\nu\sigma}+3\,Q_\mu\,Q_\nu\,\eta_{\alpha\beta}\,\eta_{\rho\sigma}+3\,Q_\alpha\,Q_\beta\,\eta_{\mu\nu}\,\eta_{\rho\sigma}\,,
\end{aligned}
\tag{B.4}
$$

The sum of these contributions gives, with the help of the eikonal identity (3.2), the following result:

$$
\begin{aligned}
&\frac{1}{\kappa^3}\hat{\Gamma}^3_{(\alpha\beta)(\mu\nu)(\rho\sigma)}(k_1,k_2,k_3,Q) = \\
&2\Big[\frac{k_{1\rho}\,k_{2\sigma}\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu}{k_1\cdot Q\,k_2\cdot Q}+\frac{k_{1\rho}\,k_{2\mu}\,Q_\alpha\,Q_\beta\,Q_\nu\,Q_\sigma}{k_1\cdot Q\,k_2\cdot Q}-\frac{k_{2\mu}\,k_{2\rho}\,Q_\alpha\,Q_\beta\,Q_\nu\,Q_\sigma}{k_1\cdot Q\,k_2\cdot Q} \\
&-\frac{k_{1\alpha}\,k_{1\rho}\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\sigma}{k_1\cdot Q\,k_2\cdot Q}+\frac{k_{1\alpha}\,k_{2\rho}\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\sigma}{k_1\cdot Q\,k_2\cdot Q}-\frac{k_{1\rho}\,k_2^2\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\sigma}{2\,k_1\cdot Q\,(k_2\cdot Q)^2} \\
&+\frac{k_1^2\,k_{1\rho}\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\sigma}{2\,(k_1\cdot Q)^2\,k_2\cdot Q}+\frac{k_2^2\,k_{2\rho}\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\sigma}{2\,k_1\cdot Q\,(k_2\cdot Q)^2}-\frac{k_1^2\,k_{2\rho}\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\sigma}{2\,(k_1\cdot Q)^2\,k_2\cdot Q} \\
&+\frac{k_{1\alpha}\,k_{2\mu}\,Q_\beta\,Q_\nu\,Q_\rho\,Q_\sigma}{k_1\cdot Q\,k_2\cdot Q}+\frac{k_2^2\,k_{2\mu}\,Q_\alpha\,Q_\beta\,Q_\nu\,Q_\rho\,Q_\sigma}{2\,k_1\cdot Q\,(k_2\cdot Q)^2}-\frac{k_1^2\,k_{2\mu}\,Q_\alpha\,Q_\beta\,Q_\nu\,Q_\rho\,Q_\sigma}{2\,(k_1\cdot Q)^2\,k_2\cdot Q} \\
&-\frac{k_{1\alpha}\,k_2^2\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\rho\,Q_\sigma}{2\,k_1\cdot Q\,(k_2\cdot Q)^2}+\frac{k_1^2\,k_{1\alpha}\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\rho\,Q_\sigma}{2\,(k_1\cdot Q)^2\,k_2\cdot Q}-\frac{k_2^4\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\rho\,Q_\sigma}{4\,k_1\cdot Q\,(k_2\cdot Q)^3} \\
&+\frac{k_1^2\,k_2^2\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\rho\,Q_\sigma}{4\,(k_1\cdot Q)^2\,(k_2\cdot Q)^2}-\frac{k_1^4\,Q_\alpha\,Q_\beta\,Q_\mu\,Q_\nu\,Q_\rho\,Q_\sigma}{4\,(k_1\cdot Q)^3\,k_2\cdot Q} \\
&+(\textit{symmetrization under } (\alpha\leftrightarrow\beta),\ (\mu\leftrightarrow\nu),\ (\rho\leftrightarrow\sigma))\,\Big] \\
&+2[(k_1,\alpha,\beta)\leftrightarrow(k_3,\rho,\sigma)]+2[(k_2,\mu,\nu)\leftrightarrow(k_3,\rho,\sigma)]\,.
\end{aligned}
\tag{B.5}
$$

Using this result, together with the eikonal identity, we arrive after a straightforward calculation at the expression given by equation (3.5).

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FIGURE CAPTIONS

Fig.1 – Lowest order contributions to the thermal graviton self-energy. Wavy lines denote gravitons and dashed lines represent ghost particles.

Fig.2 – The forward scattering diagrams corresponding to Fig. 1. Crossed graphs with $k \to -k$ are to be understood.

Fig.3 – Feynman diagrams contributing to the thermal 3-graviton vertex function. Graphs obtained by permutations of external gravitons in (c) are to be understood.

Fig.4 – Examples of forward scattering graphs connected with Fig. 3. Diagrams obtained by permutations of external gravitons should be understood.

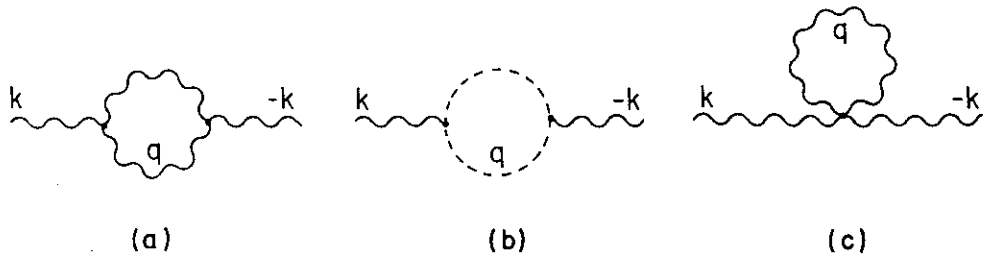

Figure 1

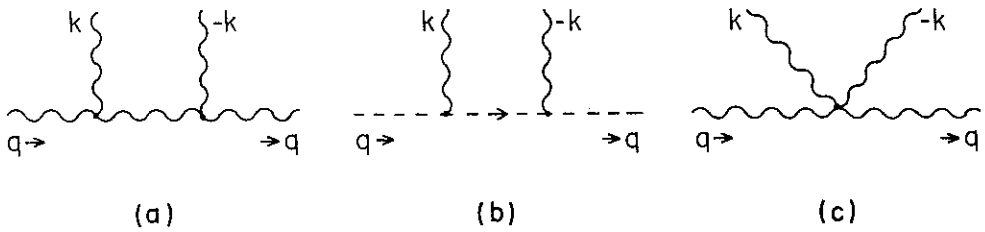

Figure 2

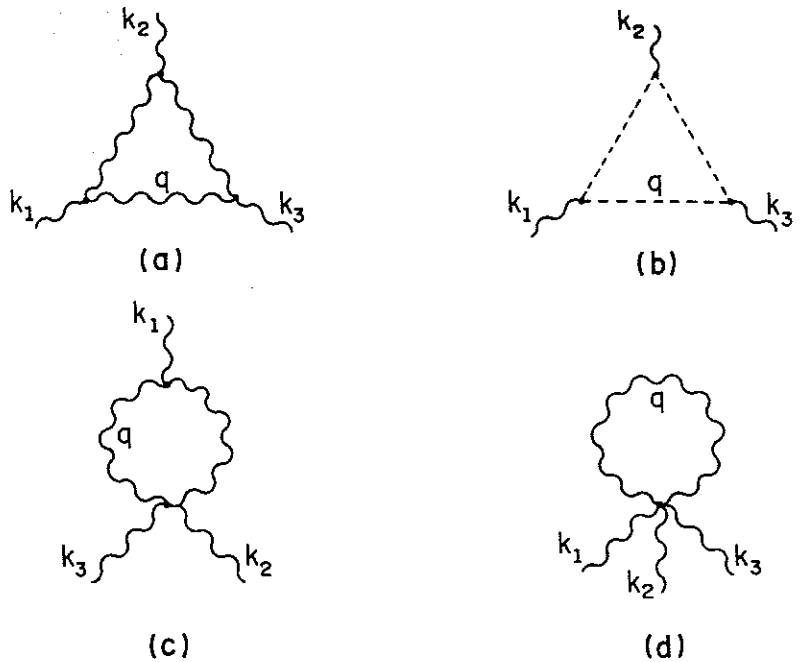

Figure 3

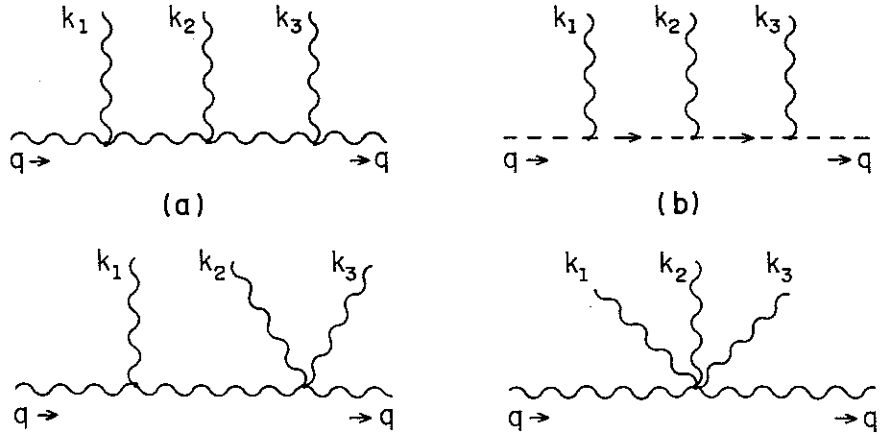

Figure 4