

UNIVERSIDADE DE SÃO PAULO

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INSTITUTO DE FÍSICA
CAIXA POSTAL 20516
01498-970 SÃO PAULO - SP
BRASIL

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TREE AMPLITUDES IN NONCRITICAL $N=2$
STRINGS

D. Dalmazi

Instituto de Física, Universidade de São Paulo

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D.Dalmazi* †

Instituto de Física, Univ. São Paulo, CP 20516, São Paulo, Brazil

Abstract

Recent results for tree amplitudes for the $N = 2$ noncritical strings are presented and compared with the critical case. Arguments are given which indicate a certain discontinuity in passing from the $\hat{c} < 1$ model (in a Coulomb gas representation) to the $\hat{c} = 1$ critical case.

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† E-mail: dalmazi@uspif.if.usp.br

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D.Dalmazi

Instituto de Física, Univ. São Paulo, CP 20516, São Paulo, Brazil

1. Introduction

In the last few years noncritical strings have been extensively studied from many different viewpoints. Specially exciting is the nonperturbative aspect of the results furnished by the discrete approach based on matrix models. The continuous approach on the other hand has proven to be less powerful for higher genus calculations but the tree level (genus zero) correlators¹⁾ of physical operators agree with the matrix model results for the sphere for $c = 1$ (see [2] for a comparison). For $c < 1$ the spectrum of physical operators is not exactly the same (see comments in [3]) but the scaling behaviour of the correlators agree⁴⁾ with the results predicted by matrix models. Although less powerful, the continuous approach is most easily generalized to the supersymmetric strings. We have studied such generalization in [5] for the case of the $N = 1$ noncritical string by calculating correlation functions in a $\hat{c} \leq 1$ ($c_m = \frac{3\hat{c}}{2}$) $N = 1$ matter in a Coulomb gas representation conformally coupled to a $N = 1$ superliouville theory (see also [1,6]). The results obtained in the NS-sector are very simple and similar to the bosonic case ($N = 0$). In the $N = 0$ and $N = 1$ (NS-sector) cases the only propagating particle is a massless scalar. Such particle is a remnant of the tachyon ground state of the respective critical dimensions $d = 26, 10$. Due to the low dimensionality ($d = \hat{c} + 1 \leq 2$) of the noncritical theories the remnants of the excited states can only appear as poles in the amplitudes for certain discrete values of the momentum. From this point of view the situation is similar to the critical $N = 2$ string which contains only a massless scalar field (a deformation of the Kähler potential) in its spectrum in the NS-sector. Tree scattering amplitudes of this particle have been calculated in [7] and the expected simplicity was confirmed by the vanishing of the four-point function. Those results suggest the study of a possible $N = 2$ noncritical string. In particular, due to the finiteness of the spectrum of the critical case, we do not expect an infinite tower of discrete states in the $d \rightarrow 2^-$ limit like the $N = 0, 1$ noncritical strings. In a recent work³⁾ we made an attempt to understand noncritical $N = 2$ strings and the preliminary results that we found will be presented and commented in the third and final section of this talk. For sake of comparison the calculations of [7] for the critical case are reviewed in some detail in the next section. We finish by mentioning some open problems and perspectives of the $N = 2$ noncritical string.

2. The Critical Case

The critical $N = 2$ string lives in a space time with $d = 2$ complex dimensions described by the $N = 2$ chiral (anti-chiral) superfields $X^i(\bar{X}^i, i = 1, 2)$. On shell we have the decomposition⁷⁾ (analogously for \bar{X}^i):

$$X^i = x^i(z, \bar{z}) + \psi_R^i(z)\theta^- + \psi_L^i(\bar{z})\bar{\theta}^- - \partial x^i\theta^+\theta^- - \bar{\partial}x^i\bar{\theta}^+\bar{\theta}^- \quad (1)$$

where $(\theta^\pm)^\dagger = \bar{\theta}^\mp$. The component fields have in our notation the following propagators:

$$\begin{aligned} \langle x^i(z)\bar{x}^j(w) \rangle &= \eta^{ij} \ln |z - w|^{-2} \\ \langle \psi_R^i(z)\bar{\psi}_R^j(w) \rangle &= \langle \psi_L^i(\bar{z})\bar{\psi}_L^j(\bar{w}) \rangle^* = 2\eta^{ij}(z - w)^{-1} \end{aligned} \quad (2)$$

where $\eta^{ij} = (+, -)$. The vertex operator below represents the massless scalar particle mentioned in the introduction:

$$V(k) = \int d^2z d^4\theta e^{ik \cdot \bar{X}(z) + i\bar{k} \cdot X(z)} \quad (3)$$

In order that $V(k)$ be a physical operator its $U(1)$ charge (q) and conformal weight (Δ) must vanish. The first requirement is automatically satisfied by (3) and the second one imply the on shell condition:

$$k \cdot \bar{k} = k_1 \bar{k}_1 - k_2 \bar{k}_2 = 0. \quad (4)$$

Now we can calculate n -particle amplitudes:

$$\mathcal{A}_n = \langle V_{k_1} \cdots V_{k_n} \rangle = \prod_{i=1}^n \int d^2z_i d^4\theta_i \left\langle \prod_{i=1}^N e^{i(k_i \bar{X}_i + \bar{k}_i X_i)} \right\rangle \quad (5)$$

Integration over the zero-modes x_0^i, y_0^j of the first component of the supercoordinate ($x^j = x^j + iy^j$) leads to the momentum and energy conservation:

$$\sum_{j=1}^n k_j = 0 = \sum_{j=1}^n \bar{k}_j \quad (6)$$

Furthermore, the residual $OSP(2, 2)$ symmetry of the superconformal gauge permits us to fix, e.g., $\theta_1^{(\pm)} = \bar{\theta}_3^{(\pm)} = 0$ and $z_1 = \infty, z_2 = 1, z_3 = 0$. In this case we have for the 3-particle scattering:

$$\begin{aligned} \mathcal{A}_3 = & \left\langle e^{i(k_3 \cdot \bar{x}(0) + \bar{k}_3 \cdot x(0))} e^{i(k_2 \cdot \bar{x}(1) + \bar{k}_2 \cdot x(1))} [i k_2 \cdot \bar{\partial} \bar{x} - i \bar{k}_2 \cdot \partial x - (k_2 \cdot \bar{\psi}_R)(\bar{k}_2 \cdot \psi_R)] \right. \\ & \left. \times [i k_2 \cdot \bar{\partial} \bar{x} - i \bar{k}_2 \cdot \partial x - (k_2 \cdot \bar{\psi}_L)(\bar{k}_2 \cdot \psi_L)] \right\rangle \end{aligned} \quad (7)$$

Using the propagators (2) we reproduce Ooguri and Vafa's result⁷⁾:

$$\mathcal{A}_3 = (c_{23})^2 \quad (8)$$

where $c_{ij} = k_i \cdot \bar{k}_j - \bar{k}_i \cdot k_j$.

For the four-particle scattering Ooguri and Vafa have obtained⁷⁾ (in the gauge $\theta_1^{(\pm)} = \bar{\theta}_4^{(\pm)} = 0, z_1 = \infty, z_2 = 1, z_3 = z, z_4 = 0$):

$$\mathcal{A}_4 = \int d^2z |z|^{2s_{34}} |1 - z|^{2s_{32}} \left| \frac{s_{32}(s_{32} - 1)}{(1 - z)^2} + \frac{c_{12}c_{34}}{z} + \frac{c_{23}c_{41}}{1 - z} \right|^2 \quad (9)$$

where $s_{ij} = k_i \cdot \bar{k}_j + \bar{k}_i \cdot k_j$. The above integral can be calculated by generalizing the technique of analytic continuation of Dotsenko⁹⁾ implemented originally in the calculation of the simpler integral $\int d^2z |z|^{2\alpha} |1 - z|^{2\beta}$ (see also [10]). Finally one obtains:

$$\mathcal{A}_4 = -\frac{\pi F^2}{16} \Delta(1 + s_{34}) \Delta(1 + s_{14}) \Delta(1 + s_{24}) \quad (10)$$

where $\Delta(x) = \Gamma(x)/\Gamma(1 - x)$ and

$$F = 1 - \frac{c_{23}c_{41}}{s_{14}s_{24}} - \frac{c_{34}c_{12}}{s_{34}s_{24}} \quad (11)$$

Above, we have used the identities:

$$s_{32} = s_{14}, \quad s_{14} + s_{24} + s_{34} = 0. \quad (12)$$

It turns out that after use of the on shell condition (4) one gets $F = 0$ identically, and therefore:

$$\mathcal{A}_4 = 0 \quad (13)$$

It is expected that higher point amplitudes also vanish in the same fashion⁷⁾.

Before we finish this section note that formula (10) may be checked quite quickly by looking at the residues of (9) even though we did not know how to calculate exactly the complicated integrals (9). For instance, let us calculate the residue (R) of \mathcal{A}_4 at the first pole of the (34)-channel. Supposing $s_{34} = -1 + \epsilon$ we have:

$$S = S_{\Phi} + S_X$$

$$S_{\Phi} = \frac{1}{4\pi} \int d^2w d^4\theta E (\Phi \bar{\Phi} - \partial_Y \Phi + \bar{\Phi})$$

$$S_X = \frac{1}{4\pi} \int d^2w d^4\theta \bar{E} (X \bar{X} + 2i\alpha_0 Y(X + \bar{X}))$$
(18)

Similar to the $N = 0, 1$ noncritical 2d-strings studied in [1,5] it is natural to consider the coupling of a chiral (anti-chiral) $N = 2$ superfield $X(X)$ in a Coulomb gas representation with $\tilde{\epsilon} \leq 1$ ($\epsilon X = 3\tilde{\epsilon}$) to a superconformal chiral (anti-chiral) superfield $\Phi(\Phi)$ such that the total action (suppressing the cosmological terms⁸) is given by:

3. The Noncritical Case

confirms formula (10). This is the result expected from (10). The analysis of other poles and channels also

$$R = \pi [F(s_{34} = -1)]^2 (1 - s_{14})^2 (s_{14})^2 = \pi [F(s_{34} = -1)]^2 \Delta(1 + s_{14}) \Delta(1 + s_{24})$$
(17)

Using the identities (12) we have:

$$\frac{\pi}{R} = (s_{32}(s_{32} - 1) + c_{23}c_{41})^2$$

$$+ (c_{34}c_{12})^2 \int d^2z d^2\theta \delta^{(2)}(z) |1 - z|^{2s_{32}}$$

$$+ c_{34}c_{12}c_{23}c_{41} \int d^2z d^2\theta \delta^{(2)}(z) (1 - z)^{s_{32}} (1 - \bar{z})^{s_{32}-1}$$

$$+ c_{34}c_{12}c_{23}c_{41} \int d^2z d^2\theta \delta^{(2)}(z) (1 - z)^{s_{32}} (1 - \bar{z})^{s_{32}-2}$$

$$+ c_{34}c_{12}c_{23}c_{41} \int d^2z d^2\theta \delta^{(2)}(z) |1 - z|^{2s_{32}}$$

$$+ [s_{32}(s_{32} - 1)]^2 \left(1 + \frac{s_{32}(s_{32} - 1)}{c_{23}c_{41}} + \frac{c_{34}c_{12}}{c_{23}c_{41}} \right)^2$$
(16)

Thus we obtain,

$$|z|^{-2+\epsilon} = \frac{\epsilon}{\pi} \delta^{(2)}(z).$$
(15)

Next we use the following representation for the distribution $|z|^{-2+\epsilon}$:

$$\frac{\epsilon}{R} = \int d^2z |z|^{-2+\epsilon} |1 - z|^{2s_{32}} \left| \frac{s_{32}(s_{32} - 1)}{c_{12}c_{34}} + \frac{z}{1 - z} + \frac{z}{c_{23}c_{41}} \right|^2$$
(14)

$$\Delta \left(e^{i(k\bar{X} + \bar{k}X + \beta\bar{\Phi} + \beta\Phi)} \right) = 2 \left[(k - \alpha_0)(\bar{k} - \alpha_0) - (\beta - \alpha_0) - (\beta + \frac{\alpha_0}{2}) \right]$$

$$\bar{\Delta} \left(e^{i(k\bar{X} + \bar{k}X + \beta\bar{\Phi} + \beta\Phi)} \right) = 2\alpha_0(k - \bar{k}) + \partial(\beta - \bar{\beta}) = 0$$
(25)

the dressings β and $\bar{\beta}$:

Imposing vanishing conformal weight and $U(1)$ charge we have two equations for

$$V(k, \bar{k}) = \int d^2z d^4\theta e^{i(k\bar{X} + \bar{k}X + \beta\bar{\Phi} + \beta\Phi)}$$
(24)

Still following [11] the noncritical version of the vertex (3) is defined by:

$$\partial = 2|\alpha_0|.$$
(23)

Following [11] we require a vanishing total central charge, fixing ∂ to be :

$$\langle \phi(z) \bar{\phi}(w) \rangle = \ln |z - w|^{-2}$$

$$\langle \psi^R(z) \bar{\psi}^R(w) \rangle = \langle \zeta^R(z) \bar{\zeta}^R(w) \rangle = 2(z - w)^{-1}$$

$$\langle \psi^L(z) \bar{\psi}^L(w) \rangle = \langle \zeta^L(z) \bar{\zeta}^L(w) \rangle = 2(\bar{z} - \bar{w})^{-1}$$
(22)

Analogously to (2) we have the following propagators:

$$J_X = \frac{1}{4} : \bar{\psi}^R \psi^R \phi^R : - i\alpha_0 \partial(x - \bar{x})$$

$$J_{\bar{\Phi}} = \frac{1}{4} : \bar{\zeta}^R \zeta^R : + \frac{\partial}{2} \partial(\phi - \bar{\phi})$$

$$J = J_{\bar{\Phi}} + J_X$$
(21)

$$T_X = - : \partial \bar{x} \partial x : + \frac{1}{4} : \bar{\psi}^R \partial \psi^R : + \frac{1}{4} : \bar{\psi}^R \partial \psi^R : + i\alpha_0 \partial^2(x + \bar{x})$$

$$T_{\bar{\Phi}} = - : \partial \phi \partial \bar{\phi} : + \frac{1}{4} : \bar{\zeta}^R \partial \zeta^R : + \frac{1}{4} : \bar{\zeta}^R \partial \zeta^R : - \frac{\partial}{2} \partial^2(\phi + \bar{\phi})$$

$$T = T_{\bar{\Phi}} + T_X$$
(20)

The total energy momentum tensor and $U(1)$ current are given respectively by:

$$\Phi = \phi(z, \bar{z}) + \zeta^R(z) \bar{\theta} - \partial \phi \partial \bar{\theta} - \partial \phi \partial \bar{\theta}$$
(19)

The quantity \bar{Y} stands for the $N = 2$ supercurvature superfield and \bar{E} for the superdeterminant of the superzweibein. In order to introduce the notation we give the on shell decomposition (analogous for $\bar{\Phi}$, see also (1)) :

The second equation of (25) determines the imaginary part of the dressing (for $\alpha_0 \neq 0$) and by plugging it in the first one we obtain the real part of the dressing (up to a sign):

$$E_{\pm} = \left(\frac{\beta + \bar{\beta}}{2} \right)_{\pm} + \frac{Q}{2} = \pm \left| \frac{k + \bar{k}}{2} - \alpha_0 \right| \quad (26)$$

where E_{\pm} is the energy associated with the time direction ($\Phi + \bar{\Phi}$). Following Seiberg¹²⁾ we take henceforth only positive energy solutions (E_+).

Having defined the vertex operator we can start calculating n -point correlation functions $\mathcal{A}_n = \langle V_{k_1} \cdots V_{k_n} \rangle$. Integrating over the double zero-modes of x and Φ we obtain the momentum and energy conservation laws, respectively*:

$$\sum_{j=1}^n k_j = 2\alpha_0 = \sum_{j=1}^n \bar{k}_j \quad (27)$$

$$\sum_1^n \beta_i + Q = 0 = \sum_1^n \bar{\beta}_i + Q \quad (28)$$

The calculation of the amplitudes is very similar to the critical case and we obtain for the 3-point function:

$$\mathcal{A}_3 = (\ln \mu)^2 (c_{23})^2 \quad (29)$$

where now,

$$\begin{aligned} c_{ij} &= k_i \cdot \bar{k}_j - \bar{k}_i \cdot k_j \\ &= k_i \bar{k}_j - \beta_i \bar{\beta}_j - \bar{k}_i k_j + \bar{\beta}_i \beta_j \end{aligned} \quad (30)$$

The overall factor $(\ln \mu)^2$ comes from the volume of the Liouville zero-modes (see [8]) with μ interpreted as a cosmological constant. Now the important difference with respect to the critical case comes from the non analytical structure of the dispersion relation (26) which allows us to eliminate completely, in a given kinematic region, the real part of the momentum of one of the scattered particles and to rewrite \mathcal{A}_3 in a factorized form. For instance, in the region** $\Re k_2, \Re k_3 \leq \alpha_0, \Re k_1 \geq \alpha_0 < 0$ we have :

$$\Re k_1 = 0 \quad ; \quad k_1 \cdot \bar{k}_1 = 0 \quad (31)$$

and (29) can be written as³⁾:

$$\mathcal{A}_3 = (\ln \mu)^2 \left(\frac{|k_1|}{\alpha_0} \right)^2 (k_2 \cdot \bar{k}_2)(k_3 \cdot \bar{k}_3) \quad (32)$$

In the same way if we repeat now the calculation for \mathcal{A}_4 we get at the first sight the same result of the critical case, i.e. (10), multiplied by the factor $(\ln \mu)^2$. But in a given kinematic region, e.g., $\Re k_1, \Re k_2, \Re k_3 \leq \alpha_0, \Re k_4 \geq \alpha_0 < 0$, we have:

$$\Re k_4 = -\alpha_0 \quad ; \quad k_4 \cdot \bar{k}_4 = 0 \quad (33)$$

and after some algebra,

$$F = \frac{1}{4} \left(\frac{|k_4|}{\alpha_0} \right)^2 \prod_{i=1}^3 \frac{(k_i \cdot \bar{k}_i)}{s_{i4}} \quad (34)$$

By further noticing that, in the above kinematic region, $s_{i4} = -k_i \cdot \bar{k}_i$; we can write \mathcal{A}_4 (see (10)) in a completely factorized form³⁾:

$$\mathcal{A}_4 = \frac{\pi (\ln \mu)^2}{16} \left(\frac{|k_4|}{\alpha_0} \right)^4 \prod_1^3 \Delta(1 - k_i \cdot \bar{k}_i). \quad (35)$$

It's important to remark that in any kinematic region where at least two particles satisfy $\Re k_i \geq \alpha_0 < 0$ ($\rightarrow k_i \cdot \bar{k}_i = 0$) both amplitudes \mathcal{A}_3 and \mathcal{A}_4 vanish.

We can start now the analysis of the results (32) and (35) observing that when we take $\alpha_0 = 0$ ($\hat{c} = 1$) the $U(1)$ charge of the vertex operator vanishes identically (see (25)) and we have no restrictions on the imaginary part of the dressing ($\beta - \bar{\beta}$). In particular, this means that the dispersion relation (26) does not apply to the $\alpha_0 = 0$ case. Therefore the factorized results that we have obtained so far are only true, strictly speaking, for $\hat{c} < 1$ ($\alpha_0 \neq 0$). For $\alpha_0 = 0$ the requirement of vanishing conformal weight (see (25)) just reproduces the on shell condition $k \cdot \bar{k} = k\bar{k} - \beta\bar{\beta} = 0$ and we recover the critical case whose amplitudes have been already calculated by Ooguri and Vafa⁷⁾. The discontinuous nature of the $\hat{c} \rightarrow 1^-$ ($\alpha_0 \rightarrow 0$) limit can be also seen from other points of view. Note, for instance, that contrary to the $N = 0, 1$ noncritical strings, in the $N = 2$ case it is impossible to obtain the $\hat{c} = 1$ noncritical theory by an appropriate rotation of the $\hat{c} < 1$ model (see (20), (21) and (23)). One can also take the factorized expression (35) for \mathcal{A}_4 in the limit $\alpha_0 \rightarrow 0$ to see that the result diverges with $\frac{1}{\alpha_0}$ which shows the non existence of discrete states in the $\hat{c} \rightarrow 1^-$ limit, as expected. For $\hat{c} < 1$ the interesting models are the minimal ones for which the functions $\Delta(1 - k \cdot \bar{k})$ have no poles or zeroes. Thus, as in the $N = 0, 1$ noncritical strings, these functions have a mild effect and can be absorbed through redefinitions of the vertices.

Concluding we must say that there are still many aspects of $N = 2$ noncritical strings to be understood in the continuous which might be useful in developing super matrix models. In particular we do not know how to continue for other kinematic regions the results that we have derived in a given region. For this aim it may be useful to calculate higher point functions (work in progress), as well as, to properly

* We have used that on the sphere $\int d^2z \sqrt{\hat{g}} \hat{r} = 8\pi$.

** Calculations for $\alpha_0 > 0$ are completely analogous.

include the cosmological terms^{8),13)} to understand the space time picture behind the amplitudes that we have obtained.

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