

UNIVERSIDADE DE SÃO PAULO

**PUBLICAÇÕES**

**INSTITUTO DE FÍSICA  
CAIXA POSTAL 20516  
01498-970 SÃO PAULO - SP  
BRASIL**

IFUSP/P-1024

**ON MINIMAL COUPLING IN EINSTEIN-CARTAN  
SPACE-TIMES**

**Alberto Saa**

Instituto de Física, Universidade de São Paulo

Janeiro/1993

## On Minimal Coupling in Einstein-Cartan Space-Times

Alberto Saa

*Instituto de Física*  
*Universidade de São Paulo, Caixa Postal 20516*  
*01498-970 São Paulo, S.P., Brazil*

### Abstract

The usual minimal coupling procedure is investigated in Einstein-Cartan space-times, and motivated by some problems with scalar fields a new procedure is proposed. By this procedure new effects are predicted.

It is a statement in the literature that scalar fields should neither feel nor produce torsion as a consequence of minimal coupling procedure (MCP) used at the action level [1]. This procedure consists in, starting with a Lorentz invariant action in Minkowski space-time, substituting the metric tensor  $\eta_{\mu\nu}$  by its generalization  $g_{\mu\nu}(x)$ , partial derivatives  $\partial_\mu$  by covariant ones and the measure of integration, the "volume element"  $d^4x$ , by its covariant form  $\sqrt{-g}d^4x$ . The purpose of this note is to argue that the usual MCP cannot be used in a general Einstein-Cartan space-time at the action level, and to propose a new procedure. By this procedure, new effects are possible, and in particular, scalar fields are sensitive to the non-Riemannian structure of space-time.

The Einstein-Cartan space-time  $U_4$  is characterized by its metric  $g_{\mu\nu}(x)$  and by its metric-compatible connection  $\Gamma_{\alpha\beta}^\mu$ , which is used to define the covariant derivative

$$D_\nu A^\mu = \partial_\nu A^\mu + \Gamma_{\nu\rho}^\mu A^\rho. \quad (1)$$

The  $U_4$  connection is non-symmetric in its lower indices, and from its anti-symmetric part can be defined the torsion tensor

$$S_{\alpha\beta}^\gamma = \frac{1}{2} (\Gamma_{\alpha\beta}^\gamma - \Gamma_{\beta\alpha}^\gamma), \quad (2)$$

and one can write the connection as a function of the torsion tensor

$$\Gamma_{\alpha\beta}^\gamma = \left\{ \begin{matrix} \gamma \\ \alpha\beta \end{matrix} \right\} + S_{\alpha\beta}^\gamma - S_{\beta\alpha}^\gamma + S_{\alpha\beta}^\gamma. \quad (3)$$

where  $\left\{ \begin{matrix} \gamma \\ \alpha\beta \end{matrix} \right\}$  are the usual Christoffel symbols from Riemannian space-time  $V_4$ .

Using MCP, we get the following action for the case of a massless scalar field on  $U_4$  (the presence of mass makes no difference for our purposes)

$$S = \int d^4x \frac{\sqrt{-g}}{2} \partial^\mu \varphi \partial_\mu \varphi, \quad (4)$$

and from (4) we obtain for the equation of motion

$$\square_{V_4} \phi = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} \partial^\mu \phi = 0. \quad (5)$$

From (5) it is clear that the scalar field doesn't feel the non-Riemannian structure of  $U_4$ . In order to have a plausible physical behaviour for the field

$\varphi$ , one must be able to define a scalar product for the space of solutions of (5), and this can be attained if the operator  $\square_{V_4}$  is hermitian in the sense that [2]

$$\int d^4x \sqrt{-g} (\square_{V_4} \varphi_1)^* \varphi_2 = \int d^4x \sqrt{-g} \varphi_1^* \square_{V_4} \varphi_2. \quad (6)$$

By manipulations of (5) we get

$$\int d^4x \sqrt{-g} \{(\square_{V_4} \varphi_1)^* \varphi_2 - \varphi_1^* \square_{V_4} \varphi_2\} = \int d^4x \partial_\mu \sqrt{-g} ((\partial^\mu \varphi_1)^* \varphi_2 - \varphi_1^* \partial^\mu \varphi_2), \quad (7)$$

and to verify (6) one needs that the right-hand side of (7) to vanish. This is guaranteed by invoking the generalized Gauss' formula

$$\int_{\mathcal{M}} dv \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} J^\mu = \int_{\partial \mathcal{M}} d\sigma_\mu J^\mu \quad (8)$$

and by the assumption that the current vector  $J^\mu = (\partial^\mu \varphi_1)^* \varphi_2 - \varphi_1^* \partial^\mu \varphi_2$  vanishes on the boundary of the integration. In (8)  $dv$  and  $d\sigma_\mu$  are respectively the covariant volume element and the covariant surface element. The generalized Gauss' formula is a corollary of Stokes' theorem

$$\int_{\mathcal{M}} d\omega = \int_{\partial \mathcal{M}} \omega, \quad (9)$$

as one can see choosing, for example [3]  $\omega = \pi_\alpha J^\alpha$ , where  $\pi_\alpha$  is the 3-form

$$\pi_\alpha = \frac{1}{6} \sqrt{-g} \varepsilon_{\alpha\beta\gamma\delta} dx^\beta \wedge dx^\gamma \wedge dx^\delta, \quad (10)$$

where  $\varepsilon_{\alpha\beta\gamma\delta}$  is the totally anti-symmetric symbol. Although Stokes' theorem is defined in a general differentiable manifold  $X_n$ , with the choice (10) we get only the Riemannian terms of  $U_4$  in (8).

The question that can be introduced now is that if we use MCP at the equations of motion level, instead of (5) we get for the equations of motion the  $U_4$  d'Alambertian, which is given by

$$\square \varphi = D_\mu \partial^\mu \varphi = \square_{V_4} \varphi + 2S_\mu \partial^\mu \varphi = 0, \quad (11)$$

where  $S_\mu$  is the trace of the torsion tensor (2),  $S_\mu = S_{\alpha\mu}^\alpha$ . Differently from (5) the equation (11) takes into account non-Riemannian terms. One may

ask: Why does the use of MCP at equations of motion and action level lead to different results? Is it possible to obtain (11) from an Action? If yes, is it possible from (11) to define a scalar product? The answer to the last two questions is yes, provide that the trace  $S_\mu$  can be defined from a scalar potential

$$S_\mu(x) = \partial_\mu \Theta(x). \quad (12)$$

To the first question we will see that the problem is the use of MCP at action level.

In an  $U_4$  space-time where the trace of the torsion tensor  $S_\mu$  doesn't vanish, the generalization of the measure used in the action is not covariantly constant, as one can check using the fact that  $\sqrt{-g}$  is a scalar density (weight 1). Using properties of Christoffel symbols we get

$$D_\mu \sqrt{-g} = \partial_\mu \sqrt{-g} - \Gamma_{\alpha\mu}^\alpha \sqrt{-g} = -2S_\mu \sqrt{-g}. \quad (13)$$

Under the hypothesis (12) one can check that the scalar density  $e^{2\Theta} \sqrt{-g}$  is covariantly constant, and then it can be used to define a new measure. Using this new measure as the generalization of the usual measure of Minkowski space-time and the MCP we get the following action for a massless scalar field minimally coupled to the  $U_4$  space-time

$$S = \int d^4x \frac{e^{2\Theta} \sqrt{-g}}{2} \partial_\mu \varphi \partial^\mu \varphi. \quad (14)$$

From (14) we get the following equation of motion

$$\square \varphi = \square_{V_4} \varphi + 2S_\mu \partial^\mu \varphi = \frac{e^{-2\Theta}}{\sqrt{-g}} \partial_\mu e^{2\Theta} \sqrt{-g} \partial^\mu \varphi = 0. \quad (15)$$

We can check that (15) is hermitean in the sense of (6), provide that we generalize the volume and surface element also in the Gauss' formula (8), and this can be obtained from Stokes' theorem if we change the 3-form (10) by

$$\pi_\alpha = \frac{1}{6} e^{2\Theta} \sqrt{-g} \varepsilon_{\alpha\beta\gamma\delta} dx^\beta \wedge dx^\gamma \wedge dx^\delta. \quad (16)$$

It is well known that the equations of motion for scalar fields obtained via MCP are not conformally invariant, and to restore the conformal invariance one needs to add a term proportional to the scalar of curvature  $R$ . This new

term will allow the interaction of the scalar field with the torsion even in the case of traceless torsion [4]. The conformally invariant generalization of (14) is given by (using the conventions of ref. [5])

$$S = \int d^4x \frac{e^{2\Theta} \sqrt{-g}}{2} \left\{ \partial_\mu \varphi \partial^\mu \varphi - \frac{R}{6} \varphi^2 \right\}, \quad (17)$$

and the correspondent equation of motion is

$$\left( \square + \frac{R}{6} \right) \varphi = \frac{e^{-2\Theta}}{\sqrt{-g}} \partial_\mu e^{2\Theta} \sqrt{-g} \partial^\mu \varphi + \frac{R}{6} \varphi = 0. \quad (18)$$

The action (17) and the equations of motion (18) are invariant under the conformal transformation

$$\begin{aligned} g_{\mu\nu}(x) &\rightarrow \Omega^2(x) g_{\mu\nu}(x) \\ \varphi(x) &\rightarrow \Omega^{-1}(x) \varphi(x) \\ S_{\alpha\beta}{}^\gamma(x) &\rightarrow S_{\alpha\beta}{}^\gamma(x) \\ \Theta(x) &\rightarrow \Theta(x) \end{aligned} \quad (19)$$

The generalization to other space-time dimensions is straightforward. The coefficient of the term  $R\varphi^2$  and the conformal weight of the field  $\varphi$  for the  $n$  dimensional case are the same of the Riemann space-time  $V_n$ , and can be found for example in the ref. [5]. However, it is important to note that the torsion tensor  $S_{\alpha\beta}{}^\gamma(x)$  and the potential  $\Theta(x)$  are conformal invariant quantities in any space-time dimension.

As the conclusion, it must be stressed that the new measure  $e^{2\Theta} \sqrt{-g} d^4x$  must be used whenever we wish to describe a field on an  $U_4$  space-time that obeys (12), using MCP at the action level. In the same way that this new measure allows the interaction of scalar fields with torsion, it can allow also the interaction of others fields, like for example Maxwell field. Even the dynamical equations for the non-Riemannian geometrical quantities of  $U_4$ , that are obtained from an action principle [1], can be modified. These topics are now under investigation.

The author is grateful to Professor Josif Frenkel and Fundação de Amparo à Pesquisa do Estado de São Paulo.

## References

- [1] F.W. Hehl, P. von der Heyde, G.D. Kerlick, and J.M. Nester, *Rev. Mod. Phys.* **48**, 393 (1976)
- [2] B.S. DeWitt, *Phys. Rep.* **19C**, 295 (1975)
- [3] D. Lovelock and H. Rund, *Tensors, Differential Forms, and Variational Principles*, Dover Publications, 1989
- [4] M. Novello, *Phys. Lett.* **59A**, 105 (1976)
- [5] N.D. Birrel and P.C.W. Davies, *Quantum Fields in Curved Space*, Cambridge University Press, 1982