UNIVERSIDADE DE SÃO PAULO

INSTITUTO DE FÍSICA
CAIXA POSTAL 20516
01498-970 SÃO PAULO - SP
BRASIL

PUBLICAÇÕES

IFUSP/P-1031

GAUGE FIELDS ON EINSTEIN-CARTAN SPACE-TIMES

Alberto Saa
Instituto de Física, Universidade de São Paulo

Fevereiro/1993

IFUSP/P-1031

Gauge fields on Einstein-Cartan space-times

Alberto Saa

Instituto de Física,

Universidade de São Paulo, Caixa Postal 20516

01498-970 São Paulo, S.P.

Brazil

Gauge fields are described on an Einstein-Cartan space-time by means of tensor-valued differential forms and exterior calculus. It is shown that minimal coupling procedure leads to a gauge invariant theory where gauge fields interact with torsion, and that consistency conditions for the gauge fields impose restrictions in the non-Riemannian structure of space-time. These results contradict the well known statement that gauge fields do not couple minimally to torsion. The sources of these differences are pointed out and discussed.

I. INTRODUCTION

The Einstein-Cartan theory is the natural theory of gravity that emerges from the local gauge theory for the Poincaré group, and it is in accordance with the present day experimental data[1, 2]. This theory has been discussed in recent years, and in particular the problem of coupling gauge fields to Einstein-Cartan space-time U_4 has been studied (see for example [3] and references therein). The wide spread conclusion that gauge fields don't couple minimally to the non-Riemannian structure of space-time arises from an analysis using minimal coupling procedure (MCP) at the Action level.

In this work it is shown that actually gauge fields couple to torsion without breaking gauge invariance, and, of course, this contradicts the above mentioned wide spread conclusion.

The equations of motion are written by means of tensor-valued differential forms and exterior calculus in Minkowski space-time and by using MCP one gets the U_4 equations of motion, which will allow the interaction between gauge fields and the torsion. In order to have consistent equations we are lead to the restriction that the trace of the torsion tensor must be derived from a scalar potential. With this condition it is possible to get the U_4 equations of motion by using MCP at the Action level, provided that we introduce the invariant and covariantly constant U_4 volume element [4].

The work is organized in 5 sections, where the first is this introduction. In section 2, basic facts on Einstein-Cartan geometry are briefly presented. Maxwell fields are described on an U_4 manifold in Section 3. In Section 4, the results of Section 3 are generalized to the non-abelian case. In the last section, it is shown that all problems with gauge fields on U_4 are connected with the Hodge star operator (*), which in U_4 space-time must be different from the usual one of V_4 space-time. Yet in the last section, further developments are discussed.

II. THE U_4 MANIFOLD

The Einstein-Cartan space-time U_4 is characterized by its metric $g_{\alpha\beta}(x)$ and by its metric-compatible connection $\Gamma^{\mu}_{\alpha\beta}$, which is used to define the covariant derivative of a vector as

$$D_\nu A^\mu = \partial_\nu A^\mu + \Gamma^{\mu}_{\nu\rho} A^\rho. \tag{1}$$

The U_4 connection is non-symmetric in its lower indices, and from its anti-symmetric part can be defined the torsion tensor

$$S_{\alpha\beta}{}^{\gamma} = \frac{1}{2}\left(\Gamma^{\gamma}_{\alpha\beta} - \Gamma^{\gamma}_{\beta\alpha}\right). \tag{2}$$

One can write the connection as a function of the torsion tensor

$$\Gamma^{\gamma}_{\alpha\beta} = \left\{{}^{\gamma}_{\alpha\beta}\right\} + S_{\alpha\beta}{}^{\gamma} - S_{\beta}{}^{\gamma}{}_{\alpha} + S^{\gamma}{}_{\alpha\beta}, \tag{3}$$

where $\left\{{}^{\gamma}_{\alpha\beta}\right\}$ are the usual Christoffel symbols from Riemannian space-time V_4. A quantity that will be particularly useful is the trace of the connection (3), and using properties of Christoffel symbols we get the following expression for it

$$\Gamma^{\alpha}_{\alpha\beta} = \frac{1}{\sqrt{-g}}\partial_\beta \sqrt{-g} + 2S_\beta, \tag{4}$$

where g is the determinant of the metric tensor, and S_β is the trace of the torsion tensor $S_\beta = S_{\alpha\beta}{}^{\alpha}$.

The case where the trace S_β can be obtained from a scalar potential

$$S_\beta(x) = \partial_\beta \Theta(x), \tag{5}$$

will be crucial in our discussion. Under the condition (5) we have the following expression for (4)

$$\Gamma^{\alpha}_{\alpha\beta} = \frac{e^{-2\Theta}}{\sqrt{-g}}\partial_\beta e^{2\Theta}\sqrt{-g}. \tag{6}$$

It is important to note that the often used V_4 relation between the exterior derivative of an 1-form and the covariant derivative

$$dA = \partial_\alpha A_\beta dx^\alpha \wedge dx^\beta = D_\alpha A_\beta dx^\alpha \wedge dx^\beta, \tag{7}$$

is not valid in U_4, where instead of (7) we have[5]

$$D_\alpha A_\beta dx^\alpha \wedge dx^\beta \neq dA = \left(D_\alpha A_\beta + \frac{1}{2} S_{\alpha\beta}{}^{\rho} A_\rho\right) dx^\alpha \wedge dx^\beta. \tag{8}$$

We will see that this difference between V_4 and U_4 exterior derivatives is the origin of the problems with the naive use of MCP in U_4.

III. ABELIAN FIELDS

It is well known that Maxwell's equations can be expressed by means of differential forms and exterior calculus. This description is the most "economical", in the sense that it requires the minimal from the geometry of the manifold. Differential forms and their exterior derivatives are covariant objects in any differentiable manifold, in spite of the manifold is endowed with a connection or not. We will see that this description can be considered as the most fundamental one, not due to aesthetic arguments, but by physical reasons.

In order to study Maxwell's equations in a metric differentiable manifold, we introduce a fundamental quantity, the electromagnetic potential 1-form

$$A = A_\alpha dx^\alpha, \tag{9}$$

and from the potential 1-form we can define the Faraday's 2-form

$$F = dA = \frac{1}{2} F_{\alpha\beta}\, dx^\alpha \wedge dx^\beta, \tag{10}$$

where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the usual electromagnetic tensor.

It should be noted that (9) plays the role of a connection in the principal bundle $\mathcal{P}(\mathcal{M}, U(1))$, where the base space $\mathcal{M}$ is the space-time and the electromagnetic gauge group $U(1)$ is the fiber. If the bundle $\mathcal{P}(\mathcal{M}, U(1))$ is trivial, as for example for $\mathcal{M} = R^4$, we can assure that the gauge connection (9) is defined globally. However, for a non-trivial bundle we can define only locally the gauge connection. This is similar to the Dirac monopole case where, due to the $\mathcal{P}(S^2, U(1))$ non-triviality, we need at least two gauge potentials to describe it [6]. We will ignore by now these problems.

The homogenous Maxwell's equations arise naturally due to the definition (10) as a consequence of Poincaré's lemma [5]

$$dF = d(dA) = \frac{1}{2}\partial_\gamma F_{\alpha\beta}\, dx^\gamma \wedge dx^\alpha \wedge dx^\beta = 0, \tag{11}$$

and in terms of components we have

$$\partial_{[\gamma} F_{\alpha\beta]} = 0, \tag{12}$$

where $[\ \ ]$ means antisymmetrization.

The non-homogenous equations in Minkowski space-time are given by

$$d{}^*F = 4\pi {}^*J, \tag{13}$$

where ${}^*J = \frac{1}{3!}\varepsilon_{\alpha\beta\gamma\delta} J^\delta dx^\alpha \wedge dx^\beta \wedge dx^\gamma$ is the current 3-form constructed from the current vector J^δ and

$${}^*F = \frac{1}{4}\varepsilon_{\alpha\beta\gamma\delta} F^{\gamma\delta}\, dx^\alpha \wedge dx^\beta, \tag{14}$$

is the dual of Faraday's 2-form, constructed from it by using $\varepsilon_{\alpha\beta\gamma\delta}$, the totally anti-symmetric symbol, and the metric tensor.

By an accurate analysis of (13) one can see that it is not covariant in a curved space-time, because of ${}^*F$ is not a scalar 2-form, but it is a relative scalar 2-form with weight -1, due to the anti-symmetrical symbol. Now we assume that the manifold is endowed with a connection to use it in order to cast (13) in a covariant way. This is done by substituting the exterior derivative by the covariant one

$$d{}^*F \to \mathcal{D}{}^*F = \frac{1}{3!}\left(\partial_\alpha {}^*F_{\beta\gamma} + \Gamma^\rho_{\rho\alpha} {}^*F_{\beta\gamma}\right) \delta^{\alpha\beta\gamma}_{\mu\nu\omega}\, dx^\mu \wedge dx^\nu \wedge dx^\omega. \tag{15}$$

where $\delta^{\alpha\beta\gamma}_{\mu\nu\omega}$ is the generalized Kronecker delta. The covariant exterior derivative in (15) takes into account that ${}^*F_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta}$ is a relative $(0,2)$ tensor with weight -1. One can check that $\mathcal{D}{}^*F$ is a relative scalar 3-form with weight -1. We have then the following covariant generalization of (13)

$$\mathcal{D}{}^*F = 4\pi {}^*J. \tag{16}$$

Equations (11) are already in a covariant form in any differentiable manifold.

The informations about the geometry of the manifold are contained in the metric tensor $g_{\alpha\beta}(x)$ used in the construction of ${}^*F$ (14), and in the trace of the connection used to define the covariant exterior derivative (15). Therefore we can think that equations (16) and (11) were obtained from Minkowskian ones by means of MCP used at differential forms level. The components expression for (16) in an Einstein-Cartan space is

$$\frac{1}{\sqrt{-g}}\partial_\mu \sqrt{-g} F^{\nu\mu} + 2S_\mu F^{\nu\mu} = 4\pi J^\nu. \tag{17}$$

One can see that equation (17) allows the interaction of electromagnetism with torsion of space-time without destroying gauge invariance and using MCP at differential forms level.

Taking the covariant exterior derivative in both sides of (16) we get

$$4\pi \mathcal{D}^*J = \frac{1}{4!}\left(\partial_\lambda \Gamma^\rho_{\rho\mu}\right) F_{\nu\omega}\delta^{\lambda\mu\nu\omega}_{\alpha\beta\gamma\delta}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta, \tag{18}$$

and to have a generalized conservation condition for the current we need that

$$\partial_\lambda \Gamma^\rho_{\rho\mu} - \partial_\mu \Gamma^\rho_{\rho\lambda} = 0, \tag{19}$$

which has, at least locally, as general solution [5]

$$\Gamma^\rho_{\rho\mu} = \partial_\mu f(x). \tag{20}$$

Using that $\left\{{}^{\rho}_{\rho\mu}\right\} = \partial_\mu \ln\sqrt{-g}$, equation (20) will have general solution only if the trace of the torsion tensor obeys (5). In this case $f(x) = \ln\left(e^{2\Theta}\sqrt{-g}\right)$. When $J = 0$, the condition (20) is a consistency condition for equation (16). Under the condition (5) we have the following components expression for (16)

$$\frac{e^{-2\Theta}}{\sqrt{-g}}\partial_\mu e^{2\Theta}\sqrt{-g}F^{\nu\mu} = 4\pi J^\nu, \tag{21}$$

and for the generalized conservation condition we have

$$\frac{e^{-2\Theta}}{\sqrt{-g}}\partial_\mu e^{2\Theta}\sqrt{-g}J^\mu = 0. \tag{22}$$

It must be stressed that if the trace of the torsion tensor does not obey (5) we cannot obtain a generalized conservation condition.

One can ask now if it is possible to obtain the non-homogeneous equations (21) from an Action principle. We know that in Minkowski space-time, the non-homogeneous equations are gotten from the following action

$$S = \int \left(4\pi {}^*J \wedge A - F \wedge {}^*F\right). \tag{23}$$

Besides the metrical tensor, the unique non-covariant term in (23) is the implicit measure

$$dv = \frac{1}{4!}\varepsilon_{\alpha\beta\gamma\delta}dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta. \tag{24}$$

In order to get a covariant measure we need to introduce a scalar density. In this case the choice

$$dv = \frac{1}{4!}e^{2\Theta}\sqrt{-g}\varepsilon_{\alpha\beta\gamma\delta}dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta, \tag{25}$$

leads to a covariantly constant measure [4]. With this new measure one gets the following coordinate expression for (23)

$$S = \int d^4x\, e^{2\Theta}\sqrt{-g}\left(-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + 4\pi J^\alpha A_\alpha\right). \tag{26}$$

It is easy to check that we can obtain equations (21) from the action (26). We can check also that equations (12) and (21) are invariant under the usual $U(1)$ gauge transformation

$$A_\mu \to A_\mu + \partial_\mu\varphi. \tag{27}$$

We would like to stress the importance of the generalized conservation condition (22) to guarantee the gauge invariance of the action (26).

Now we can convince ourselves that the usual way of coupling Maxwell fields to torsion is fallacious, not only because we loose gauge invariance, but because we also loose the homogenous equation (12). In the usual way, one applies MCP at the level of tensorial equations. Applying usual MCP to the equation (12) we get

$$\partial_{[\alpha}\tilde{F}_{\beta\gamma]} + 2S_{[\alpha\beta}{}^\rho\tilde{F}_{\gamma]\rho} = 0, \tag{28}$$

where $\tilde{F}_{\alpha\beta} = F_{\alpha\beta} - 2S_{\alpha\beta}{}^\rho A_\rho$. One can check that (28) has no general solution. The origin of the problem is the difference between exterior derivatives and covariant ones pointed out in section 2. In V_4 there exist an "equivalence" between exterior derivatives and covariant ones, and due to this in V_4 it makes no difference if one applies MCP at differential forms or at tensorial levels. In U_4 we have another situation, and due to the "inequivalence" between exterior and covariant derivatives we don't get the same result applying MCP at different levels. Based in these facts one claims that the differential forms representation for Maxwell equations are the most fundamental one.

IV. NON-ABELIAN FIELDS

In order to generalize the results of section 3 for the non-abelian case one needs to introduce the non-abelian potential 1-form

$$A = A^a_\mu \lambda^a dx^\mu, \tag{29}$$

where λ^a are the generators of the gauge Lie group $\mathcal{G}$ satisfying

$$\left[\lambda^a, \lambda^b\right] = f^{abc}\lambda^c. \tag{30}$$

Latin indices are reserved to the group manifold, and the summation convention for repeated indices is adopted. Let us restrict ourselves to compact semisimple Lie groups, so that the structure constants are anti-symmetrical under the change of any couple of indices. Any element $g \in \mathcal{G}$ can be written as

$$g(x) = \exp i\theta^a(x)\lambda^a, \tag{31}$$

where $\theta^a(x)$ are the group continuous parameters.

Here it's important the same comment already made in Section 2. The gauge potential 1-form (29) plays the role of a connection in the principal bundle $\mathcal{P}(\mathcal{M},\mathcal{G})$, and we can assure the global validity of a single gauge potential only for trivial bundles. [6]

From (29) we can define the 2-form equivalent to (10),

$$F = DA = dA + A \wedge A = \frac{1}{2} F^a_{\mu\nu} \lambda^a dx^\mu \wedge dx^\nu, \tag{32}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu$ is the usual non-abelian strength tensor. The derivative D is the covariant derivative that has the appropriated transformation law under gauge transformations.

As in the abelian case, the homogeneous non-abelian equations are a consequence of Bianchi identity

$$DF = dF + A \wedge F - F \wedge A = \frac{1}{2}\left(\partial_\mu F^a_{\omega\nu} + A^b_\mu F^c_{\omega\nu} f^{abc}\right) \lambda^a dx^\mu \wedge dx^\omega \wedge dx^\nu = 0. \tag{33}$$

The non-homogenous equation for non-abelian gauge fields are written as the Maxwell ones (13). For simplicity and without generality loss, we will treat the case without sources:

$$D^*F = d^*F + A \wedge {}^*F - {}^*F \wedge A = \frac{1}{2}\left(\partial_\mu {}^*F^a_{\omega\nu} + A^b_\mu {}^*F^c_{\omega\nu} f^{abc}\right) \lambda^a dx^\mu \wedge dx^\omega \wedge dx^\nu = 0, \tag{34}$$

where the dual of the non-abelian strength tensor is defined as in (14). In the same way of abelian case, equation (33) is already in a covariant form, but due to the term $^*F$, equation (34) must be generalized in a curved space-time. To cast (34) in a covariant way, one needs to substitute $d^*F \to \mathcal{D}^*F$ in (34) as we did in (15). The derivative $\mathcal{D}$ is defined as

$$\mathcal{D}^*F = d^*F + \omega \wedge {}^*F, \tag{35}$$

where $\omega = \Gamma^\rho_{\rho\alpha} dx^\alpha$. We can check that (35) is equivalent to (15). Using the derivative $\mathcal{D}$ we get the following generalization for (34)

$$\begin{aligned} D^*F &= d^*F + \omega \wedge {}^*F + A \wedge {}^*F - {}^*F \wedge A = \\ &= \frac{1}{2}\left(\partial_\mu {}^*F^a_{\omega\nu} + \Gamma^\rho_{\rho\alpha} {}^*F^a_{\omega\nu} + A^b_\mu {}^*F^c_{\omega\nu} f^{abc}\right) \lambda^a dx^\mu \wedge dx^\omega \wedge dx^\nu = 0. \end{aligned} \tag{36}$$

In order to equations (33) and (36) have non trivial solutions one needs that $D(DF) = D({}^*F) = 0$. Using the fact that DF and $D{}^*F$ are respectively an 3-form and a relative 3-form with weight -1, we can obtain

$$D(DF) = d(DF) + A \wedge (DF) + (DF) \wedge A = 0, \tag{37}$$

for the homogenous equation (33). For the case of the non-homogeneous equation (36) we have

$$D(D{}^*F) = d(D{}^*F) + \omega \wedge D{}^*F + A \wedge (D{}^*F) + (D{}^*F) \wedge A = d\omega \wedge {}^*F, \tag{38}$$

and to get the desired condition $D(D{}^*F) = 0$, we are enforced to have $d\omega = 0$. Since ω is an 1-form and it is closed, by the converse of Poincaré lemma, we have at least in a star-shaped region that ω is exact, $\omega = df$, what is the same result that we got in the abelian case.

Under the hypothesis (5) we have the usual coordinate expression for (33)

$$\varepsilon^{\alpha\beta\gamma\delta}\left(\partial_\beta F^a_{\gamma\delta} + A^b_\beta F^c_{\gamma\delta} f^{abc}\right) = 0, \tag{39}$$

and the following expression for the generalized non-homogeneous equation (34)

$$\frac{e^{-2\Theta}}{\sqrt{-g}}\partial_\mu e^{2\Theta}\sqrt{-g}F^{a\nu\mu} + A^b_\mu F^{c\nu\mu} f^{abc} = 0. \tag{40}$$

One can check that the equations (39) and (40) are invariant under the usual non-abelian gauge transformation

$$A_\mu \to g A_\mu g^{-1} + g\partial_\mu g^{-1}, \tag{41}$$

where $A_\mu = A^a_\mu \lambda^a$. It is clear from (40) that non-abelian gauge fields are sensitive to the non-Riemannian structure of space-time.

As in the abelian case, one can try to get equation (40) from an Action principle. We know that in Minkowski space-time the non-homogeneous equations are gotten form the action

$$S = \int \text{trace}\,(F \wedge {}^*F), \tag{42}$$

which has the following coordinate expression

$$S = \frac{1}{4}\int d^4x\, \text{trace}\left(F^a_{\mu\nu}F^{b\,\mu\nu}\lambda^a\lambda^b\right) = \frac{1}{4}\int d^4x F^a_{\mu\nu}F^{a\,\mu\nu}, \tag{43}$$

where the normalization condition: $\text{trace}(\lambda^a\lambda^b) = \delta^{ab}$, was assumed for the group generators.

In order to cast (42) in a covariant way, one needs to substitute the metric tensor used in the definition of the dual and to modify the measure of integration. We pick the same measure used in the abelian case (25), and get

$$S = \int e^{2\Theta}\sqrt{-g}\,\text{trace}(F \wedge {}^*F), \tag{44}$$

which has the following coordinate expression

$$S = \frac{1}{4}\int d^4x\, e^{2\Theta}\sqrt{-g}F^a_{\mu\nu}F^{a\,\mu\nu}. \tag{45}$$

Equations (40) follow from minimization of (45). It's easy to realize that the action (45) is invariant under non-abelian gauge transformation (41).

V. FINAL CONSIDERATIONS

Let us summarize the results of previous sections. The Action for gauge fields in U_4 space time is given by

$$S = \int e^{2\Theta}\sqrt{-g}\,\text{trace}(F \wedge {}^*F) = \frac{1}{4}\int d^4x\, e^{2\Theta}\sqrt{-g}F^a_{\mu\nu}F^{a\,\mu\nu}.$$

The U_4 equations of motion are

$$\varepsilon^{\alpha\beta\gamma\delta}\left(\partial_\beta F^a_{\gamma\delta} + A^b_\beta F^c_{\gamma\delta} f^{abc}\right) = 0,$$
$$\frac{e^{-2\Theta}}{\sqrt{-g}}\partial_\mu e^{2\Theta}\sqrt{-g}F^{a\nu\mu} + A^b_\mu F^{c\nu\mu} f^{abc} = 0,$$

where the last equations follow from the Action principle. It's easy to check that the abelian limit of these results corresponds to the Maxwell model of Section 3. These results are valid in an Einstein-Cartan space-time where the trace of the torsion tensor can be derived from a scalar potential, $S_\alpha = \partial_\alpha \Theta$. On the other hand, we cannot get consistent equations if the trace cannot be obtained from a scalar potential.

It should be noted that the existence of the scalar Θ, such that $S_\alpha = \partial_\alpha \Theta$, was assured by the converse of the Poincaré lemma, and then, one cannot assure that only one scalar Θ is enough to define globally the trace S_α. This will depend on the topology of the space-time manifold $\mathcal{M}$. As an example, for $\mathcal{M} = R^4$ we can define the trace S_α globally from an unique scalar Θ.

In our discussion, the space-time manifold is considered as independent from the gauge fields. Gauge fields impose some restrictions on the geometry, but its dynamics are not affected by the gauge fields, as an external field. But we know, from General Relativity, that the dynamics of the space-time geometry must be governed by the non-gravitational fields, in this case the gauge fields. An interesting point is to introduce the geometry of the space-time in the discussion, by adding an Action for it in the Action principle. These topics are now under investigation.

The problems of covariance of the equations of motion always was connected with the duals of the strength tensors, and we would like to dedicate the last subsection to this topic.

A. Hodge star operator

The mathematical essence of the problems with the covariance presented in the last two sections, is the duality transformations, i.e. Hodge star (*) operation[6]. The problems with covariance could be avoided if one changes appropriately the Hodge star operator for an U_4 manifold. For this purpose, we introduce the * operator

following [6].

Be $\mathcal{M}$ a n-dimensional differentiable manifold endowed with a metric $g_{\mu\nu}$ and with a metric-compatible connection $\Gamma^\alpha_{\gamma\beta}$, and $\Omega^m(\mathcal{M})$ the space of differential m-forms on it. The Hodge * operator is a linear operator

$$^* : \Omega^m(\mathcal{M}) \to \Omega^{n-m}(\mathcal{M}), \tag{46}$$

which for a Riemannian manifold has the following action on a basis vector of $\Omega^m(\mathcal{M})$

$$^*(dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \ldots \wedge dx^{\alpha_m}) = \frac{\sqrt{g}}{(n-m)!} \varepsilon^{\alpha_1 \ldots \alpha_m}{}_{\beta_{m+1} \ldots \beta_n} dx^{\beta_{m+1}} \wedge \ldots \wedge dx^{\beta_n}, \tag{47}$$

where $\varepsilon_{\alpha_1 \ldots \alpha_n}$ is the totally anti-symmetrical symbol. The action of (47) on the basis vector for $\Omega^0(\mathcal{M})$ gives

$$^*1 = \frac{\sqrt{g}}{n!} \varepsilon_{\alpha_1 \ldots \alpha_n} dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_n} = \sqrt{g} d^4x, \tag{48}$$

that is the invariant and covariantly constant volume element for a Riemannian manifold.

In an Einstein-Cartan space-time, the volume element (48) is not covariantly constant, in contrast to the Riemannian case, as one can check using the fact that $\sqrt{g}$ is a scalar density

$$D_\mu \sqrt{g} = \partial_\mu \sqrt{g} - \Gamma^\alpha_{\alpha\mu} \sqrt{g} = -2S_\mu \sqrt{g}. \tag{49}$$

To get an invariant and covariantly constant volume element for an Einstein-Cartan space-time that obeys (5) one needs to modify the Hodge * operator by

$$^*(dx^{\alpha_1} \wedge dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_1}) = \frac{h(x)}{(n-m)!} \varepsilon^{\alpha_1 \ldots \alpha_m}{}_{\beta_{m+1} \ldots \beta_n} dx^{\beta_{m+1}} \wedge \ldots \wedge dx^{\beta_n}, \tag{50}$$

where the scalar density $h(x)$ is such that $\partial_\mu h = \Gamma^\nu_{\nu\mu} h$. We already know that $h(x) = e^{2\Theta}\sqrt{g}$. Using (50) to define the duals used in the equations of motion and in the Lagrangian we will get automatically the U_4 covariant equations.

It is not clear if one can define a Hodge * operator in order to obtain a invariant and covariantly constant volume element for the case of Einstein-Cartan space-times not obeying (5).

ACKNOWLEDGMENTS

The author is grateful to Professor Josif Frenkel and Fundação de Amparo à Pesquisa do Estado de São Paulo for support.

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