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## ENERGY-MOMENTUM TENSOR: METRICAL AND CANONICAL

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# Energy-momentum tensor : Metrical and Canonical

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**Abstract**

The relation between these two types of energy-momentum tensor is explained in a way that is easily appended to most text-book treatments.

Resumo : demonstra-se a equivalencia entre esses dois tipos de tensor de momento-energia procurando-se seguir o estilo dos tratamentos usuais nos textos.

# 1 INTRODUCTION

It is well known that the canonical energy-momentum tensor [1] of a classical field theory is not symmetric in its indices, except for zero-spin fields, and that this spoils the elegance of the formalism by requiring an ugly expression for the angular momentum density of the field. This is clearly exposed in many places, like, for instance, [2], and the solution is given in the classical works of Belinfante [5] and of Rosenfeld [6]. All the matter is scholarly settled there, albeit in a research report style, not appropriate for inclusion in a set of lectures based on standard texts, like Landau, Lifshitz [1] or Jackson [4].The rules for replacing the canonical energy-momentum tensor by a symmetrical, equivalent one, the so-called Belinfante-Rosenfeld tensor, however, are quite simple, and deservedly well-known. A possible alternative is the use of the metrical energy-momentum tensor, introduced by Hilbert in his classical paper [7]. In this note we intend to elucidate in a simple way when these two kinds of energy-momentum tensor are equivalent and when they are not. We will introduce our treatment in the very simple case of a scalar field $\phi$. Then we will consider the case of a vector meson, which already exhibits the most general features, and outline the extension to other spins. The paper purports to be a pedagogical one.

# 2 QUESTIONS OF EQUIVALENCE

In order to properly introduce the metrical energy-momentum tensor we must work in curvilinear coordinates. Let $\mathcal{L}(g^{ij}, \frac{\partial g_{ij}}{\partial x^l}, \phi, \partial_i\phi)$ be a Lagrangian density.

The action is given by

$$S = \int d^4x \sqrt{(-g)}\mathcal{L} \tag{1}$$

The metrical tensor is obtained [1] by exploiting the fact that $S$ must be invariant under infinitesimal coordinate transformations $x^i \to x'^i$, with

$$x'^i = x^i + \xi^i(x). \tag{2}$$

Fields and the metric respond to this transformation in the following way [1]:

$$\delta\phi(x) \equiv \phi'(x) - \phi(x) = -\xi^i(x)\partial_i\phi \tag{3}$$

$$\delta g^{ik}(x) = \xi^{i;k} + \xi^{k;i}. \tag{4}$$



This induces in the action $S$ the variation

$$\delta S = \int d^4x\delta(\sqrt{(-g)}\mathcal{L}) + \int d\sigma_l\xi^l\sqrt{(-g)}\mathcal{L} \tag{5}$$

where the second integral is essential, as a general coordinate transformation doesn't have to vanish at the boundaries of the integration domain. For a nice derivation of this term see [3] . It is his equation (170). Actually, this surface term is the key to the proof, as will be shortly seen. More explicitly,

$$\begin{aligned}\delta S \;=\; & \int d^4x\sqrt{(-g)}\{\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_l\phi)}\partial_l\delta\phi\} \\ & + \int d^4x[\frac{\partial(\sqrt{(-g)}\mathcal{L})}{\partial g^{ij}}\delta g^{ij} + \frac{\partial(\sqrt{(-g)}\mathcal{L})}{\partial(\frac{\partial g^{ij}}{\partial x^l})}\frac{\partial}{\partial x^l}\delta g^{ij}] \\ & + \int d\sigma_l\xi^l\sqrt{(-g)}\mathcal{L}.\end{aligned} \tag{6}$$

The usual partial integrations lead to

$$\begin{aligned}\delta S \;=\; & \int d^4x\sqrt{(-g)}\{\frac{\partial\mathcal{L}}{\partial\phi} - (1/\sqrt{(-g)})\partial_l[\sqrt{(-g)}\frac{\partial\mathcal{L}}{\partial(\partial_l\phi)}]\}\delta\phi + \\ & + \int d^4x[\frac{\partial(\sqrt{(-g)}\mathcal{L})}{\partial g^{ij}} - \partial_l\frac{\partial(\sqrt{(-g)}\mathcal{L})}{\partial(\partial_l g^{ij})}]\delta g^{ij} + \\ & + \int d\sigma_l[\sqrt{(-g)}\frac{\partial\mathcal{L}}{\partial(\partial_l\phi)}]\delta\phi \\ & + \int d\sigma_l\frac{\partial(\sqrt{(-g)}\mathcal{L})}{\partial(\partial_l g^{ij})}\delta g^{ij} + \int d\sigma_l\xi^l\sqrt{(-g)}\mathcal{L}.\end{aligned} \tag{7}$$

For $\phi(x)$ satisfying the equations of motion the first integral vanishes. Defining [1] the metrical energy-momentum tensor $T_{ij}$ by

$$\frac{1}{2}T_{ij}\sqrt{(-g)} \equiv \frac{\partial(\sqrt{(-g)}\mathcal{L})}{\partial g^{ij}} - \partial_l\frac{\partial(\sqrt{(-g)}\mathcal{L})}{\partial(\partial_l g^{ij})} \tag{8}$$

one has

$$\begin{aligned}\delta S \;=\; & \frac{1}{2}\int d^4x\sqrt{(-g)}T_{ij}\delta g^{ij} + \int d\sigma_l\sqrt{(-g)}\frac{\partial\mathcal{L}}{\partial(\partial_l\phi)}\delta\phi + \\ & + \int d\sigma_l\frac{\partial(\sqrt{(-g)}\mathcal{L})}{\partial(\frac{\partial g^{ij}}{\partial x^l})}\delta g^{ij} + \int d\sigma_l\xi^l\sqrt{(-g)}\mathcal{L}\end{aligned} \tag{9}$$

Assume for a moment that $\mathcal{L}$ does not depend on the derivatives of $g^{ij}$. This means that the connection coefficients $\Gamma^i_{jk}$are not present, either explicitly or inside curvature tensors.(Of course this is always the case in Minkowski spacetime described by "cartesian" coordinates).Inserting into (9) the values of $\delta\phi$ and $\delta g^{ij}$ one has

$$\delta S = \frac{1}{2}\int d^4x\sqrt{(-g)}T_{ij}(\xi^{i;j}+\xi^{j;i}) + \int d\sigma_l\sqrt{(-g)}\xi^m\{-\frac{\partial\mathcal{L}}{\partial(\partial_l\phi)}\partial_m\phi + \delta^l_m\mathcal{L}\}, \tag{10}$$

that is,

$$\delta S = \frac{1}{2}\int d^4x\sqrt{(-g)}T_{ij}(\xi^{i;j}+\xi^{j;i}) - \int d\sigma_l\sqrt{(-g)}\xi^m)\Theta^l_m \tag{11}$$

where we recognize

$$\Theta^l_m = \frac{\partial\mathcal{L}}{\partial(\partial_l\phi)}\partial_m\phi - \delta^l_m\mathcal{L}$$

as the canonical energy-momentum tensor. Now, as shown in detail by [1],

$$\frac{1}{2}\int d^4x\sqrt{(-g)}T_{ij}(\xi^{i;j}+\xi^{j;i}) = -\int d^4x\sqrt{(-g)}T^k_{i;k}\xi^i + \int d\sigma_l\sqrt{(-g)}T^l_m\xi^m \tag{12}$$

Taking (12) into (11),

$$\delta S = -\int d^4x\sqrt{(-g)}T^k_{i;k}\xi^i + \int d\sigma_l\sqrt{(-g)}\xi^m(T^l_m - \Theta^l_m). \tag{13}$$

As $\delta S$ should vanish for arbitrary $\xi^i$, one has

$$T^k_{i;k} = 0 \tag{14}$$

and

$$\int d\sigma_l\sqrt{(-g)}(T^l_m - \Theta^l_m) = 0$$

or

$$T^l_m = \Theta^l_m. \tag{15}$$

showing the equivalence of the two tensors.

## 3 ELECTRODYNAMICS

Scalar mesons are a bit too simple, however. We now treat Electrodynamics, where the main features of the general case are already apparent. Besides, the method applies to vector mesons as well.



Consider Eq.(3). It exhibits the form variation of a scalar field, an essential ingredient in the previous discussion. The form variation of a vector field $A_s$ is given by [8]

$$\delta A_s = -\xi^m\partial_m A_s - A_m(\partial_s\xi^m). \tag{16}$$

This induces on the action $S$ the variation

$$\begin{aligned}\delta S &= \int d^4x\sqrt{(-g)}\{\frac{\partial\mathcal{L}}{\partial A_s}\delta A_s + \frac{\partial\mathcal{L}}{\partial(\partial_l A_s)}\partial_l\delta A_s\} \\ &+ \int d^4x[\frac{\partial(\sqrt{(-g)}\mathcal{L})}{\partial g^{ij}}\delta g^{ij} + \frac{\partial(\sqrt{(-g)}\mathcal{L})}{\partial(\frac{\partial g^{ij}}{\partial x^l})}\frac{\partial}{\partial x^l}\delta g^{ij}] \\ &+ \int d\sigma_l\xi^l\sqrt{(-g)}\mathcal{L}.\end{aligned} \tag{17}$$

Proceeding as before,

$$\begin{aligned}\delta S = \\ &\int d^4x\sqrt{(-g)}\{\frac{\partial\mathcal{L}}{\partial A_s} - (1/\sqrt{(-g)})\partial_l[\sqrt{(-g)}\frac{\partial\mathcal{L}}{\partial(\partial_l A_s)}]\}\delta A_s + \\ &\int d^4x[\frac{\partial(\sqrt{(-g)}\mathcal{L})}{\partial g^{ij}} - \partial_l\frac{\partial(\sqrt{(-g)}\mathcal{L})}{\partial(\partial_l g^{ij})}]\delta g^{ij} + \\ &+ \int d\sigma_l[\sqrt{(-g)}\frac{\partial\mathcal{L}}{\partial(\partial_l A_s)}]\delta A_s \\ &+ \int d\sigma_l\frac{\partial(\sqrt{(-g)}\mathcal{L})}{\partial(\partial_l g^{ij})}\delta g^{ij} + \int d\sigma_l\xi^l\sqrt{(-g)}\mathcal{L}.\end{aligned} \tag{18}$$

Suppose the Lagrangian does not depend on $\partial_l g^{ij}$ . Then, using the equations of motion for $A_s$ and the definition of the metrical energy-momentum tensor,

$$\begin{aligned}\delta S &= \frac{1}{2}\int d^4x\sqrt{(-g)}T_{ij}\delta g^{ij} + \int d\sigma_l\sqrt{(-g)}\frac{\partial\mathcal{L}}{\partial(\partial_l A_s)}\delta A_s + \\ &+ \int d\sigma_l\xi^l\sqrt{(-g)}\mathcal{L}\end{aligned} \tag{19}$$

We now study the second term in some detail. It is better to write it in the form

$$\int d^4x \partial_l \{\sqrt(-g) \frac{\partial \mathcal{L}}{\partial(\partial_l A_s)} \delta A_s\} =$$
$$- \int d^4x \partial_l \{\sqrt(-g) \frac{\partial \mathcal{L}}{\partial(\partial_l A_s)} (\partial_m A_s) \xi^m\} -$$
$$\int d^4x \partial_l \{\sqrt(-g) \frac{\partial \mathcal{L}}{\partial(\partial_l A_s)} A_m \partial_s \xi^m\} \tag{20}$$

where use was made of the form variation of $A_s$. Taking this into Eq.(),

$$\begin{aligned} \delta S = & \int d^4x \sqrt(-g) T_{ij} \delta^{ij} - \\ & \int d^4x \partial_l \{\sqrt(-g) \frac{\partial \mathcal{L}}{\partial(\partial_l A_s)} (\partial_m A_s) \xi^m\} - \\ & \int d^4x \partial_l \{\sqrt(-g) \frac{\partial \mathcal{L}}{\partial(\partial_l A_s)} A_m \partial_s \xi^m\} + \\ & \int d\sigma_l \xi^l \sqrt(-g) \mathcal{L}. \end{aligned} \tag{21}$$

Transforming the second integral into a surface one and using Eq.(),

$$\begin{aligned} \delta S = & \\ & \int d^4x \sqrt(-g) T^k_{i\ ;k} \xi^i + \\ & \int d\sigma_l \sqrt(-g) T^l_m \xi^m - \int d\sigma_l \sqrt(-g) \Theta^l_m \xi^m - \\ & \int d^4x \partial_l \{\sqrt(-g) \frac{\partial \mathcal{L}}{\partial(\partial_l A_s)} A_m \partial_s \xi^m\} \end{aligned} \tag{22}$$

where we used

$$\Theta^l_m = \frac{\partial \mathcal{L}}{\partial(\partial_l A_s)} \partial_m A_s - \delta^l_m \mathcal{L} \tag{23}$$

So,

$$\begin{aligned} \delta S = & \\ & \int d^4x \sqrt(-g) T^k_{i\ ;k} \xi^i + \\ & \int d\sigma_l \sqrt(-g) \{T^l_m - \Theta^l_m\} \xi^m - \\ & \int d^4x \partial_l \{\sqrt(-g) \frac{\partial \mathcal{L}}{\partial(\partial_l A_s)} A_m \partial_s \xi^m\}. \end{aligned} \tag{24}$$

Using the fact that $\frac{\partial \mathcal{L}}{\partial(\partial_l A_s)}$ is antisymmetric in $(r, s)$ , we have



$$\begin{aligned} & \int d^4x \partial_l \{\sqrt(-g) \frac{\partial \mathcal{L}}{\partial(\partial_l A_s)} A_m \partial_s \xi^m\} = \\ & \int d^4x \partial_l \partial_s \{\sqrt(-g) \frac{\partial \mathcal{L}}{\partial(\partial_l A_s)} A_m \xi^m\} \\ & - \int d^4x \partial_l \{\xi^m \partial_s (\sqrt(-g) \frac{\partial \mathcal{L}}{\partial(\partial_l A_s)} A_m)\} = \\ & = \int d\sigma_l \xi^m \partial_s (\sqrt(-g) \frac{\partial \mathcal{L}}{\partial(\partial_l A_s)} A_m) \end{aligned}$$

Together with Eq.(24) this gives

$$\begin{aligned} \delta S = & \int d^4x \sqrt(-g) T^k_{i\ ;k} \xi^i + \\ & \int d\sigma_l \sqrt(-g) \{T^l_m - \Theta^l_m - \frac{1}{\sqrt(-g)} \partial_s (\sqrt(-g) \frac{\partial \mathcal{L}}{\partial(\partial_l A_s)} A_m)\} \xi^m. \end{aligned} \tag{25}$$

This must vanish for arbitrary $\xi^i$ and integration domain. Therefore,

$$T^k_{i\ ;k} = 0$$

and

$$T^l_m - \Theta^l_m = \frac{1}{\sqrt(-g)} \partial_s (\sqrt(-g) \frac{\partial \mathcal{L}}{\partial(\partial_l A_s)} A_m) \tag{26}$$

In cartesian coordinates this tells us that the difference between the two tensors is a tensor

$$\partial_s G^{lms}$$

which is antisymmetric in the indices $(l, s)$. The two energy-momentum tensors are, therefore, equivalent (See [1] , §32 ).

## 4 CONCLUSION

In the vector meson case the proof of the equivalence made use of two facts : the Lagrangian did not depend on $\partial_l g^{ij}$, and $\frac{\partial \mathcal{L}}{\partial(\partial_l A_s)}$ was antisymmetric in $(l, s)$. The latter property is common to all Lagrangians which describe particles of integral spin, in the Fierz-Pauli formalism [10]. For a brief and lucid review of this theory, see [9]. As for the dependence on $\partial_l g^{ij}$ , we shall see that it poses no problem in the case of Minkowski spacetime.

Consider a field which is a tensor of rank $s$, symmetrical in all its indices, vanishing on contraction with respect to any pair of indices, satisfying the condition of 4-transversality,

$$\partial^i \psi_{i...} = 0,$$

and let its dynamics be described by the Lagrangian density

$$\mathcal{L} = -(\partial_i \psi_{j_1...j_s})(\partial^i \psi^{j_1...j_s}) + (\partial_{j_1} \psi_{i...j_s})(\partial^i \psi^{j_1...j_s}) + m^2 \psi_{j_1...j_s} \psi^{j_1...j_s}. \quad (27)$$

It represents a particle of spin $s$ [9] and has the form variation

$$\delta\psi_{j_1...j_s} = -\xi^m \partial_m \psi_{j_1...j_s} - (\partial_{j_1} \xi^m)\psi_{m j_2...j_s} - ... - (\partial_{j_s} \xi^m)\psi_{j_1...m} \quad (28)$$

Notice that $\frac{\partial \mathcal{L}}{\partial(\partial_i \psi_{j_1...j_s})}$ is antisymmetrical in $(i, j_k)$ for all $k$.

We can now reproduce every step of the previous demonstration.The term $-\xi^m \partial_m \psi_{j_1...j_s}$ of the form variation will participate in the expression of $\Theta^l_m$. The remaining terms of Eq.() will, as in Eq.(25),compose the terms $G^{lmf}{}_{;f}$, which will be the sum of $s$ terms, all of them with the same symmetries in the indices. Eventually we will reach the following expression:

$$\begin{aligned} \delta S = & \int d^4x \sqrt{(-g)} T^k_{i\ ;k} \xi^i + \\ & \int d\sigma_l \sqrt{(-g)} \{T^l_m - \Theta^l_m - G^{l\ f}_{m;f}\} \xi^m + \\ & + \int d\sigma_l \frac{\partial(\sqrt{(-g)}\mathcal{L})}{\partial(\partial_l g^{ij})} \delta g^{ij} \end{aligned} \quad (29)$$

But Minkowski spacetime has maximum symmetry, meaning that we can choose the $\xi^m$ so that

$$\delta g^{ij} = \xi^{i;j} + \xi^{j;i}$$

is vanishing, and still have a family of 10 parameters of vectors. The vanishing of $\delta S$ for every such $\xi^i$ therefore secures the result

$$T^{lm} - \Theta^{lm} = \partial_f G^{lmf}, \quad (30)$$

(where the $G^{lmf}$ will, in general, be a new tensor with the same symmetries as before) even when the Lagrangian depends on $\partial_l g^{ij}$ (provided the spacetime is Minkowskian).

We could treat also the half integral spin case, but won't. We send the reader to [11].



We have shown, by a slight modification of the standard formalism which consists in conserving all the surface terms, that, in Minkowski spacetime, the metrical energy-momentum tensor is equivalent to the canonical one, in the sense of Belinfante-Rosenfeld, for all fields which describe particles of integral spin. As this depends on the high degree of symmetry of Minkowski spacetime, the result is not extensible to all spaces. The method can, however, be conveniently used to analyse each single case.

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