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Path Integral over Velocities for Relativistic Particle Propagator

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Abstract

A representation for the causal propagator of a relativistic spinless particle by means of a path integral over velocities is presented. For a class of the so called quasi-Gaussian functionals one can formulate universal rules of handling the path integral, similar to ones in the field theory (in the framework of perturbation theory). An advantage of the representation consist in the integration over velocities is not anymore restricted by boundary conditions, and matrices which have to be inverted in course of doing Gaussian integrals, do not contain any derivatives in time. Using the technique, an explicit expression for the propagator is gotten in arbitrary constant homogeneous electromagnetic field and its combination with a plane wave field.

1 Introduction

Already for a long time, different path integral representations for propagators of relativistic particles are discussed in the literature [1–35]. Over recent years this activity got some additional motivation to learn on these simple examples how to quantize by means of path integrals more complicated theories, such as string theory, gravity and so on. On the other hand, the representations are interesting themselves and could be used for calculations of relativistic particle propagators in external electromagnetic or gravitational fields. However, in contrast with the field theory, where path integration rules are well enough defined, at least in the frame of perturbation theory [36,37], in relativistic and nonrelativistic quantum mechanics there are some problems with uniqueness of definition of path integrals, with boundary conditions, and so on [1,2,38–41].

In this paper we present spinless relativistic particle propagator by a path integral over velocities. One can define universal Gaussian and quasi-Gaussian integrals over velocities and rules of handling them. An advantage of the representation consists in the integration over velocities is not anymore restricted by boundary conditions, and matrices which have to be inverted in course of doing Gaussian integrals, do not contain any derivatives in time. This approach is very similar to one used in the field theory (in the frame of perturbation theory [36,37]). We illustrate the convenience and advantage of our method on the most general combination of an external electromagnetic field, admissible for path integration, namely on calculations of the propagator in a constant homogeneous electromagnetic field and its combination with a plane wave field. For these cases we get closed expressions for the propagator. One ought to say that path integral methods were often applied for such kind of calculations. For example, in the works [3,4,42,43,13] the causal propagators for scalar and spinning particles in external electromagnetic field of a plane wave were found by means of path integrations. More complicated combination of electromagnetic field, consisting of parallel magnetic and electric field together with a plane wave, propagating along, was considered in [4]. In [19] they made particular functional integrations to proof a path integral representation for the causal propagator of spinning particle in an electromagnetic field.

2 Representation of scalar particle propagator by means of path integral

As known, the propagator of a scalar particle in an external electromagnetic field $A_\mu(x)$ is the causal Green's function $D^c(x,y)$ of the Klein-Gordon equation in this field,

$$\left[(i\partial - gA)^2 - m^2 + i\epsilon\right] D^c(x,y) = -\delta^4(x-y)\,, \tag{1}$$

where $x = (x^\mu)$, Minkowski tensor $\eta_{\mu\nu}$ =diag$(1,-1,-1,-1)$, and infinitesimal term $i\epsilon$ selects the causal solution.

Consider the Hamiltonian form of the path integral representation for $D^c(x,y)$. For certainty we will use notations and a definition of the integral by means of discretization procedure presented in [29]:

$$\begin{aligned} D^c &= D^c(x_{out}, x_{in}) = i\int_0^\infty d\lambda_0 \int_{\lambda_0} D\lambda \int D\pi \int_{x_{in}}^{x_{out}} Dx \int Dp \\ &\times \exp\left\{ i\int d\tau \left[\lambda\left(\mathcal{P}^2 - m^2\right) + p\dot{x} + \pi\dot{\lambda}\right]\right\}, \end{aligned} \tag{2}$$

where $\mathcal{P}_\mu = -p_\mu - gA_\mu(x)$, and the integration goes over trajectories $x^\mu(\tau)$, $p_\mu(\tau)$,$\lambda(\tau)$, $\pi(\tau)$, parameterized by some parameter $\tau \in [0,1]$. The boundary conditions supposed to hold only for $x(\tau)$ and $\lambda(\tau)$,

$$x(0) = x_{in}, \quad x(1) = x_{out}, \quad \lambda(0) = \lambda_0\,. \tag{3}$$

In (2) and in what follow we use the notation

$$\int d\tau = \int_0^1 d\tau\,.$$

Our aim is to transform the integral (2) to a form convenient, from our point of view, for calculations. First we shift the momenta,

$$-p_\mu \to p_\mu + \frac{\dot{x}_\mu}{2\lambda} + gA_\mu(x)\,,$$

make the replacement $e = 2\lambda$ and fulfil the integration over π and λ ,

$$\begin{aligned} D^c &= \frac{i}{2}\int_0^\infty \frac{de_o}{e_o^2}\int_{x_{in}}^{x_{out}} Dx \int Dp \\ &\times \exp\left\{ i\int d\tau \left[-\frac{\dot{x}^2}{2e_0} + \frac{e_0}{2}\left(p^2 - m^2\right) - g\dot{x}A(x)\right]\right\}. \end{aligned} \tag{4}$$

Then, after the replacement

$$\sqrt{e_0}p \to p\,, \quad \frac{x - x_{in} - \tau\Delta x}{\sqrt{e_0}} \to x\,, \quad \Delta x = x_{out} - x_{in}\,,$$

taking into account the definition of the integral (2) by means of discretization [29], we get the expression

$$\begin{aligned} D^c &= \frac{i}{2}\int_0^\infty \frac{de_o}{e_o^2}\exp\left[-\frac{i}{2}\left(e_0 m^2 + \frac{\Delta x^2}{e_0}\right)\right]\int_0^0 Dx \int Dp \\ &\times \exp\left\{ i\int d\tau \left[-\frac{\dot{x}^2}{2} + \frac{p^2}{2} - g(\sqrt{e_0}\dot{x} + \Delta x)A(\sqrt{e_0}x + x_{in} + \tau\Delta x)\right]\right\}, \end{aligned} \tag{5}$$

where the trajectories $x^\mu(\tau)$ obey already zero boundary conditions,

$$x(0) = x(1) = 0\,. \tag{6}$$

On this step we replace the integration over the trajectories $x^\mu(\tau)$ by one over velocities $v^\mu(\tau)$,

$$\begin{aligned} x(\tau) &= \int \theta(\tau - \tau')v(\tau')d\tau' = \int_0^\tau v(\tau')d\tau'\,, \\ v(\tau) &= \dot{x}(\tau)\,. \end{aligned} \tag{7}$$

The corresponding Jacobian can be formally written as

$$J = \text{Det}\;\theta(\tau - \tau')$$

and regularized, for example, in the frame of discretization procedure. But, our desire is as less as possible to refer to that procedure, so we will try to define that Jacobian in an independent way. Note that because of (6), the trajectories $v(\tau)$ must obey the conditions

$$\int v(\tau)\,d\tau = 0\,. \tag{8}$$

We can take it into account, inserting the corresponding four-dimensional δ-function in the path integral. Thus,

$$D^c = \frac{i}{2}\int_0^\infty \frac{de_o}{e_o^2}\exp\left[-\frac{i}{2}\left(e_0 m^2 + \frac{\Delta x^2}{e_0}\right)\right] \tag{9}$$
$$\int Dv\, J \int Dp\, \delta^4\left(\int v d\tau\right)\exp\left\{i\int d\tau\left[-\frac{v^2}{2}+\frac{p^2}{2}\right.\right.$$
$$\left.\left.-g\left(\sqrt{e_0}v+\Delta x\right)A\left(\sqrt{e_0}\int_0^\tau v(\tau')d\tau' + x_{in} + \tau\Delta x\right)\right]\right\}\,.$$

One can formally find the Jacobian J, switching off the potential $A_\mu(x)$ in (9) and using the expression for the free causal Green function D_0^c ,

$$D_0^c = D_0^c(x_{out}, x_{in}) = \frac{1}{2(2\pi)^2}\int_o^\infty \frac{de_0}{e_0^2}\exp\left[-\frac{i}{2}\left(e_0 m^2 + \frac{\Delta x^2}{e_0}\right)\right]\,.$$

So, we formally get

$$J = \frac{1}{i\,(2\pi)^2}\left[\int Dv \int Dp\;\; \delta^4\left(\int v d\tau\right)\exp\left\{i\int d\tau\left(-\frac{v^2}{2}+\frac{p^2}{2}\right)\right\}\right]^{-1}. \tag{10}$$

Gathering these results, we may write

$$D^c = \frac{1}{2(2\pi)^2}\int_o^\infty \frac{de_0}{e_0^2}\exp\left[-\frac{i}{2}\left(e_0 m^2 + \frac{\Delta x^2}{e_0}\right)\right]\Delta(e_0)\,, \tag{11}$$
$$\Delta(e_0) = \int \mathcal{D}v\, \delta^4\left(\int v d\tau\right)\exp\left\{i\int d\tau\left[-\frac{v^2}{2} - g\left(\sqrt{e_0}v+\Delta x\right)\right.\right.$$
$$\left.\left.\times A\left(\sqrt{e_0}\int_0^\tau v(\tau')d\tau' + x_{in} + \tau\Delta x\right)\right]\right\}\,, \tag{12}$$

where new measure $\mathcal{D}v$ has the form

$$\mathcal{D}v = Dv\left[\int Dv\;\; \delta^4\left(\int v d\tau\right)\exp\left\{i\int d\tau\left(-\frac{v^2}{2}\right)\right\}\right]^{-1}\,. \tag{13}$$

It is clear that $\Delta(e_0) = 1$ at $A = 0$.

A regularization of the infinite measure (13) can be, in principle, performed in the frame of the discretization procedure, but, we are going to consider in the next Section a different approach to the calculation of path integrals of type (12).

3 Gaussian and quasi-Gaussian path integrals over velocities

A calculation of path integrals of the type (12) ,

$$\int \mathcal{D}v\, \delta^4\left(\int v d\tau\right) F[v]\,, \tag{14}$$

with some functional $F[v]$, may be performed in the frame of the discretization procedure, according to the initial definition of the integral (2). However, if we restrict ourselves with a limited class of functionals $F[v]$, which are called quasi-Gaussian [37] and are defined below, then one can formulate some universal rules of their calculation without referring each time to the initial definition. Similar idea has been realized in the field theory (see [36,37]). The restriction with quasi-Gaussian functionals corresponds, in fact, to a perturbation theory, in that concrete case it corresponds to the perturbation theory in the interaction with the external potential of an electromagnetic field.

Introduce the Gaussian functional as

$$F_G[v, I] \tag{15}$$
$$= \exp\left\{-\frac{i}{2}\int d\tau d\tau' v^\mu(\tau)L_{\mu\nu}(g,\tau,\tau')v^\nu(\tau') - i\int d\tau I_\mu(\tau)v^\mu(\tau)\right\}\,,$$

and the quasi-Gaussian functional as

$$F_{qG}[v, I] = F[v]F_G[v, I]\,, \tag{16}$$

where $F[v]$ is a functional, which can be expanded in the functional series of v,

$$F[v] = \sum_{n=0} \int d\tau_1 \dots d\tau_n F_{\mu_1 \dots \mu_n}(\tau_1 \dots \tau_n)\, v^{\mu_1}(\tau_1) \dots v^{\mu_n}(\tau_n)\,, \qquad (17)$$

and I are sources associated with the velocities v. In (15) the matrix $L_{\mu\nu}(g,\tau,\tau')$ supposes to have the following form

$$L_{\mu\nu}(g,\tau,\tau') = \eta_{\mu\nu}\delta(\tau-\tau') + gM_{\mu\nu}(\tau,\tau')\,. \qquad (18)$$

Define the path integral over velocities v of the Gaussian functional as

$$\int \mathcal{D}v\, \delta^4\left(\int v d\tau\right) F_G[v,I] \qquad (19)$$
$$= \left[\frac{\text{Det } L(g) \ \det\ l(g)}{\text{Det } L(0)\ \det\ l(0)}\right]^{-1/2} \exp\left\{\frac{i}{2}\int d\tau d\tau' I(\tau)K(\tau,\tau')I(\tau')\right\}\,,$$

where

$$K(\tau,\tau') = L^{-1}(g,\tau,\tau') - Q^T(\tau)l^{-1}(g)Q(\tau')\,,$$
$$l(g) = \int d\tau d\tau' L^{-1}(g,\tau,\tau'),\ \ Q(\tau) = \int d\tau' L^{-1}(g,\tau',\tau)\,. \qquad (20)$$

It is possible to verify after straightforward calculations that the formula (19) can be derived from the discretization procedure, taking into account the origin of the measure $\mathcal{D}v$.

To calculate not quite good defined determinants of matrices $L(g)$ with continuous indices, entering in (19), one may use some convenient representation. Let us differentiate the well known formula

$$\text{Det } L(g) = \exp\left[\text{Tr}\ \ln\ L(g)\right]$$

with respect to g. So we get the equation

$$\frac{d}{dg}\text{Det } L(g) = \text{Det } L(g)\ \text{Tr } L^{-1}(g)\frac{dL(g)}{dg} = \text{Det } L(g)\ \text{Tr } L^{-1}(g)M\,,$$

which can be solved in the form

$$\frac{\text{Det } L(g)}{\text{Det } L(0)} = \exp\left\{\int_0^g dg' \text{Tr } L^{-1}(g')M\right\}\,. \qquad (21)$$

Taking into account that det $l(0) = -1$, we get for the path integral of the Gaussian functional

$$\int \mathcal{D}v\, \delta^4\left(\int v d\tau\right) F_G[v,I] \qquad (22)$$
$$= [-\det\ l(g)]^{-1/2} \exp\left\{\frac{i}{2}\int d\tau d\tau' I(\tau)K(\tau,\tau')I(\tau')\right.$$
$$\left. -\frac{1}{2}\int_0^g dg' \text{Tr } L^{-1}(g')M\right\}\,.$$

The path integral of the quasi-Gaussian functional we define through one of the Gaussian functional

$$\int \mathcal{D}v\, \delta^4\left(\int v d\tau\right) F_{qG}[v,I] = F\left(i\frac{\delta}{\delta I}\right)\int \mathcal{D}v\, \delta^4\left(\int v d\tau\right) F_G[v,I]$$
$$= [-\det\ l(g)]^{-1/2} F\left(i\frac{\delta}{\delta I}\right) \exp\left\{\frac{i}{2}\int d\tau d\tau' I(\tau)K(\tau,\tau')I(\tau')\right.$$
$$\left. -\frac{1}{2}\int_0^g dg' \text{Tr } L^{-1}(g')M\right\}\,. \qquad (23)$$

One can derive rules of handling with integrals from quasi-Gaussian functionals, using the formula (23). For example, the integral (22) is invariant under the shifts of the integration variables,

$$\int \mathcal{D}v\, \delta^4\left(\int (v+u) d\tau\right) F_{qG}[v+u,I] = \int \mathcal{D}v\, \delta^4\left(\int v d\tau\right) F_{qG}[v,I]\,. \qquad (24)$$

The validity of this assertion for the Gaussian integral (19) can be verified by a direct calculation. Then the general formula (24) follows from the (23). As another consequence of the property (24) one can derive an useful generalization of the formula (23),

$$\int \mathcal{D}v\ \delta^4\left(\int v d\tau - a\right) F_{qG}[v,I] \tag{25}$$
$$= [-\det\ l(g)]^{-1/2} F\left(i\frac{\delta}{\delta I}\right) \exp\left\{\frac{i}{2}\int d\tau d\tau' I(\tau) K(\tau,\tau') I(\tau')\right.$$
$$\left. -\frac{i}{2} a l^{-1}(g) a - i a l^{-1}(g) \int Q(\tau) I(\tau) d\tau - \frac{1}{2}\int_0^g dg' \mathrm{Tr}\ L^{-1}(g') M\right\} ,$$

where a ia a constant vector. The integral of the total functional derivative over $v^\mu(\tau)$ is equal to zero,

$$\int \mathcal{D}v\ \frac{\delta}{\delta v^\mu(\tau)}\ \delta^4\left(\int v d\tau\right) F_{qG}[v,I] = 0\ . \tag{26}$$

This property may also be obtained as a consequence of the functional integral invariance under the shift of variables, as well as by direct calculations of integral (26). As a consequence of the property (26), one can derive formulas of integration by parts, which we do not present here. If a quasi-Gaussian functional depends on a parameter α , then the derivative with respect to this parameter is commutative with the integral sign,

$$\frac{\partial}{\partial \alpha}\int \mathcal{D}v\ \delta^4\left(\int v d\tau\right) F_{qG}[v,I,\alpha] = \int \mathcal{D}v\ \delta^4\left(\int v d\tau\right) \frac{\partial}{\partial\alpha} F_{qG}[v,I,\alpha]\ . \tag{27}$$

Finally, the formula for the change of the variables holds:

$$\int \mathcal{D}v\ \delta^4\left(\int v d\tau\right) F_{qG}[v,I] = \int \mathcal{D}v\ \delta^4\left(\int \phi d\tau\right) F_{qG}[\phi,I]\ \mathrm{Det}\ \frac{\delta\phi_\tau(v)}{\delta v(\tau')}\ , \tag{28}$$

where $\phi_\tau(v)$ is a set of analytical functionals in v , parameterized by τ . The ideas of the proofs of the formulas (27,28) are similar to ones used in [36,37] for the corresponding proofs in the field theory.

Thus, in the relativistic quantum mechanics in the frame of perturbation theory, one can define path integrals over velocities and rules of handling them. These definitions are very close to ones in field theory, the analogy is stressed by the circumstance that, as in the field theory, the integrals over velocities do not contain explicitly any boundary condition for trajectories of

the integration. After the rules of integration are formulated, one can forget about the origin of the integrals over velocities and fulfil integrations, using the rules only. In the next Section we demonstrate this technique on concrete calculations.

4 Calculation of propagator in external electromagnetic fields

Here we are going to calculate the propagator of a scalar particle in an external electromagnetic field, using representation (11) and rules of integrations, presented in the previous Sections. We consider a combination of a constant homogeneous field and a plane wave field. The potentials for this field may be taken to be

$$A_\mu(x) = -\frac{1}{2} F_{\mu\nu} x^\nu + f_\mu(nx)\ , \tag{29}$$

where $F_{\mu\nu}$ is the field strength tensor of the constant homogeneous field with nonzero invariantes

$$\mathcal{F} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \neq 0\ ,\ \ \mathcal{G} = -\frac{1}{4} F^*_{\mu\nu} F^{\mu\nu} \neq 0\ ,$$

($F^*_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}$, $\epsilon_{\mu\nu\alpha\beta}$ is totally antisymmetric tensor), in terms of which its eigenvalues $\mathcal{E}$ and $\mathcal{H}$ are expressed

$$\begin{aligned} &F_{\mu\nu} n^\nu = -\mathcal{E} n_\mu\ ,\ \ F_{\mu\nu}\bar{n}^\nu = \mathcal{E}\bar{n}_\mu\ , \\ &F_{\mu\nu}\ell^\nu = i\mathcal{H}\ell_\mu\ ,\ \ F_{\mu\nu}\bar{\ell}^\nu = -i\mathcal{H}\bar{\ell}_\mu\ , \\ &\mathcal{E} = \left[(\mathcal{F}^2+\mathcal{G}^2)^{\frac{1}{2}} - \mathcal{F}\right]^{\frac{1}{2}}\ ,\ \mathcal{H} = \left[(\mathcal{F}^2+\mathcal{G}^2)^{\frac{1}{2}} + \mathcal{F}\right]^{\frac{1}{2}}\ . \end{aligned} \tag{30}$$

The eigenvectors n, $\bar{n}$, ℓ, $\bar{\ell}$ are isotropic and obey the conditions

$$\begin{aligned} &n^2 = \bar{n}^2 = \ell^2 = \bar{\ell}^2 = 0\ , \\ &n\bar{n} = 2\ ,\ \ell\bar{\ell} = -2\ ,\ n\ell = \bar{n}\ell = n\bar{\ell} = \bar{n}\bar{\ell} = 0\ . \end{aligned} \tag{31}$$

The functions $f_\mu(nx)$ are arbitrary, except for the fact that they are subject to the conditions

$$f_\mu(nx)n^\mu = f_\mu(nx)\bar{n}^\mu = 0\,. \tag{32}$$

The total field strength tensor for the potential (29) is

$$F_{\mu\nu}(x) = F_{\mu\nu} + \Psi_{\mu\nu}(nx),\ \Psi_{\mu\nu}(nx) = n_\mu f'_\nu(nx) - n_\nu f'_\mu(nx)\,. \tag{33}$$

Since the invariantes $\mathcal{F}$, $\mathcal{G}$ of the tensor $F_{\mu\nu}$ are nonzero, there exists a special reference frame, where the electric and magnetic fields, corresponding to this tensor, are collinear with respect to one another and to the spatial part **n** of the four-vector n. In this reference frame, the total field $F_{\mu\nu}(x)$ corresponds to a constant homogeneous and collinear electric and magnetic fields together with a plane wave, propagating along them; $\mathcal{E}$, $\mathcal{H}$, being equal to the strengths of a constant homogeneous electric and magnetic fields, respectively. In terms of the defined eigenvectors the tensor $F_{\mu\nu}$ can be written as

$$F_{\mu\nu} = \frac{\mathcal{E}}{2}\left(\bar{n}_\mu n_\nu - n_\mu\bar{n}_\nu\right) + \frac{i\mathcal{H}}{2}\left(\bar{\ell}_\mu\ell_\nu - \ell_\mu\bar{\ell}_\nu\right)\,, \tag{34}$$

and the completeness relation holds

$$\eta_{\mu\nu} = \frac{1}{2}\left(\bar{n}_\mu n_\nu + n_\mu\bar{n}_\nu - \bar{\ell}_\mu\ell_\nu - \ell_\mu\bar{\ell}_\nu\right)\,. \tag{35}$$

The latter allows one to express any four-vector u in terms of the eigenvectors (30),

$$u^\mu = n^\mu u^{(1)} + \bar{n}^\mu u^{(2)} + \ell^\mu u^{(3)} + \bar{\ell}^\mu u^{(4)}\,,$$
$$u^{(1)} = \frac{1}{2}\bar{n}u\,,\ u^{(2)} = \frac{1}{2}nu\,,\ u^{(3)} = -\frac{1}{2}\bar{\ell}u\,,\ u^{(4)} = -\frac{1}{2}\ell u\,. \tag{36}$$

In these concrete calculations it is convenient for us to make a shift of variables in the formula (12), to rewrite it in the following form

$$\Delta(e_0) = \exp\left(i\frac{\Delta x^2}{2e_0}\right)\int \mathcal{D}v\ \delta^4\left(\int v d\tau - \frac{\Delta x}{\sqrt{e_0}}\right)$$
$$\times\exp\left\{i\int d\tau\left[-\frac{v^2}{2} - g\sqrt{e_0}vA\left(\sqrt{e_0}\int_0^\tau v(\tau')d\tau' + x_{in}\right)\right]\right\}\,. \tag{37}$$

The calculations will be made in two steps: first in a constant homogeneous field only, and then in the total combination (29), using some results of the first problem. Thus, on the first step the potentials of the electromagnetic field are

$$A_\mu(x) = -\frac{1}{2}F_{\mu\nu}x^\nu\,. \tag{38}$$

Substituting the external field (38) into (37), one can find

$$\Delta(e_0) = \exp\left(i\frac{\Delta x^2}{2e_0}\right)\int \mathcal{D}v\ \delta^4\left(\int v d\tau - \frac{\Delta x}{\sqrt{e_0}}\right)$$
$$\times\exp\left\{-\frac{i}{2}\int d\tau d\tau' v(\tau)L(g,\tau,\tau')v(\tau') - i\int\frac{g\sqrt{e_0}}{2}x_{in}F\,v d\tau\right\}\,, \tag{39}$$

where

$$L_{\mu\nu}(g,\tau,\tau') = \eta_{\mu\nu}\delta(\tau-\tau') - \frac{ge_0}{2}F_{\mu\nu}\epsilon(\tau-\tau')\,. \tag{40}$$

The path integral (39) is the Gaussian one (see (25)). To get an answer, one needs to find the inverse matrix $L^{-1}(g,\tau,\tau')$, which satisfies the equation

$$\int L(g,\tau,\tau'')L^{-1}(g,\tau'',\tau')d\tau'' = \delta(\tau-\tau')\,.$$

One can demonstrate, that this equation is equivalent to a differential one,

$$\frac{\partial}{\partial\tau}L^{-1}(g,\tau,\tau') - ge_0F\,L^{-1}(g,\tau,\tau') = \delta'(\tau-\tau')\,, \tag{41}$$

with initial condition

$$L^{-1}(g,0,\tau') + \frac{ge_0F}{2}\int L^{-1}(g,\tau'',\tau')d\tau'' = \delta(\tau')\,.$$

Its solution has the form

$$L^{-1}(g,\tau,\tau')$$
$$= \delta(\tau-\tau') + \frac{ge_0F}{2}\exp\left\{ge_0(\tau-\tau')F\right\}\left[\epsilon(\tau-\tau') - \tanh\left(\frac{ge_0F}{2}\right)\right]\,. \tag{42}$$

Using (42), one can find all ingredients of the general formula (25), taking into account that

$$a = -\frac{\Delta x}{\sqrt{e_0}}\,,\quad I(\tau) = \frac{g\sqrt{e_0}}{2} x_{in} F\,.$$

Thus,

$$\begin{aligned} &K(\tau,\tau') \\ &= \delta(\tau-\tau') + \frac{ge_0F}{2}\exp\{ge_0(\tau-\tau')F\}\left[\epsilon(\tau-\tau') - \coth\left(\frac{ge_0F}{2}\right)\right]\,, \\ &\int d\tau d\tau' K(\tau,\tau') = 0\,,\ \int d\tau Q(\tau) = l(g)\,,\ l(g) = \frac{\tanh ge_0F/2}{ge_0F/2}\,, \\ &M(\tau,\tau') = -\frac{e_0}{2}F\epsilon(\tau-\tau')\,,\ \int_0^g dg' \mathrm{Tr} L^{-1}(g')M = \mathrm{tr}\ln(\cosh ge_0F/2)\,, \end{aligned} \tag{43}$$

where the symbol "tr" is being taken over four dimensional indices only. Then

$$\begin{aligned} \Delta(e_0) = &\left[-\det\left(\frac{\sinh ge_0F/2}{gF/2}\right)\right]^{-1/2} \\ &\times\exp\left\{\frac{i}{2}\left[\frac{\Delta x^2}{e_0} + g\,x_{out}Fx_{in} - \frac{1}{2}\Delta x\; gF\coth\left(\frac{ge_0F}{2}\right)\Delta x\right]\right\}\,. \end{aligned} \tag{44}$$

Substituting (44) into (11), we get the final expression for the causal propagator of a scalar particle in a constant homogeneous electromagnetic field

$$\begin{aligned} D^c(x_{out},x_{in}) = &\frac{1}{2(2\pi)^2}\int_0^\infty de_o\left[-\det\left(\frac{\sinh ge_0F/2}{gF/2}\right)\right]^{-1/2} \\ &\times\exp\left\{\frac{i}{2}\left[gx_{out}Fx_{in} - e_0m^2 - \frac{1}{2}\Delta x\; gF\coth(\frac{ge_0F}{2})\Delta x\right]\right\}\,. \end{aligned} \tag{45}$$

This result was first derived by Schwinger, using his proper time method [44].

Now we return to the total electromagnetic field (29). Let us substitute the potential (29) into (37),

$$\begin{aligned} \Delta(e_0) = &\exp\left(i\frac{\Delta x^2}{2e_0}\right)\int \mathcal{D}v\,\delta^4\left(\int v d\tau - \frac{\Delta x}{\sqrt{e_0}}\right) \\ &\times\exp\left\{-\frac{i}{2}\int d\tau d\tau' v(\tau)L(g,\tau,\tau')v(\tau') - i\int\frac{g\sqrt{e_0}}{2}\,x_{in}F\,v d\tau\right. \\ &\left.-ig\sqrt{e_0}\int d\tau v(\tau) f\left(nx_{in} + \sqrt{e_0}\int_0^\tau nv(\tau')d\tau'\right)\right\}\,, \end{aligned} \tag{46}$$

with $L(g,\tau,\tau')$ defined in (40). One can take the integral (46) as quasi-Gaussian, in accordance with the formula (25). So, one can write

$$\begin{aligned} &\Delta(e_0) \\ &= \exp\left\{g\sqrt{e_0}\int d\tau f\left(nx_{in} + i\sqrt{e_0}\int_0^\tau n\frac{\delta}{\delta I(\tau')}d\tau'\right)\frac{\delta}{\delta I(\tau)}\right\}B(I)\,|_{I=0}\,, \end{aligned} \tag{47}$$

where

$$\begin{aligned} B(I) = &\exp\left(i\frac{\Delta x^2}{2e_0}\right)\int \mathcal{D}v\,\delta^4\left(\int v d\tau - \frac{\Delta x}{\sqrt{e_0}}\right) \\ &\times\exp\left\{-\frac{i}{2}\int d\tau d\tau' v(\tau)L(g,\tau,\tau')v(\tau')\right. \\ &\left.-i\int\left(\frac{g\sqrt{e_0}}{2}x_{in}F + I(\tau)\right)v(\tau)d\tau\right\}\,. \end{aligned} \tag{48}$$

The integral can be found similar to (39). As a result we get

$$\begin{aligned} &B(I) \\ &= \exp\left\{\frac{i}{2}\int d\tau d\tau' I(\tau)K(\tau,\tau')I(\tau') - i\int I(\tau)a(\tau)d\tau\right\}\Delta(e_0)|_{\Psi=0}\,, \end{aligned} \tag{49}$$

where $\Delta(e_0)|_{\Psi=0}$ is the expression given by (44), $K(\tau,\tau')$ is defined in (43), and

$$a(\tau) = \frac{\Delta x}{2\sqrt{e_0}}\left(1 + \coth(ge_0F/2)\right)ge_0F\exp(-ge_0F\tau)\,.$$

To obtain the action of the operator, involved in (47), on the functional $B(I)$, we decompose the sources $I^\mu(\tau)$ in the eigenvectors (30), using (36)

$$I^\mu(\tau) = \frac{1}{2}\left(n^\mu\, \bar{n}I(\tau) + \bar{n}^\mu\, nI(\tau) - \ell^\mu\, \bar{\ell}I(\tau) - \bar{\ell}^\mu\, \ell I(\tau)\right) .$$

Then, it is possible to write

$$\begin{aligned} n\tfrac{\delta}{\delta I(\tau)} &= 2\tfrac{\delta}{\delta \bar{n} I(\tau)} , \\ f\tfrac{\delta}{\delta I(\tau)} &= \bar{\ell} f \tfrac{\delta}{\delta \bar{\ell} I(\tau)} + \ell f \tfrac{\delta}{\delta \ell I(\tau)} . \end{aligned}$$

Using this, we get

$$\begin{aligned} & f\left(nx_{in} + i\sqrt{e_0}\int_0^\tau n\frac{\delta}{\delta I(\tau')}d\tau'\right)\frac{\delta}{\delta I(\tau)} \\ & = \bar{\ell} f\left(nx_{in} + i\sqrt{e_0}\int_0^\tau \frac{\delta}{\delta \bar{n} I(\tau')}d\tau'\right)\frac{\delta}{\delta \bar{\ell} I(\tau)} \\ & + \ell f\left(nx_{in} + i\sqrt{e_0}\int_0^\tau \frac{\delta}{\delta \bar{n} I(\tau')}d\tau'\right)\frac{\delta}{\delta \ell I(\tau)} , \\ & \int d\tau d\tau' I(\tau) K(\tau,\tau') I(\tau') \\ & = \int d\tau d\tau' \left[\bar{n}I(\tau)\, nI(\tau') K(\tau,\tau',\mathcal{E}) - \bar{\ell}I(\tau)\, \ell I(\tau') K(\tau,\tau',i\mathcal{H})\right] , \\ & \int I(\tau) a(\tau) d\tau \\ & = \frac{1}{2}\int \left[\bar{n}I(\tau)\, na(\tau) + nI(\tau)\, \bar{n}a(\tau) - \bar{\ell}I(\tau)\, \ell a(\tau) - \ell I(\tau)\, \bar{\ell}a(\tau)\right] d\tau , \end{aligned}$$

where

$$\begin{aligned} & K(\tau,\tau',\mathcal{E}) \\ & = \delta(\tau-\tau') + \frac{ge_0\mathcal{E}}{2}\exp\{ge_0(\tau-\tau')\mathcal{E}\}\left[\epsilon(\tau-\tau') - \coth\left(\frac{ge_0\mathcal{E}}{2}\right)\right] , \\ & K(\tau,\tau',i\mathcal{H}) \\ & = \delta(\tau-\tau') + \frac{ige_0\mathcal{H}}{2}\exp\{ige_0(\tau-\tau')\mathcal{H}\}\left[\epsilon(\tau-\tau') + i\cot\left(\frac{ge_0\mathcal{H}}{2}\right)\right] . \end{aligned}$$

Now the exponent of the functional $B[I]$ is linear in $nI(\tau)$, $\bar{n}I(\tau)$, $\ell I(\tau)$, $\bar{\ell}I(\tau)$. Thus, one can easy to get a result

$$\begin{aligned} \Delta(e_0) &= \exp\left\{\frac{i}{2}g^2 e_0 \int d\tau d\tau' f(nx_{cl}(\tau)) K(\tau,\tau') f(nx_{cl}(\tau'))\right. \\ & \left. + ig\sqrt{e_0}\int d\tau a(\tau) f(nx_{cl}(\tau))\right\}\Delta(e_0)|_{\Psi=0} , \\ nx_{cl}(\tau) &= nx_{in} + \frac{1-\exp(ge_0\mathcal{E}\tau)}{1-\exp(ge_0\mathcal{E})} n\Delta x , \\ nx_{cl}(0) &= nx_{in} , \quad nx_{cl}(1) = nx_{out} , \end{aligned} \tag{50}$$

where $x_{cl}(\tau)$ is the solution of the Lorentz equation in the external electromagnetic field (29) [28]. Gathering (50) and (44), we get

$$\begin{aligned} \Delta(e_0) &= \left[-\det\frac{\sinh(ge_0F/2)}{ge_0F/2}\right]^{-1/2}\exp\left\{\frac{i}{2}\left[gx_{out}Fx_{in}\right.\right. \\ & -\frac{1}{2}(\Delta x + l(e_0,1))\; gF\coth(ge_0F/2)\; (\Delta x + l(e_0,1)) + 2\Phi(e_0) \\ & \left.\left. + \Delta x\, gFl(e_0,1) + \frac{\Delta x^2}{e_0}\right]\right\} , \end{aligned} \tag{51}$$

where

$$\begin{aligned} \Phi(e_0) &= e_0\int gf(nx_{cl}(\tau))\left[gf(nx_{cl}(\tau)) + gFl(e_0,\tau)\right]d\tau , \\ l(e_0,\tau) &= e_0\int_0^\tau \exp\{ge_0(\tau-\tau')F\}\, gf(nx_{cl}(\tau'))\, d\tau' . \end{aligned} \tag{52}$$

Substituting (51) into (11), we arrive to the final expression for the causal propagator of a scalar particle in the external electromagnetic field (29):

$$\begin{aligned} D^c(x_{out},x_{in}) &= \frac{1}{2(2\pi)^2}\int_0^\infty de_0\left[-\det\left(\frac{\sinh ge_0F/2}{gF/2}\right)\right]^{-1/2} \\ & \times\exp\left\{\frac{i}{2}\left[gx_{out}Fx_{in} - e_0m^2 + \Delta x\, gFl(e_0,1)\right.\right. \\ & \left.\left. -\frac{1}{2}(\Delta x + l(e_0,1))\; gF\coth(ge_0F/2)\; (\Delta x + l(e_0,1)) + 2\Phi(e_0)\right]\right\} . \end{aligned} \tag{53}$$

This expression coincides with the one, obtained in [45], by means of the method of summation over exact solutions of Klein-Gordon equation in the external field (29).

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