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PUBLICAÇÕES

IFUSP/P-1061

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Julho/1993

The Algebra of Non-Local Charges in Non-Linear Sigma Models

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Abstract

We obtain the exact Dirac algebra obeyed by the conserved non-local charges in bosonic non-linear sigma models. Part of the computation is specialized for a symmetry group $O(N)$. As it turns out the algebra corresponds to a cubic deformation of the Kac-Moody algebra. The non-linear terms are computed in closed form. In each Dirac bracket we only find highest order terms (as explained in the paper), defining a saturated algebra. We generalize the results for the presence of a Wess-Zumino term. The algebra is very similar to the previous one, containing now a calculable correction of order one unit lower.

* Partially supported by CNPq. Address after August/93: CERN-Theory Division, CH-1211, Genève 23, Switzerland.
† Supported by CNPq.
* Supported by FAPESP.

1. Introduction

In general, quantum field theoretic models where non-perturbative computations are known, contain an infinite number of conservation laws [1,2]. In fact, the solvability of several exact S-matrices in two dimensional models can be traced back to the Yang-Baxter relations [3,4], which in turn are a direct consequence of the conservation of higher powers of the momentum. Alternatively, there is an infinite number of non-local conservation laws in most of these models as well [2,5]. Both sets of conserved quantities can be related to the existence of a Lax pair in the theory: demanding compatibility of the Lax pair, one arrives at conserved charges as functions of the so called spectral parameter implying, after Taylor expansion, an infinite number of conservation laws.

Another set of models containing an infinite number of conserved quantities are the two dimensional conformally invariant theories [6,7]. The Virasoro generators are a generalization of the energy momentum conserved charges. Defining a realization of the symmetry in terms of the null vectors implies a number of differential equations to be obeyed by the correlation functions which can be integrated. In other words, a further knowledge of the underlying algebra obeyed by the conserved quantities, namely the Virasoro algebra, together with the differential representation of the conserved charges, permitted one to go one step further, i.e. the complete computation of the correlators.

Our aim here is to obtain the algebra of conserved quantities for integrable theories. The algebra of local conservation laws is abelian and therefore too simple. Massive perturbations of the conformal generators are also a possibility, since they also form a non-commuting algebra, and it would be worthwhile to understand the algebra, as well as the role played by the conservation laws surviving the mass perturbation [8]. For free fermions ($k = 1$ WZW models) the results conform to our expectation [9].

Non-local conserved charges, on the other hand, are very powerful objects. The first non-trivial one alone fixes almost completely the on-shell dynamics [5,10].

Infinite algebras connected with non-trivial conserved quantities could thus be the key ingredient for the complete solvability of integrable models, and the knowledge of their correlation functions. It is thus no wonder that the problem evaded solution in spite of several attempts. Indeed, it has been claimed long ago [11] that non-local charges might build up a Kac-Moody algebra, but the appearance of cubic terms found by several authors showed that the algebraic problem was much more involved [12-14]. For non-linear sigma models with a simple gauge group the quantum non-local charges present no anomaly [15], and the monodromy matrix can be computed. Therefore the non-local charge algebra should be manageable; however, as it turns out, the complete algebra was not known, and one had hints that a possible break of the Jacobi identity might occur [12].

We show that there is a natural recombination of the standard non-local charges, whose algebra has an approachable structure, being composed of a linear part of the Kac-Moody form, and a calculable cubic term. Later we add a Wess-Zumino (WZ) term to the action, and show that both linear and cubic pieces of the algebra acquire a further contribution.

In order to find these results we adopt the following strategy. First we compute explicitly the first few conserved charges generated by the procedure of Brézin et al. [16]: the Dirac brackets of those charges are rather obscure, as we compute (there are also examples in the literature [12-14]). Therefore we subsequently define an *improved* set of charges in order to simplify the algebra. By inspection, we propose an Ansatz for the general algebra of the improved charges. Then we argue, based on the Jacobi identity proved in the subsequent section, that once we have verified the algebra up to some order, there must be a set of charges whose algebra agrees with the Ansatz.

In order to verify the Jacobi identity, we introduce a set of (non-conserved) charges whose algebra is isomorphic to the Ansatz. In that case it is useful to start from the analysis of a kind of *chain algebra*, in the sense that we commute elements defined by chains of local currents tied by a non-local function in space. In terms of these objects we define a linear algebra, albeit with a much larger set of terms. Finally, by a sort of trace projection, we recover the original algebra in terms of the *saturated* charges, proving the Jacobi identity in an indirect way.

This paper is divided as follows: in Sect. 2 we review the algebra obeyed by Noether local currents based on Refs. [17,18]. In Sect. 3 we consider the canonical construction of higher non-local conservation laws and calculate some non-local charges explicitly. Then we derive the brackets for some appropriate combinations of charges and write the Ansatz of the complete algebra. In Sect. 4 we define the algebra of saturated charges, which turns out to be isomorphic to the algebra of conserved charges. We derive the chain algebra structure, the corresponding Jacobi identity, and relate the results to the case of non-local charges. Based on the chain algebra and the consequent Jacobi identity, we complete the proof about the highest order structure of the algebra of improved charges, outlined in Sec. 3. In Sect. 5 we introduce the WZ interaction to derive the corresponding algebra. We leave Sect. 6 for conclusions.

2. Current Algebra of Non-Linear Sigma Models

The current algebra of classical non-linear sigma models on arbitrary Riemannian manifolds (M) is known [17]. Indeed, consider a non-linear sigma model on M, with metric $g_{ij}(\varphi)$, and the maps $\varphi^i(x)$ from two dimensional Minkowski space Σ to M. The sigma model action is given by

$$S = \frac{1}{2\lambda^2}\int_\Sigma d^2x\, \eta^{\mu\nu} g_{ij}(\varphi)\partial_\mu\varphi^i\partial_\nu\varphi^j \quad . \tag{1}$$

The phase space consists of pairs $(\varphi^i(x), \pi_i(x))$, where π is a section of the pull-back $\varphi^*(T^*M)$ of the cotangent bundle of M to the Minkowski space via φ, and the canonical equal-time Poisson brackets read

$$\begin{aligned}&\{\varphi^i(x), \varphi^j(y)\} = \{\pi_i(x), \pi_j(y)\} = 0\\ &\{\varphi^i(x), \pi_j(y)\} = \delta^i_j\delta(x-y) \quad .\end{aligned} \tag{2}$$

From the action (1) we find the canonically conjugated momenta, given by the expression

$$\pi_i = \frac{1}{\lambda^2} g_{ij}\dot\varphi^j \quad . \tag{3}$$

We suppose that there is a connected Lie group G acting on M by isometries, such that a generator of the Lie algebra $\mathbf{g}$ of G is represented by a fundamental vector field

$$X_M(m) = \frac{d}{dt} e^{tX}\cdot m\Big|_{t=0} \tag{4}$$

on M; the Noether current may be defined as

$$(j_\mu, X) = -\left(\frac{1}{\lambda^2} g_{ij}(\varphi)\partial_\mu\varphi^i X^j_M(\varphi)\right) \quad . \tag{5}$$

We define also the symmetric scalar field j as

$$(j, X\otimes Y) = \frac{1}{\lambda^2} g_{ij}(\varphi) X^i_M Y^j_M \quad . \tag{6}$$

In terms of a basis t^a of $\mathbf{g}$, such that $[t^a, t^b] = f^{abc}t^c$, we have

$$\begin{aligned} j_\mu &= j^a_\mu t^a\\ j &= j^{ab} t^a\otimes t^b \quad ,\end{aligned} \tag{7}$$

and we find the current algebra

$$\begin{aligned}
&\{j^a_0(x), j^b_0(y)\} = -f^{abc} j^c_0(x)\delta(x-y)\\
&\{j^a_0(x), j^b_1(y)\} = -f^{abc} j^c_1(x)\delta(x-y) + j^{ab}(y)\delta'(x-y)\\
&\{j^a_1(x), j^b_1(y)\} = 0\\
\\
&\{j^a_0(x), j^{bc}(y)\} = -(f^{abd} j^{cd}(x) + f^{acd} j^{bd}(x))\delta(x-y)\\
&\{j^a_1(x), j^{bc}(y)\} = 0\\
&\{j^{ab}(x), j^{cd}(y)\} = 0 \quad .
\end{aligned} \tag{8}$$

In order to give explicit examples, although without loss of generality, we specialize to the $O(N)$ case, with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi_i\partial^\mu\varphi_i \quad , \quad \sum_{i=1}^N \varphi_i^2 = 1 \quad , \tag{9}$$

and Hamiltonian density

$$\mathcal{H} = \frac{1}{2}(\pi_i^2 + \varphi_i'^2) \quad , \tag{10}$$

here $\pi_i = \dot{\varphi}_i$. We have to impose the constraints

$$\varphi_i^2 - 1 = 0 \qquad \text{and} \qquad \varphi_i \pi_i = 0 \quad . \tag{11}$$

irac brackets can be easily calculated and read

$$\begin{aligned} \{\varphi_i(x), \varphi_j(y)\} &= 0 \quad , \\ \{\varphi_i(x), \pi_j(y)\} &= (\delta_{ij} - \varphi_i\varphi_j)(x)\delta(x-y) \quad , \\ \{\pi_i(x), \pi_j(y)\} &= -(\varphi_i\pi_j - \varphi_j\pi_i)(x)\delta(x-y) \quad . \end{aligned} \tag{12}$$

terms of phase space variables the conserved current components may be written as

$$(j_0)_{ij} = \varphi_i\pi_j - \varphi_j\pi_i \quad , \tag{13a}$$
$$(j_1)_{ij} = \varphi_i\varphi_j' - \varphi_j\varphi_i' \quad . \tag{13b}$$

tice that j_μ is an antisymmetric matrix-valued field. On the other hand the intertwiner ld given in (7) is symmetric,

$$(j)_{ij} = \varphi_i\varphi_j \quad . \tag{13c}$$

e observe that the Hamiltonian (10) can be written in the Sugawara form

$$\mathcal{H} = -\frac{1}{4}\,\mathrm{tr}(j_0^2 + j_1^2) \quad . \tag{14}$$

is convenient to present the current algebra in terms of matrix components, which follows m the elementary brackets (12):

$$\begin{aligned} \{(j_0)_{ij}(x), (j_0)_{kl}(y)\} &= (\delta \circ j_0)_{ij,kl}(x)\delta(x-y) \\ \{(j_1)_{ij}(x), (j_0)_{kl}(y)\} &= (\delta \circ j_1)_{ij,kl}(x)\delta(x-y) + (\delta \circ j)_{ij,kl}(x)\delta'(x-y) \\ \{(j_1)_{ij}(x), (j_1)_{kl}(y)\} &= 0 \\ \\ \{(j)_{ij}(x), (j)_{kl}(y)\} &= 0 \\ \{(j)_{ij}(x), (j_1)_{kl}(y)\} &= 0 \\ \{(j)_{ij}(x), (j_0)_{kl}(y)\} &= -(\delta \star j)_{ij,kl}(x)\delta(x-y) \end{aligned} \tag{15}$$

ere

$$(\delta \circ A)_{ij,kl} \equiv \delta_{ik}A_{jl} - \delta_{il}A_{jk} + \delta_{jl}A_{ik} - \delta_{jk}A_{il} \tag{16}$$
$$(\delta \star A)_{ij,kl} \equiv \delta_{ik}A_{jl} - \delta_{il}A_{jk} - \delta_{jl}A_{ik} + \delta_{jk}A_{il} \quad . \tag{17}$$

rther useful properties of the product defined in (16) are listed in the Appendix. The ebra of components (8) can be easily rederived from (15) using the property (A.8).

3. Standard and Improved Non-local Charges

Non-local charges may be generated by a very simple algorithm [16], starting out of a current j_μ obeying

$$\begin{aligned} \partial^\mu j_\mu &= 0 \\ \partial_\mu j_\nu - \partial_\nu j_\mu + 2[j_\mu, j_\nu] &= 0 \quad . \end{aligned} \tag{18}$$

Given a conserved current $J_\mu^{(n)}$, one defines the associated non-local potential $\chi^{(n)}$ through the equation

$$J_\mu^{(n)} = \epsilon_{\mu\nu}\partial^\nu\chi^{(n)} \quad , \tag{19}$$

and build the $(n+1)^{\text{th}}$ order non-local current

$$\begin{aligned} J_\mu^{(n+1)} &\equiv D_\mu\chi^{(n)} = \partial_\mu\chi^{(n)} + 2[j_\mu, \chi^{(n)}] \quad , \\ \hat{Q}^{(n)} &\equiv \int dx J_0^{(n)} \quad . \end{aligned} \tag{20}$$

Such current is also conserved as a consequence of Eqs. (18). Here we have to mention that for the first non-local current $J_\mu^{(1)}$ the coefficient in front of commutator in (20) must be taken as 1 instead of 2.

Eq. (19) can be inverted for $\chi^{(n)} = \partial^{-1}J_0^{(n)}$, where we choose the antiderivative operator as

$$\bar{\partial}^1 A(x) = \frac{1}{2}\int dy\, \epsilon(x-y)A(y) \quad , \quad \epsilon(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ +1, & x > 0 \end{cases} \quad . \tag{21}$$

With this definition we have antisymmetric boundary conditions for $\chi^{(n)}$,

$$\chi^{(n)}(\pm\infty) = \pm\frac{1}{2}\int dx J_0^{(n)}(x) = \pm\frac{1}{2}\hat{Q}^{(n)} \quad . \tag{22}$$

Other boundary conditions could be used [13] but the above choice guarantees that the algebra of charges produces antisymmetric combinations of charges, which belong to the $O(N)$ algebra; moreover, the charge algebra closes in terms of $O(N)$ generator. We call the charges defined in Eq. (20) the *standard* charges. After some partial integrations the first few of them read

$$\hat{Q}^{(0)} = \int dx\, j_0 \tag{23}$$

$$\hat{Q}^{(1)} = \int dx \left(j_1 + 2j_0\bar{\partial}^1 j_0\right) \tag{24}$$

$$\hat{Q}^{(2)} = \hat{Q}^{(0)} + \frac{1}{2}\left(\hat{Q}^{(0)}\right)^3 + 3\int dx \left(j_1\bar{\partial}^1 j_0 - \bar{\partial}^1 j_0 j_1 - 2\bar{\partial}^1 j_0 j_0 \bar{\partial}^1 j_0\right) \tag{25}$$

$$\begin{aligned} \hat{Q}^{(3)} &= \hat{Q}^{(1)} + 2\hat{Q}^{(1)}\left(\hat{Q}^{(0)}\right)^2 + 4\hat{Q}^{(0)}\hat{Q}^{(1)}\hat{Q}^{(0)} + 2\left(\hat{Q}^{(0)}\right)^2\hat{Q}^{(1)} \\ &+4\int dx \Big[j_0\bar{\partial}^1 j_0 + j_1\bar{\partial}^1 j_1 - 2(\bar{\partial}^1 j_0 j_1 \bar{\partial}^1 j_0 + \bar{\partial}^1 j_0 j_0 \bar{\partial}^1 j_1 + \bar{\partial}^1 j_1 j_0 \bar{\partial}^1 j_0) + 4\bar{\partial}(\bar{\partial}^1 j_0 j_0) j_0 \bar{\partial}^1 j_0\Big] \quad . \end{aligned} \tag{26}$$

However, it turns out that the algebra satisfied by these standard set of charges is not transparent enough [12-14]. In the search for a more suitable basis of charges we find out an algebraic algorithm, where the charge $\hat{Q}^{(1)}$ plays a fundamental role, which generates an improved set of conserved charges ($\{Q^{(n)}\}$). We can relate this new set to the standard one: for instance, the first few *improved* charges read

$$
\begin{aligned}
Q^{(0)} &\equiv \hat{Q}^{(0)} \\
Q^{(1)} &\equiv \hat{Q}^{(1)} \\
Q^{(2)} &\equiv \frac{2}{3}\hat{Q}^{(2)} + \frac{4}{3}\hat{Q}^{(0)} - \frac{1}{3}\left(\hat{Q}^{(0)}\right)^3 \\
Q^{(3)} &\equiv \frac{1}{3}\hat{Q}^{(3)} + \frac{8}{3}\hat{Q}^{(1)} - \frac{2}{3}\hat{Q}^{(1)}\left(\hat{Q}^{(0)}\right)^2 - \frac{4}{3}\hat{Q}^{(0)}\hat{Q}^{(1)}\hat{Q}^{(0)} - \frac{2}{3}\left(\hat{Q}^{(0)}\right)^2\hat{Q}^{(1)} \quad .
\end{aligned}
\tag{27}
$$

In terms of the local currents j_μ, we write down the first six improved charges,

$$
\begin{aligned}
Q^{(0)} &= \int dx\, j_0 \\
Q^{(1)} &= \int dx\left(j_1 + 2j_0\partial^{-1}j_0\right) \\
Q^{(2)} &= \int dx\left(2j_0 + 2j_1\partial^{-1}j_0 - 2\partial^{-1}j_0 j_1 - 4\partial^{-1}j_0 j_0\partial^{-1}j_0\right) \\
Q^{(3)} &= \int dx\Big[3j_1 + 8j_0\partial^{-1}j_0 + 2j_1\partial^{-1}j_1 - 4(\partial^{-1}j_0 j_1\partial^{-1}j_0 + \partial^{-1}j_0 j_0\partial^{-1}j_1 + \partial^{-1}j_1 j_0\partial^{-1}j_0) \\
&\quad +8\partial^{-1}(\partial^{-1}j_0 j_0)j_0\partial^{-1}j_0\Big] \\
Q^{(4)} &= \int dx\Big\{6j_0 + 10j_1\partial^{-1}j_0 - 10\partial^{-1}j_0 j_1 - 24\partial^{-1}j_0 j_0\partial^{-1}j_0 \\
&\quad - 4(\partial^{-1}j_1 j_0\partial^{-1}j_1 + \partial^{-1}j_0 j_1\partial^{-1}j_1 + \partial^{-1}j_1 j_1\partial^{-1}j_0) \\
&\quad + 8[\partial^{-1}(\partial^{-1}j_0 j_0)(j_0\partial^{-1}j_1 + j_1\partial^{-1}j_0) - (\partial^{-1}j_0 j_1 + \partial^{-1}j_1 j_0)\partial^{-1}(j_0\partial^{-1}j_0)] \\
&\quad +16\partial^{-1}(\partial^{-1}j_0 j_0)j_0\partial^{-1}(j_0\partial^{-1}j_0)\Big\} \\
Q^{(5)} &= \int dx\Big\{10j_1 + 32j_0\partial^{-1}j_0 + 12j_1\partial^{-1}j_1 \\
&\quad - 28(\partial^{-1}j_0 j_1\partial^{-1}j_0 + \partial^{-1}j_0 j_0\partial^{-1}j_1 + \partial^{-1}j_1 j_0\partial^{-1}j_0) - 4\partial^{-1}j_1 j_1\partial^{-1}j_1 \\
&\quad + 64\partial^{-1}(\partial^{-1}j_0 j_0)j_0\partial^{-1}j_0 + 8[\partial^{-1}(\partial^{-1}j_1 j_0)j_0\partial^{-1}j_1 + \partial^{-1}(\partial^{-1}j_0 j_0)j_1\partial^{-1}j_1 + \partial^{-1}(\partial^{-1}j_0 j_1)j_0\partial^{-1}j_1 \\
&\quad + \partial^{-1}(\partial^{-1}j_1 j_0)j_1\partial^{-1}j_0 + \partial^{-1}(\partial^{-1}j_0 j_1)j_1\partial^{-1}j_0 + \partial^{-1}(\partial^{-1}j_1 j_1)j_0\partial^{-1}j_0] \\
&\quad + 16[\partial^{-1}(\partial^{-1}j_0 j_0)j_0\partial^{-1}(j_0\partial^{-1}j_1) + \partial^{-1}(\partial^{-1}j_0 j_0)j_0\partial^{-1}(j_1\partial^{-1}j_0) + \partial^{-1}(\partial^{-1}j_1 j_0)j_0\partial^{-1}(j_0\partial^{-1}j_0) \\
&\quad +\partial^{-1}(\partial^{-1}j_0 j_1)j_0\partial^{-1}(j_0\partial^{-1}j_0) + \partial^{-1}(\partial^{-1}j_0 j_0)j_1\partial^{-1}(j_0\partial^{-1}j_0)] - 32\partial^{-1}(\partial^{-1}(\partial^{-1}j_0 j_0)j_0)j_0\partial^{-1}(j_0\partial^{-1}j_0)\Big\} \,.
\end{aligned}
\tag{28}
$$

Using the above definitions of the improved charges and the current algebra given in

(15) we obtain after a rather tedious calculation the following algebra

$$
\begin{aligned}
\{Q^{(0)}_{ij}, Q^{(0)}_{kl}\} &= \left(\delta\circ Q^{(0)}\right)_{ij,kl} \\
\{Q^{(1)}_{ij}, Q^{(0)}_{kl}\} &= \left(\delta\circ Q^{(1)}\right)_{ij,kl} \\
\{Q^{(1)}_{ij}, Q^{(1)}_{kl}\} &= \left(\delta\circ Q^{(2)}\right)_{ij,kl} - \left(Q^{(0)}Q^{(0)}\circ Q^{(0)}\right)_{ij,kl} \\
\{Q^{(2)}_{ij}, Q^{(0)}_{kl}\} &= \left(\delta\circ Q^{(2)}\right)_{ij,kl} \\
\{Q^{(2)}_{ij}, Q^{(1)}_{kl}\} &= \left(\delta\circ Q^{(3)}\right)_{ij,kl} - \left(Q^{(0)}Q^{(0)}\circ Q^{(1)}\right)_{ij,kl} - \left(Q^{(1)}Q^{(0)}\circ Q^{(0)}\right)_{ij,kl} \\
\{Q^{(3)}_{ij}, Q^{(0)}_{kl}\} &= \left(\delta\circ Q^{(3)}\right)_{ij,kl} \\
\{Q^{(3)}_{ij}, Q^{(1)}_{kl}\} &= \left(\delta\circ Q^{(4)}\right)_{ij,kl} - \left(Q^{(0)}Q^{(0)}\circ Q^{(2)}\right)_{ij,kl} - \left(Q^{(1)}Q^{(0)}\circ Q^{(1)}\right)_{ij,kl} \\
&\quad - \left(Q^{(2)}Q^{(0)}\circ Q^{(0)}\right)_{ij,kl} \\
\{Q^{(2)}_{ij}, Q^{(2)}_{kl}\} &= \left(\delta\circ Q^{(4)}\right)_{ij,kl} - \left(Q^{(0)}Q^{(0)}\circ Q^{(2)}\right)_{ij,kl} - \left(Q^{(1)}Q^{(0)}\circ Q^{(1)}\right)_{ij,kl} \\
&\quad - \left(Q^{(0)}Q^{(1)}\circ Q^{(1)}\right)_{ij,kl} - \left(Q^{(1)}Q^{(1)}\circ Q^{(0)}\right)_{ij,kl} \\
\{Q^{(3)}_{ij}, Q^{(2)}_{kl}\} &= \left(\delta\circ Q^{(5)}\right)_{ij,kl} - \left(Q^{(0)}Q^{(0)}\circ Q^{(3)}\right)_{ij,kl} - \left(Q^{(1)}Q^{(0)}\circ Q^{(2)}\right)_{ij,kl} \\
&\quad - \left(Q^{(2)}Q^{(0)}\circ Q^{(1)}\right)_{ij,kl} - \left(Q^{(0)}Q^{(1)}\circ Q^{(2)}\right)_{ij,kl} \\
&\quad - \left(Q^{(1)}Q^{(1)}\circ Q^{(1)}\right)_{ij,kl} - \left(Q^{(2)}Q^{(1)}\circ Q^{(0)}\right)_{ij,kl} \\
\{Q^{(4)}_{ij}, Q^{(1)}_{kl}\} &= \left(\delta\circ Q^{(5)}\right)_{ij,kl} - \left(Q^{(0)}Q^{(0)}\circ Q^{(3)}\right)_{ij,kl} - \left(Q^{(1)}Q^{(0)}\circ Q^{(2)}\right)_{ij,kl} \\
&\quad - \left(Q^{(2)}Q^{(0)}\circ Q^{(1)}\right)_{ij,kl} - \left(Q^{(3)}Q^{(0)}\circ Q^{(0)}\right)_{ij,kl} \quad .
\end{aligned}
\tag{29}
$$

Indeed we have used the algebra above to *define* the improved charges: we verify that the bracket $\{Q^{(n)}, Q^{(1)}\}$ always produces a term of the form $(\delta\circ A)$ for some A, which we call linear piece; and other essentially different terms as $(B\circ C)$, with B and C different from the identity matrix (δ), coming from surface contributions, which we refer to as the non-linear piece. Therefore we can take A as a definition of the $Q^{(n+1)}$ charge,

$$
(\delta\circ Q^{(n+1)}) \equiv \{Q^{(n)}, Q^{(1)}\} - (\text{n.l.t.}) \quad ,
$$

where n.l.t. means "non-linear terms". While the standard charges are defined through an integro-differential algorithm, the improved ones are generated by an algebraic procedure (where $Q^{(1)}$ plays the role of a "step" generator).

The fact that all brackets $\{Q^{(n-i)}, Q^{(i)}\}$, $i = 0, \cdots n$, produce the same linear term $(\delta\circ Q^{(n)})$ means that the linear part of the algebra is of the Kac-Moody type.

These results and observations motivate us to write down the Ansatz

$$\{Q_{ij}^{(m)}, Q_{kl}^{(n)}\} = \left(\delta \circ Q^{(n+m)}\right)_{ij,kl} - \sum_{p=0}^{m-1}\sum_{q=0}^{n-1}\left(Q^{(p)}Q^{(q)} \circ Q^{(m+n-p-q-2)}\right)_{ij,kl} \tag{30}$$

which is the first main result of this paper. In order to prove it, we consider a set of simplified charges which obey an isomorphic algebra, and we prove the result (30) for this new set upon deriving a chain algebra structure in a sense to be defined in Sect. 4. We prove the Jacobi identity for the chain algebra and finally project it back into the non-local charge algebra.

Let us call n the order of the charge $Q^{(n)}$ and $n+m$ the order of the bracket $\{Q^{(n)}, Q^{(m)}\}$. Having proved up to some order $N > 1$ that the algebra is composed of a linear part which is the Kac-Moody algebra plus some non-linear piece, the question is whether the charges can always be defined in such a way that this structure still holds for some linear combination of the previous charges.

In order to solve this problem, we first argue that if one solves the linear part of the algebra in such a way that only the highest order (or genus) term survives (i.e., one has a single term $f^{abc}Q_c^{(m+n)}$), the same will be true concerning the non-linear part. Indeed, relying upon the chain algebra (which will be introduced and discussed in Sect. 4) we notice that, in general, a non-local charge can be defined in terms of a group theoretical factor times some integrals of *chains*, constructed with both components of the current j_μ, which we can denote as in Fig. 1 where the full dots denote the insertion of the component j_1 and the empty ones j_0. The "longest" chain of a charge constitutes its highest order term. The Dirac brackets of two chains might generate lower order terms in the algebra of charges; however, the presence of a lower order term in the linear part of the algebra implies a lower order term in the non-linear part as well, with the same coefficient.

Therefore we argue with the linear algebra only, and this must be enough. We further suppose that when $n+m < N$ for some $N > 1$ the linear part is of the Kac-Moody type (as we verified in (29) for $N = 6$), namely

$$\left\{Q_a^{(n)}, Q_b^{(m)}\right\} = -f^{abc}Q_c^{(n+m)} + \text{(n.l.t.)} \quad . \tag{31}$$

Let us suppose that for $n' + m' \geq N$ the algebra is non-saturated, i.e. it contains lower order terms,

$$\left\{Q_a^{(n')}, Q_b^{(m')}\right\} = -f^{abc}\left[Q_c^{(n'+m')} + a_{n',m'}Q_c^{(n'+m'-1)} + b_{n',m'}Q_c^{(n'+m'-2)}\cdots\right] + \text{(n.l.t.)} \tag{32}$$

We consider the Jacobi identity

$$\{\{Q_a^{(n)}, Q_b^{(m)}\}, Q_c^{(p)}\} + \{\{Q_c^{(p)}, Q_a^{(n)}\}, Q_b^{(m)}\} + \{\{Q_b^{(m)}, Q_c^{(p)}\}, Q_a^{(n)}\} = 0 \quad . \tag{33}$$

Using (32) and (31) we have

$$\begin{aligned} f^{abd}\{Q_d^{(n+m)}, Q_c^{(p)}\} &+ f^{cad}\{Q_d^{(n+p)} + a_{n,p}Q_d^{(n+p-1)} + b_{n,p}Q_d^{(n+p-2)} + \cdots, Q_b^{(m)}\} \\ &+ f^{bcd}\{Q_d^{(m+p)} + a_{m,p}Q_d^{(m+p-1)} + b_{m,p}Q_d^{(m+p-2)} + \cdots, Q_a^{(n)}\} = 0 \end{aligned} \tag{34}$$

which implies, upon use of the relation

$$f^{abd}f^{dce} + f^{cad}f^{dbe} + f^{bcd}f^{dae} = 0 \quad , \tag{35}$$

the result

$$a_{n+m,p} = a_{n+p,m} + a_{n,p} = a_{m+p,n} + a_{m,p} \quad . \tag{36}$$

If $p < n, m$, we have $a_{n,p} = a_{m,p} = 0$ by the induction hypothesis, therefore

$$a_{n+m,p} = a_{n+p,m} = a_{m+p,n} \quad . \tag{37}$$

Since the l.h.s. only depends on the contribution $n+m$, we conclude that $a_{n,m}$ only depends on $n+m$. It is a simple exercise to show that the same is true for the coefficients b, c etc, therefore the linear part of $\{Q_a^n, Q_b^m\}$ only depends on $n+m$, and we can redefine the r.h.s.

$$f^{abc}\left[Q_c^{n+m} + a(n+m)Q_c^{n+m-1} + \cdots\right] + \text{(n.l.t.)} = f^{abc}\tilde{Q}_c^{n+m} + \text{(n.l.t.)} \tag{38}$$

in such a way that it is of the Kac Moody type after the above redefinition. Therefore there must exist a basis of charges satisfying the algebra (30).

4. Saturated Charges

Consider the improved basis of charges: from the examples listed in (28) we see that each one of them has a higher order piece, containing the maximum number of current components (the component j_0) in the integrand, depicted below,

$$\begin{aligned} Q^{(0)} &= \int dx\, j_0(x) \\ Q^{(1)} &= \cdots + 2\int dx\, j_0\bar{\partial}^{1} j_0 = \cdots + \int dxdy\, j_0(x)\epsilon(x-y)j_0(y) \\ Q^{(2)} &= \cdots + 4\int dx j_0\bar{\partial}^{1}(j_0\bar{\partial}^{1} j_0) \\ &= \cdots + \int dxdydz\, j_0(x)\epsilon(x-y)j_0(y)\epsilon(y-z)j_0(z) \\ Q^{(3)} &= \cdots + 8\int dx\, j_0\bar{\partial}^{1}(j_0\bar{\partial}^{1}(j_0\bar{\partial}^{1} j_0)) \\ &= \cdots + \int dxdydzdw\, j_0(x)\epsilon(x-y)j_0(y)\epsilon(y-z)j_0(z)\epsilon(z-w)j_0(w) \\ \cdots & \qquad\qquad \cdots \\ Q^{(n)} &= \cdots + \int \prod_{i=0}^{n} dx_i\, j_0(x_0)\epsilon(x_0-x_1)j_0(x_1)\cdots\epsilon(x_{n-1}-x_n)j_0(x_n)\,. \end{aligned} \tag{39}$$

Inspired by the saturated character of the algebra (29) (i.e., the presence of highest order erms only) and the expressions above we propose the definition of the *saturated* charges

$$\overline{Q}^{(n)} \equiv \int \prod_{i=0}^{n} dx_i \, \mathcal{J}(x_0,\cdots,x_n) \tag{40}$$

here the non-local densities

$$\mathcal{J}(x_0,\cdots,x_n) \equiv j_0(x_0)\epsilon(x_0-x_1)j_0(x_1)\cdots\epsilon(x_{n-1}-x_n)j_0(x_n) \tag{41}$$

n be seen as *chains* of current components $j_0(x_i)$ connected by non-local ϵ functions. We nphasize that the saturated charges $\overline{Q}^{(n)}$ are not conserved quantities. Nevertheless we rove that they realize the algebra (30) and use this fact to verify that the Ansatz satisfies e Jacobi identity. The fact that Jacobi identity holds for the general case is discussed at e end of the section.

In a given basis $\{t^a\}$ for the $O(N)$ algebra, the components of a saturated charge can built up from the integral of a linear chain of components j_0^a times a group theoretical race) factor:

$$\overline{Q}_a^{(n)} = -\frac{1}{2}\mathrm{tr}(t^a t^{a_0}\cdots t^{a_n}) \int \prod_{i=0}^{n} dx_i \, \mathcal{J}^{a_0\cdots a_n}(x_0,\cdots,x_n) \tag{42}$$

here

$$\mathcal{J}^{a_0\cdots a_n}(x_0,\cdots,x_n) \equiv j_0^{a_0}(x_0)\epsilon(x_0-x_1)j_0^{a_1}(x_1)\cdots\epsilon(x_{n-1}-x_n)j_0^{a_n}(x_n) \quad . \tag{43}$$

The algebra of saturated charges follows from the algebra of chains. In order to underand this relation we first consider the case of the simplest brackets and later generalize e results. Consider the first non-trivial saturated charge matrix

$$\overline{Q}_{ij}^{(1)} = \int dxdy \, (j_0)_{ik}(x)\,\epsilon(x-y)\,(j_0)_{kj}(y) \quad . \tag{44}$$

efining the components

$$\overline{Q}_a^{(1)} = -\frac{1}{2}\,\mathrm{tr}\left(t^a \overline{Q}^{(1)}\right) \quad , \tag{45}$$

have

$$\begin{aligned} \overline{Q}_a^{(1)} &= -\frac{1}{2}\,\mathrm{tr}(t^a t^b t^c) \int dxdy \, \mathcal{J}^{bc}(x,y) \\ \mathcal{J}^{ab}(x,y) &= j_0^a(x)\epsilon(x-y)j_0^b(y) \quad . \end{aligned} \tag{46}$$

The algebra obeyed by the chains in (46) is easily derived upon use of (8),

$$\begin{aligned} \{\mathcal{J}^{ab}(x,y),\mathcal{J}^{cd}(z,w)\} =& \{j_0^a(x)\epsilon(x-y)j_0^b(y)\,,\, j_0^c(z)\epsilon(z-w)j_0^d(w)\} \\ =& -f^{bce}\mathcal{J}^{aed}(x,y,w)\delta(y-z) + f^{bde}\mathcal{J}^{aec}(x,y,z)\delta(y-w) \\ &+ f^{ace}\mathcal{J}^{bed}(y,x,w)\delta(x-z) - f^{ade}\mathcal{J}^{bec}(y,x,z)\delta(x-w) \end{aligned} \tag{47}$$

where $\mathcal{J}^{abc}(x,y,z) = j_0^a(x)\epsilon(x-y)j_0^b(y)\epsilon(y-z)j_0^c(z)$ is a 3-current chain. Therefore we obtain for the algebra involving the charges (45) the expression

$$\{\overline{Q}_a^{(1)},\overline{Q}_b^{(1)}\} = -\,\mathrm{tr}(t^a t^c t^d) f^{deg}\,\mathrm{tr}(t^b t^e t^f) \int dxdydz \, \mathcal{J}^{cgf}(x,y,z) \quad . \tag{48}$$

Taking a basis $\{t^a\}$ for the $O(N)$ algebra one verifies that the traces in (48) merge into four traces,

$$\begin{aligned} &-\mathrm{tr}(t^a t^c t^d) f^{deg}\,\mathrm{tr}(t^b t^e t^f) = \\ &= \mathrm{tr}(t^g t^a t^c t^f t^b) - \mathrm{tr}(t^g t^c t^a t^f t^b) - \mathrm{tr}(t^a t^c t^g t^f t^b) + \mathrm{tr}(t^c t^a t^g t^f t^b) \quad . \end{aligned} \tag{49}$$

The third trace leads to

$$\mathrm{tr}(t^a t^b t^f t^g t^c) \int dxdydz \, \mathcal{J}^{cgf}(x,y,z) = \mathrm{tr}\left(t^a t^b \overline{Q}^{(2)}\right) \quad , \tag{50}$$

which is the Kac-Moody part of the algebra (30). As for the other three terms, we use the expression

$$\begin{aligned} \mathcal{J}^{abc}(x,y,z) &= j_0^a(x)\epsilon(x-y)j_0^b(y)\epsilon(y-z)j_0^c(z) \\ &= -\frac{1}{2}\int dw[\epsilon(y-x)j_0^a(x)]\frac{\partial}{\partial y}[\epsilon(y-w)j_0^b(w)][\epsilon(y-z)j_0^c(z)] \end{aligned} \tag{51}$$

and verify that upon contraction with the remaining three terms in (49) we get the integral of a total derivative which, due to the antisymmetric boundary condition (22), gives the cubic term

$$-\,\mathrm{tr}\left(t^a \overline{Q}^{(0)}\overline{Q}^{(0)} t^b \overline{Q}^{(0)}\right) \quad , \tag{52}$$

and we arrive at

$$\{\overline{Q}_a^{(1)},\overline{Q}_b^{(1)}\} = \mathrm{tr}\left(t^a t^b \overline{Q}^{(2)}\right) - \mathrm{tr}\left(t^a \overline{Q}^{(0)}\overline{Q}^{(0)} t^b \overline{Q}^{(0)}\right) \quad . \tag{53}$$

Further algebra, and use of the boundary condition to show the vanishing of terms of the type $\int dx\partial\,\mathrm{tr}(At^a A^t t^b)$, lead to the next (Dirac) brackets

$$\{\overline{Q}_a^{(1)},\overline{Q}_b^{(2)}\} = \mathrm{tr}\left(t^a t^b \overline{Q}^{(3)}\right) - \mathrm{tr}\left(t^a \overline{Q}^{(0)}\overline{Q}^{(0)} t^b \overline{Q}^{(1)}\right) - \mathrm{tr}\left(t^a \overline{Q}^{(0)}\overline{Q}^{(1)} t^b \overline{Q}^{(0)}\right) \quad . \tag{54}$$

We are now in position to generalize the procedure for arbitrary chains and obtain the full algebra of the saturated charges.

The 2-chain brackets can be expanded as a sum of crossing chains:

$$
\begin{aligned}
&\{\mathcal{J}^{a_0\cdots a_m}(x_0,\cdots,x_m),\mathcal{J}^{b_0\cdots b_n}(y_0,\cdots,y_n)\} = \\
&= -\int \prod_{k=0}^{m} dx_k \prod_{l=0}^{n} dy_l \times \sum_{i=0}^{m}\sum_{j=0}^{n} \delta(x_i - y_j)\ f^{a_i b_j c} \\
&\times j_0^{a_0}(x_0)\epsilon(x_0-x_1)\cdots j_0^{a_{i-1}}(x_{i-1})\epsilon(x_{i-1}-x_i) \quad \times \quad \epsilon(x_i-x_{i+1})j_0^{a_{i+1}}(x_{i+1})\cdots\epsilon(x_{m-1}-x_m)j_0^{a_m}(x_m) \\
&\times j_0^{c}(x_i) \\
&\times j_0^{b_0}(y_0)\epsilon(y_0-y_1)\cdots j_0^{b_{j-1}}(y_{j-1})\epsilon(y_{j-1}-x_i) \quad \times \quad \epsilon(x_i-y_{j+1})j_0^{b_{j+1}}(y_{j+1})\cdots\epsilon(y_{n-1}-y_n)j_0^{b_n}(y_n)
\end{aligned}
\tag{55}
$$

A typical crossing of chains is represented by Fig. 2, followed by the respective $f^{a_i b_j c}$ structure constant factor.

The algebra of two linear chains has produced another kind of non-local density: a 1-crossing chain. Computing the algebra of this extended family of chains, one generates multiple crossings. We therefore consider the infinite space of n-crossing chains, whose Dirac brackets define an infinite dimensional linear algebra. This is the underlying linear structure behind the non-local charge algebra, which we compute now.

In order to project the brackets above into $\{\overline{Q}_a^{(m)},\overline{Q}_b^{(n)}\}$ we must multiply both sides of Eq. (55) by the corresponding traces of t-matrices, as indicated by (42). Those traces contracted by a factor $f^{a_i b_j c}$ merge as follows

$$
\begin{aligned}
&-\frac{1}{4}\,\mathrm{tr}(t^{a_{i+1}}\cdots t^{a_m}t^{a}t^{a_0}\cdots t^{a_i})f^{a_i b_j c}\,\mathrm{tr}(t^{b_j}\cdots t^{b_n}t^{b}t^{b_0}\cdots t^{b_{j-1}}) \\
&= \frac{1}{4}\,\mathrm{tr}\Big(t^c \times (t^{a_{i+1}}\cdots t^{a_m}t^{a}t^{a_0}\cdots t^{a_{i-1}} + (-)^m t^{a_{i-1}}\cdots t^{a_0}t^{a}t^{a_m}\cdots t^{a_{i+1}}) \\
&\qquad \times (t^{b_{j+1}}\cdots t^{b_n}t^{b}t^{b_0}\cdots t^{b_{j-1}} + (-)^n t^{b_{j-1}}\cdots t^{b_0}t^{b}t^{b_n}\cdots t^{b_{j+1}})\Big) \quad ,
\end{aligned}
\tag{56}
$$

which generalizes Eq. (49). Once again, each contraction of chains leads to four contributions, as in Fig. 3. It is easy to verify that each contribution shows up four times, as exemplified by Fig. 4: the factors $(-)^n$ and $(-)^m$ compensate the implied inversions of arguments of the ϵ-functions. Thus we can concentrate on the 3rd representative of Fig. 3 and drop the 1/4 factor before the trace: that figure represents a typical partition of the chains, and the sum of all partitions in the bracket $\{\overline{Q}_a^{(m)},\overline{Q}_b^{(n)}\}$ reads

$$
\begin{aligned}
\{\overline{Q}_a^{(m)},\overline{Q}_b^{(n)}\} &= \int \prod_{k=0}^{m} dx_k \prod_{l=0}^{n} dy_l \sum_{i=0}^{m}\sum_{j=0}^{n} \delta(x_i - y_i) \times (-)^m \times (-)^{m-i-1} \times (-)^{i-1} \\
&\times \mathrm{tr}[t^a \mathcal{J}(x_m\cdots x_{i+1})\epsilon(x_i - x_{i+1})\epsilon(x_i - y_{j+1})\mathcal{J}(y_{j+1}\cdots y_n)t^b\mathcal{J}(y_0\cdots y_{j-1}x_i x_{i-1}\cdots x_0)]
\end{aligned}
$$

$$
\begin{aligned}
&= \int \prod_{k=0}^{m} dx_k \prod_{l=0}^{n} dy_l \sum_{i=0}^{m}\sum_{j=0}^{n} \delta(x_i - y_j) \\
&\times \mathrm{tr}\Big[t^a[\epsilon(x_i-x_{i+1})\mathcal{J}(x_m\cdots x_{i+1})][\epsilon(x_i-y_{j+1})\mathcal{J}(y_{j+1}\cdots y_n)]t^b[\mathcal{J}(y_0\cdots y_{j-1}x_i x_{i-1}\cdots x_0)]\Big] \\
&= \int \mathrm{tr}[t^a t^b \mathcal{J}(y_0\cdots y_{n-1}x_m x_{m-1}\cdots x_0)] \\
&- \sum_{i=0}^{m-1}\int dx\,\mathrm{tr}\left[t^a\Big[\int \epsilon(x - x_{i+1})\mathcal{J}(x_m\cdots x_{i+1})\Big]t^b\Big[\int \mathcal{J}(y_0\cdots y_{n-1}x x_{i-1}\cdots x_0)\Big]\right] \\
&+ \sum_{j=0}^{n-1}\int dx\,\mathrm{tr}\left[t^a\Big[\int \epsilon(x - y_{j+1})\mathcal{J}(y_{j+1}\cdots y_n)\Big]t^b\Big[\int \mathcal{J}(y_0\cdots y_{j-1}x x_{m-1}\cdots x_0)\Big]\right] \\
&- \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\int dx\,\mathrm{tr}\left[t^a\Big[\int \epsilon(x - x_{i+1})\mathcal{J}(x_m\cdots x_{i+1})\Big]\Big[\int \epsilon(x - y_{j+1})\mathcal{J}(y_{j+1}\cdots y_n)\Big]t^b\right. \\
&\qquad \left.\times\Big[\int \mathcal{J}(y_0\cdots y_{j-1}x x_{i-1}\cdots x_0)\Big]\right] \quad .
\end{aligned}
\tag{57}
$$

Above we have omitted integration measures of the labelled variables, which we assume to be resumed under the integral symbols.

Examining the four sums above, we recognize the Kac-Moody piece in the first term

$$
\int \mathrm{tr}\,(t^a t^b \mathcal{J}(y_0\cdots y_{n-1}x_m x_{m-1}\cdots x_0)) = \mathrm{tr}\,(t^a t^b Q^{(n+m)}) \quad . \tag{58}
$$

The cubic piece of the algebra can be obtained from the last sum in (57), which we rewrite as follows

$$
\begin{aligned}
&-\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\int dx \frac{\partial}{\partial x}\frac{1}{2}\,\mathrm{tr}\Big[t^a\Big[\int\epsilon(x-x_{i+1})\mathcal{J}(x_m\cdots x_{i+1})\Big]\Big[\int\epsilon(x-y_{j+1})\mathcal{J}(y_{j+1}\cdots y_n)\Big]t^b \\
&\qquad \times\Big[\int \epsilon(x - x_i)\mathcal{J}(y_0\cdots y_{j-1}x_i x_{i-1}\cdots x_0)\Big]\Big] \\
&+\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\int dx_{i+1}\,\mathrm{tr}\Big[t^a\Big[\int\mathcal{J}(x_m\cdots x_{i+1}y_{j+1}\cdots y_n)\Big]t^b\Big[\int\epsilon(x_{i+1}-x_i)\mathcal{J}(y_0\cdots y_{j-1}x_i x_{i-1}\cdots x_0)\Big]\Big] \\
&-\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\int dy_{j+1}\,\mathrm{tr}\Big[t^a\Big[\int\mathcal{J}(x_m\cdots x_{i+1}y_{j+1}\cdots y_n)\Big]t^b\Big[\int\epsilon(y_{j+1}-x_i)\mathcal{J}(y_0\cdots y_{j-1}x_i x_{i-1}\cdots x_0)\Big]\Big]
\end{aligned}
$$

$$
= - \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \mathrm{tr}(t^a \overline{Q}^{(m-1-i)} \overline{Q}^{(n-1-j)} t^b \overline{Q}^{(i+j)}) + \cdots
$$

$$
= - \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \mathrm{tr}(t^a \overline{Q}^{(i)} \overline{Q}^{(j)} t^b \overline{Q}^{(m+n-i-j-2)}) + \cdots \quad . \tag{59}
$$

The terms represented by right dots above, together with the 2nd and 3rd sums left er in (57), may be (formally) summarized as

$$
\sum_{i=0}^{m-1} \sum_{j=0}^{n} \int dx_{i+1} \mathrm{tr}\Big[t^a \Big[\int \mathcal{J}(x_m \cdots x_{i+1} y_{j+1} \cdots y_n)\Big] t^b \Big[\int \epsilon(x_{i+1} - x_i) \mathcal{J}(y_0 \cdots y_{j-1} x_i x_{i-1} \cdots x_0)\Big]\Big]
$$

$$
\sum_{i=0}^{m} \sum_{j=0}^{n-1} \int dy_{j+1} \mathrm{tr}\Big[t^a \Big[\int \mathcal{J}(x_m \cdots x_{i+1} y_{j+1} \cdots y_n)\Big] t^b \Big[\int \epsilon(y_{j+1} - x_i) \mathcal{J}(y_0 \cdots y_{j-1} x_i x_{i-1} \cdots x_0)\Big]\Big] \tag{60}
$$

exemplified by Fig. 5, the above sum generates surface terms of the form

$$
\int dx \frac{\partial}{\partial x} \mathrm{tr}(t^a A(x) t^b B(x)) \tag{61}
$$

ich vanishes due to our antiperiodic boundary conditions. When $n + m$ is odd, we also d single contributions like

$$
\int dx \, \mathrm{tr}(t^a A(x) t^b \frac{\partial}{\partial x} A^t(x)) = \frac{1}{2} \int dx \frac{\partial}{\partial x} \mathrm{tr}(t^a A(x) t^b A^t(x)) \tag{62}
$$

ich is zero for the same reasons. Therefore we have proved that the traces of t-matrices ject the chain algebra into the non-local charge algebra

$$
\{\overline{Q}_a^{(m)}, \overline{Q}_b^{(n)}\} = \mathrm{tr}\left(t^a t^b \overline{Q}^{(m+n)}\right) - \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \mathrm{tr}\left(t^a \overline{Q}^{(i)} \overline{Q}^{(j)} t^b \overline{Q}^{(m+n-i-j-2)}\right) \tag{63}
$$

By means of the formula $(A.8)$ one recognizes that the above algebra is isomorphic he Ansatz (30). Although this algebra has been derived for the $O(N)$ model, one can rite the traces appearing on the r.h.s. of (63) in terms of the structure constants of the up and therefore generalize that algebra for other groups.

Concerning the Jacobi identity, we begin by stressing that the above realization of Ansatz was built up from the elementary current component j_0: as the Dirac brackets $x), j_0^b(y)\}$, given in Eq. (8), obey the Jacobi identity by hypothesis, and since the ns are defined in Eq. (41) as products of j_0-components, it follows that the algebra hains also satisfies the Jacobi identity. On the other hand, the saturated charges are structed by simple integrations and linear combinations of chains, therefore implying the algebra (63) obeys the Jabobi identity too.

If we had considered the algebra of all chains, including those having the component j_1, the corresponding integrations and trace-projections would lead us to the algebra of improved charges. The Jacobi identity of the algebra thus obtained would follow from the algebraic properties of (8) too. The role of the intertwiner field is marginal due to its character as a projector.

From the relation between chains and saturated charges, we also understand how the linear and cubic parts of the algebra (30) are constrained: both are constructed from the same chains, with the same number of current components (implying the highest order terms) and same multiplicative coefficients, as mentioned in Sec. 3.

5. Algebra of Non-Local Charges in WZNW Model

We first reanalyze the current algebra for the principal chiral model with a Wess-Zumino term. This model [19] contains a free coupling constant λ and, for special values of λ, is equivalent to the conformally invariant WZNW model while being the ordinary chiral model for $\lambda \to 0$. Therefore, the current algebra derived below is a generalization of the current algebras for these two special cases. For the WZNW model, the current algebra is known to consist of two commuting Kac-Moody algebras, while for the ordinary chiral model, it has been presented previously.

We begin by fixing our conventions. The target space for the chiral models to be considered here will be a simple Lie group G (which is usually, though not necessarily, assumed to be compact) with Lie algebra **g**, and we use the trace tr in some irreducible representation to define the invariant scalar product $(.,.)$ on **g**, normalized so that the long roots have length $\sqrt{2}$, as well as the invariant closed three-form ω on **g** giving rise to the Wess-Zumino term. Explicitly, for $X, Y, Z \in \mathbf{g}$, we have,

$$
(X, Y) \;=\; -\,\mathrm{tr}\,(XY) \quad , \tag{64}
$$

while

$$
\omega(X, Y, Z) \;=\; \frac{1}{4\pi}\,\mathrm{tr}\,(X[Y, Z]) \quad . \tag{65}
$$

Obviously, $(.,.)$ and ω extend to a biinvariant metric $(.,.)$ on G and to a biinvariant three-form ω on G, respectively: the latter can alternatively be represented in terms of the left invariant Maurer-Cartan form $g^{-1}dg$ or right invariant Maurer-Cartan form $dg\,g^{-1}$ on G, as follows:

$$
\omega \;=\; \frac{1}{12\pi}\,\mathrm{tr}\,(g^{-1}dg)^3 \;=\; \frac{1}{12\pi}\,\mathrm{tr}\,(dg\,g^{-1})^3 \quad . \tag{66}
$$

(Due to the Maurer-Cartan structure equation, this representation implies that ω is indeed a closed three-form on G, and the normalization in Eqs. (65) and (66) is chosen so that $\omega/2\pi$ generates the third de Rham cohomology group $H^3(G, \mathbb{Z})$ of G over the integers, at least when G is simply connected; cf. Ref. [20]. The minus sign in Eq. (64) is introduced to ensure positive definiteness when G is compact.)

In part of what follows, we work in terms of (arbitrary) local coordinates u^i on G, epresenting the metric $(.,.)$ by its components g_{ij} and the three-form ω by its components $_{ijk}$. Then the total action of the so called Wess-Zumino-Novikov-Witten (WZNW) theory the sum

$$S \;=\; S_{CH} + nS_{WZ} \quad , \tag{67}$$

here the action for the ordinary chiral model, S_{CH} is given by (1), and the Wess-Zumino rm is

$$S_{WZ} \;=\; \frac{1}{6}\int_B d^3x\;\epsilon^{\kappa\lambda\mu}\,\omega_{ijk}(\tilde{\varphi})\,\partial_\kappa\tilde{\varphi}^i\,\partial_\lambda\tilde{\varphi}^j\,\partial_\mu\tilde{\varphi}^k \;=\; \int_B \tilde{\varphi}^*\omega \quad . \tag{68}$$

ere, φ and $\tilde{\varphi}$ are the basic field and the extended field of the theory, respectively, i.e., is a (smooth) map from a fixed two-dimensional Lorentz manifold Σ to G and $\tilde{\varphi}$ is a mooth) map from an appropriate three-dimensional manifold B to G, chosen so that Σ the boundary of B and φ is the restriction of $\tilde{\varphi}$ to that boundary: $\Sigma = \partial B$, $\varphi = \tilde{\varphi}|_\Sigma$. he conformally invariant WZNW model is obtained at $\lambda = \sqrt{4\pi/|n|}$, while the ordinary hiral model can be recovered in the limit $\lambda \to 0$. Note that if ω were exact, we could write $= d\alpha$ to obtain

$$S_{WZ} \;=\; \frac{1}{2}\int_\Sigma d^2x\;\epsilon^{\mu\nu}\,\alpha_{ij}(\varphi)\,\partial_\mu\varphi^i\,\partial_\nu\varphi^j \;=\; \int_\Sigma \varphi^*\alpha \quad . \tag{69}$$

ut of course this is not possible globally, i.e., the α_{ij} appearing in this formula are neither nique nor can they be chosen so as to become the components of a globally well-defined wo-form on G with respect to the u^i. Still, calculations involving quantities that arise om local variations of the action can be performed as if this were the case, and may lead results that do not depend on any artificial choices. For example, recall that in the rdinary chiral model, the canonically conjugate momenta π_i derived from the action S_{CH} re simply given by Eq. (3) and satisfy the canonical commutation relations (2). Similarly, the chiral model with a Wess-Zumino term, written in the form (69), the canonically onjugate momenta $\hat{\pi}_i$ derived from the action S are given by

$$\hat{\pi}_i \;=\; \pi_i + n\,\alpha_{ij}(\varphi)\,\varphi'^{\,j} \tag{70}$$

nd satisfy the canonical commutation relations (2) with π substituted by $\hat{\pi}$. Note, however, that in contrast to the π_i, the $\hat{\pi}_i$ do not behave naturally under local coordinate ransformations on G, so that the canonical commutation relations (2) between the φ^i and he $\hat{\pi}_j$ look non-covariant. This suggests to consider instead the commutation relations etween the fields φ^i and the π_j, which are covariant, but exhibit non-vanishing Poisson rackets between the momenta π_i. Indeed, it follows from (2) that

$$\begin{aligned}\{\pi_i(x),\pi_j(y)\} &= +n(\partial_i\alpha_{jk}+\partial_j\alpha_{ki})(\varphi(x))\varphi'^k(x)\,\delta(x-y)\\ &\quad + n\partial_k\alpha_{ij}(\varphi(x))\varphi'^k(x)\delta(x-y) \quad ,\end{aligned} \tag{71}$$

so in the presence of the Wess-Zumino term, the commutation relations between φ^i and π_j read

$$\begin{aligned}&\{\varphi^i(x),\varphi^j(y)\} = 0 \quad , \quad \{\varphi^i(x),\pi_j(y)\} = \delta^i_j\,\delta(x-y) \quad ,\\ &\{\pi_i(x),\pi_j(y)\} = n\omega_{ijk}(\varphi(x))\varphi'^k(x)\delta(x-y) \quad .\end{aligned} \tag{72}$$

They are obviously covariant (all expressions behave naturally under local coordinate transformations on G), since ω is a globally well-defined three-form on G.

To derive the desired current algebra, we recall next that the model under consideration has an obvious global invariance under the product group $G_L \times G_R$, which acts on G according to

$$g \longrightarrow (g_L, g_R)\cdot g = g_L\, g\, g_R^{-1} \quad . \tag{73}$$

This action of the Lie group $G_L \times G_R$ induces a representation of the corresponding Lie algebra $\mathbf{g}_L \oplus \mathbf{g}_R$ by vector fields, associating to each generator $X = (X_L, X_R)$ in $\mathbf{g}_L \oplus \mathbf{g}_R$ the fundamental vector field X_G on G given by

$$X_G(g) \;=\; X_L g - gX_R \qquad \text{for } g \in G \quad . \tag{74}$$

As usual, invariance of the action leads to conserved Noether currents taking values in $\mathbf{g}_L \oplus \mathbf{g}_R$ and denoted by j_μ for the ordinary chiral model and by $\hat{j}_\mu$ for the chiral model with a Wess-Zumino term. Explicitly, we have, for $X = (X_L, X_R)$ in $\mathbf{g}_L \oplus \mathbf{g}_R$,

$$(j_\mu, X) \;=\; -\frac{1}{\lambda^2}\,g_{ij}(\varphi)\,\partial_\mu\varphi^i\,X^j_G(\varphi) \quad , \tag{75}$$

while

$$(\hat{j}_\mu, X) \;=\; -\left(\frac{1}{\lambda^2}\,g_{ij}(\varphi)\,\partial_\mu\varphi^i + n\,\alpha_{ij}(\varphi)\,\epsilon_{\mu\nu}\,\partial^\nu\varphi^i\right) X^j_G(\varphi) \quad . \tag{76}$$

In addition, an important role is played by the scalar field j introduced in Ref. [17], defined by

$$(j, X \otimes Y) \;=\; \frac{1}{\lambda^2}\,g_{ij}(\varphi)\,X^i_G(\varphi)\,Y^j_G(\varphi) \quad . \tag{77}$$

The commutation relations of the Noether currents j_μ and $\hat{j}_\mu$ under Poisson brackets can now be computed directly. Note again, however, that in contrast to j_μ, $\hat{j}_\mu$ do not behave naturally under local coordinate transformations on G, so that their Dirac brackets look non-covariant. This suggests to replace them by appropriate covariant currents J_μ which, as it turns out, can be written entirely in terms of the Noether currents j_μ for the ordinary chiral model (the exact definition will be given below): it is the commutation relations of these covariant currents J_μ that form the current algebra we wish to compute (or at least an important part thereof). The most efficient way of arriving at the desired result is therefore to calculate, as an intermediate step, the brackets of the Noether currents j_μ, using the brackets (63); we arrive at the results

$$\begin{aligned}\{j^a_0(x), j^b_0(y)\} &= -f^{abc} j^c_0(x)\delta(x-y)\\ &\quad + n\omega(\varphi(x))\,\big(\varphi'(x), t^a_L\varphi(x) - \varphi(x)t^a_R, t^b_L\varphi(x) - \varphi(x)t^b_R\big)\,\delta(x-y)\end{aligned} \tag{78}$$

while other relations remain unchanged. The additional factors $\frac{1}{\lambda^2}$ have been absorbed into the normalizations of the j_μ and j).

Before proceeding further, we find it convenient to pass to a more standard notation, writing g and $\tilde{g}$, rather than φ and $\tilde{\varphi}$, for the basic field and the extended field of the heory, respectively, and using the explicit definitions (64) of the metric $(.,.)$ on G and (65) of the three-form ω on G. Then

$$S_{CH} \;=\; -\frac{1}{2\lambda^2}\int d^2x\;\eta^{\mu\nu}\;\mathrm{tr}\left(g^{-1}\partial_\mu g\,g^{-1}\partial_\nu g\right) \quad , \tag{79}$$

vhile

$$S_{WZ} \;=\; \frac{1}{4\pi}\int_0^1 dr\int d^2x\;\epsilon^{\mu\nu}\;\mathrm{tr}\left(\tilde{g}^{-1}\partial_r\tilde{g}\,\tilde{g}^{-1}\partial_\mu\tilde{g}\,\tilde{g}^{-1}\partial_\nu\tilde{g}\right) \quad . \tag{80}$$

Here, the extended field $\tilde{g}$ is assumed to be constant outside a tubular neighborhood $\Sigma\times[0,1]$ of the boundary Σ of B, and r is the coordinate normal to the boundary.) Next, ve decompose the currents j_μ and J_μ, both of which take values in $\mathbf{g}_L\oplus\mathbf{g}_R$, into left and ight currents, all of which take values in $\mathbf{g}$: $j_\mu=(j^L_\mu,j^R_\mu)$, $J_\mu=(J^L_\mu,J^R_\mu)$. Explicitly,

$$\begin{aligned} j^L_\mu &= -\frac{1}{\lambda^2}\,\partial_\mu g\,g^{-1} \quad , \\ j^R_\mu &= +\frac{1}{\lambda^2}\,g^{-1}\partial_\mu g \quad , \end{aligned} \tag{81}$$

nd, by definition,

$$\begin{aligned} J^L_\mu &= (\eta_{\mu\nu}+\alpha\epsilon_{\mu\nu})\,j^{L\,\nu} = -\frac{1}{\lambda^2}\,(\eta_{\mu\nu}+\alpha\epsilon_{\mu\nu})\,\partial^\nu g\,g^{-1} \quad , \\ J^R_\mu &= (\eta_{\mu\nu}-\alpha\epsilon_{\mu\nu})\,j^{R\,\nu} = +\frac{1}{\lambda^2}\,(\eta_{\mu\nu}-\alpha\epsilon_{\mu\nu})\,g^{-1}\partial^\nu g \quad , \end{aligned} \tag{82}$$

vhere $\alpha=\frac{n\lambda^2}{4\pi}$. The scalar field j, when viewed as taking values in the space of endomorhisms of $\mathbf{g}_L\oplus\mathbf{g}_R$, is given by the (2×2)-block matrix

$$j \;=\; \frac{1}{\lambda^2}\begin{pmatrix} 1 & -\mathrm{Ad}(g) \\ -\mathrm{Ad}(g)^{-1} & 1 \end{pmatrix} \quad . \tag{83}$$

n other words, for $X=(X_L,X_R)$ in $\mathbf{g}_L\oplus\mathbf{g}_R$,

$$j(X) \;=\; \frac{1}{\lambda^2}\Big(X_L-\mathrm{Ad}(g)X_R\,,\,X_R-\mathrm{Ad}(g)^{-1}X_L\Big) \quad . \tag{84}$$

t can be shown that the covariant currents J_μ defined by Eqs. (82) differ from the Noether urrents $\hat{j}_\mu$ for the chiral model with a Wess-Zumino term by a total curl, and that current onservation (which for both types of currents has the same physical content, because total curl is automatically conserved) is identical with the equations of motion of the heory.

Now in terms of an arbitrary basis $\{t_a\}$ of $\mathbf{g}$, with structure constants f^{abc} defined by $[t^a,t^b]=f^{abc}t^c$, the various currents are represented by their components

$$\begin{aligned} j^{La}_\mu &= (j_\mu,t^{La}) = -\mathrm{tr}(j^L_\mu\,t^a) \quad , \\ j^{Ra}_\mu &= (j_\mu,t^{Ra}) = -\mathrm{tr}(j^R_\mu\,t^a) \quad , \\ J^{La}_\mu &= (J_\mu,t^{La}) = -\mathrm{tr}(J^L_\mu\,t^a) \quad , \\ J^{Ra}_\mu &= (J_\mu,t^{Ra}) = -\mathrm{tr}(J^R_\mu\,t^a) \quad , \end{aligned} \tag{85}$$

and the scalar field j by its components

$$\eta_{ab} \;=\; (j,t^{La}\otimes t^{Lb}) \;=\; (j,t^{Ra}\otimes t^{Rb}) \;=\; -\frac{1}{\lambda^2}\,\mathrm{tr}\,(t^a t^a) \quad , \tag{86}$$

$$j^{ab} = (j,t^{La}\otimes t^{Rb}) = \frac{1}{\lambda^2}\mathrm{tr}\left(g^{-1}t^a g\,t^b\right) \tag{87}$$

where

$$t^{La} \;=\; (t^a,0) \quad , \quad t^{Ra}=(0,t^a) \quad . \tag{88}$$

With this notation, we see that the current Dirac brackets imply the following brackets relations for the components of the currents j^a_μ:

$$\begin{aligned} \{j^{La}_0(x),j^{Lb}_0(y)\} &= -f^{abc}j^{Lc}_0(x)\delta(x-y)+\alpha f^{abc}j^{Lc}_1(x)\delta(x-y) \quad , \\ \{j^{La}_0(x),j^{Lb}_1(y)\} &= -f^{abc}j^{Lc}_1(x)\delta(x-y)+\eta_{ab}\,\delta'(x-y) \quad , \\ \{j^{La}_1(x),j^{Lb}_1(y)\} &= 0 \quad , \\[1em] \{j^{Ra}_0(x),j^{Rb}_0(y)\} &= -f^{abc}j^{Rc}_0(x)\delta(x-y)-\alpha f^{abc}j^{Rc}_1(x)\delta(x-y) \quad , \\ \{j^{Ra}_0(x),j^{Rb}_1(y)\} &= -f^{abc}j^{Rc}_1(x)\delta(x-y)+\eta_{ab}\delta'(x-y) \quad , \\ \{j^{Ra}_1(x),j^{Rb}_1(y)\} &= 0 \quad , \\[1em] \{j^{La}_0(x),j^{Rb}_0(y)\} &= \alpha j'^{ab}(x)\delta(x-y) \quad , \\ \{j^{La}_0(x),j^{Rb}_1(y)\} &= j^{ba}(y)\delta'(x-y) \quad , \\ \{j^{Ra}_0(x),j^{Lb}_1(y)\} &= j^{ab}(y)\delta'(x-y) \quad , \\ \{j^L_{1,a}(x),j^R_{1,b}(y)\} &= 0 \quad . \end{aligned} \tag{89}$$

They must be supplemented by the commutation relations between the components of the currents j_μ and those of the field j.

$$\begin{aligned} \{j^{La}_0(x),j^{bc}(y)\} &= -f^{abd}j^{dc}(x)\delta(x-y) \quad , \\ \{j^{Ra}_0(x),j^{bc}(y)\} &= -f^{acd}j^{bd}(x)\delta(x-y) \quad , \\ \{j^{La}_1(x),j^{bc}(y)\} &= 0 \quad , \\ \{j^{Ra}_1(x),j^{bc}(y)\} &= 0 \quad . \end{aligned} \tag{90}$$

Finally, the components of the field j commute among themselves: $\{j^{ab}(x), j^{cd}(y)\} = 0$.

Using the explicit representation of the theory in terms of group valued fields, it is very simple to check the results using the decomposition of the momentum in terms of a local and a non-local piece as it has been done in Ref. [5].

We are now in position to generalize the previous results for the WZNW model. Classically, the equations of motion are given by the conservation laws

$$\begin{aligned} \partial_\mu \left(j^{R\mu} - \alpha\epsilon^{\mu\nu} j^R_\nu\right) = 0 \quad , \\ \partial_\mu \left(j^{L\mu} + \alpha\epsilon^{\mu\nu} j^L_\nu\right) = 0 \quad . \end{aligned} \tag{91}$$

The currents $j^{R,L}_\mu$ satisfy the zero-curvature conditions

$$\begin{aligned} \partial_\mu j^R_\nu - \partial_\nu j^R_\mu + \lambda^2 [j^R_\mu, j^R_\nu] = 0 \quad , \\ \partial_\mu j^L_\nu - \partial_\nu j^L_\mu + \lambda^2 [j^L_\mu, j^L_\nu] = 0 \quad . \end{aligned} \tag{92}$$

Concerning the covariant currents $J^{R,L}_\mu$ the above equations imply

$$\begin{aligned} \partial^\mu J^{R,L}_\mu = 0 \quad , \\ \partial_\mu J^{R,L}_\nu - \partial_\nu J^{R,L}_\mu + [J^{R,L}_\mu, J^{R,L}_\nu] = 0 \quad . \end{aligned} \tag{93}$$

The last equation states [4] that the combination $j^R_\mu - \alpha\epsilon_{\mu\nu} j^{R\nu}$ has also zero curvature and the construction (19) and (20) follows immediately, replacing $j^R_0 \to J^R_0$ and $j^R_1 \to J^R_1$. Moreover a similar construction holds for J^L_μ, replacing α by $-\alpha$. In particular, the first non-local conserved charges read

$$\begin{aligned} Q^{R(1)} &= \int dy_1 dy_2 \epsilon(y_1 - y_2)(j^R_0 + \alpha j^R_1)(t, y_1)(j^R_0 + \alpha j^R_1)(t, y_2) + 2\lambda^{-2}(1-\alpha^2)\int dy j^R_1(t,y) , \\ Q^{L(1)} &= \int dy_1 dy_2 \epsilon(y_1 - y_2)(j^L_0 - \alpha j^L_1)(t, y_1)(j^L_0 - \alpha j^L_1)(t, y_2) + 2\lambda^{-2}(1-\alpha^2)\int dy j^L_1(t,y) . \end{aligned} \tag{94}$$

The construction of previous sections can thus be performed for the WZ case with few modifications. The chain algebra construction is not touched as well as the saturated charge, but with the above replacement of currents. In this way we need to use the following Dirac bracket for the current J_0 (written in matrix components)

$$\{(J^L_0)_{ij}(x), (J^L_0)_{kl}(y)\} = (\delta \circ J^L_0)_{ij,kl}(x)\delta(x-y) + \alpha(\delta\circ\delta)_{ij,kl}\delta'(x-y) \tag{95}$$

and we are led to the following Dirac brackets for the first few (left sector) charges

$$\begin{aligned}
\{Q^{(0)}_{ij}, Q^{(0)}_{kl}\} =& \left(\delta\circ Q^{(0)}\right)_{ij,kl} \\
\{Q^{(n)}_{ij}, Q^{(0)}_{kl}\} =& \left(\delta\circ Q^{(n)}\right)_{ij,kl} + 4\alpha\left(\delta\circ Q^{(n-1)}\right)_{ij,kl} \\
\{Q^{(1)}_{ij}, Q^{(1)}_{kl}\} =& \left(\delta\circ Q^{(2)}\right)_{ij,kl} - \left(Q^{(0)}Q^{(0)}\circ Q^{(0)}\right)_{ij,kl} + 4\alpha\left(\delta\circ Q^{(1)}\right)_{ij,kl} \\
\{Q^{(2)}_{ij}, Q^{(1)}_{kl}\} =& \left(\delta\circ Q^{(3)}\right)_{ij,kl} - \left(Q^{(0)}Q^{(0)}\circ Q^{(1)}\right)_{ij,kl} - \left(Q^{(1)}Q^{(0)}\circ Q^{(0)}\right)_{ij,kl} \\
& +4\alpha\left(\delta\circ Q^{(2)}\right)_{ij,kl} - 4\alpha\left(Q^{(0)}Q^{(0)}\circ Q^{(0)}\right)_{ij,kl} \\
\{Q^{(3)}_{ij}, Q^{(1)}_{kl}\} =& \left(\delta\circ Q^{(4)}\right)_{ij,kl} - \left(Q^{(0)}Q^{(0)}\circ Q^{(2)}\right)_{ij,kl} - \left(Q^{(1)}Q^{(0)}\circ Q^{(1)}\right)_{ij,kl} \\
& - \left(Q^{(2)}Q^{(0)}\circ Q^{(0)}\right)_{ij,kl} \\
& +4\alpha\left(\delta\circ Q^{(3)}\right)_{ij,kl} - 4\alpha\left(Q^{(0)}Q^{(0)}\circ Q^{(1)}\right)_{ij,kl} - 4\alpha\left(Q^{(1)}Q^{(0)}\circ Q^{(0)}\right)_{ij,kl} \\
\{Q^{(2)}_{ij}, Q^{(2)}_{kl}\} =& \left(\delta\circ Q^{(4)}\right)_{ij,kl} - \left(Q^{(0)}Q^{(0)}\circ Q^{(2)}\right)_{ij,kl} - \left(Q^{(1)}Q^{(0)}\circ Q^{(1)}\right)_{ij,kl} \\
& - \left(Q^{(0)}Q^{(1)}\circ Q^{(1)}\right)_{ij,kl} - \left(Q^{(1)}Q^{(0)}\circ Q^{(0)}\right)_{ij,kl} \\
& +4\alpha\left(\delta\circ Q^{(3)}\right)_{ij,kl} - 4\alpha\left(Q^{(0)}Q^{(0)}\circ Q^{(1)}\right)_{ij,kl} - 4\alpha\left(Q^{(0)}\circ Q^{(0)}Q^{(1)}\right)_{ij,kl} \\
\{Q^{(3)}_{ij}, Q^{(2)}_{kl}\} =& \left(\delta\circ Q^{(5)}\right)_{ij,kl} - \left(Q^{(0)}Q^{(0)}\circ Q^{(3)}\right)_{ij,kl} - \left(Q^{(1)}Q^{(0)}\circ Q^{(2)}\right)_{ij,kl} \\
& - \left(Q^{(2)}Q^{(0)}\circ Q^{(1)}\right)_{ij,kl} - \left(Q^{(0)}Q^{(1)}\circ Q^{(2)}\right)_{ij,kl} \\
& - \left(Q^{(1)}Q^{(1)}\circ Q^{(1)}\right)_{ij,kl} - \left(Q^{(2)}Q^{(1)}\circ Q^{(0)}\right)_{ij,kl} \\
& +4\alpha\left(\delta\circ Q^{(4)}\right)_{ij,kl} - 4\alpha\left(Q^{(1)}Q^{(0)}\circ Q^{(0)}\right)_{ij,kl} - 4\alpha\left(Q^{(1)}Q^{(0)}\circ Q^{(1)}\right)_{ij,kl} \\
& - 4\alpha\left(Q^{(0)}Q^{(1)}\circ Q^{(1)}\right)_{ij,kl} - 4\alpha\left(Q^{(0)}Q^{(0)}\circ Q^{(2)}\right)_{ij,kl}
\end{aligned} \tag{96}$$

Therefore we write the following Ansatz for the algebra (we suppose $m \geq n$ with no loss of generality)

$$\begin{aligned}
\{Q^{(m)}_{ij}, Q^{(n)}_{kl}\} =& \left(\delta\circ Q^{(n+m)}\right)_{ij,kl} - \sum_{p=0}^{m-1}\sum_{q=0}^{n-1}\left(Q^{(p)}Q^{(q)}\circ Q^{(m+n-p-q-2)}\right)_{ij,kl} \\
& +4\alpha\left(\left(\delta\circ Q^{(n+m-1)}\right)_{ij,kl} - \sum_{p=0}^{m-2}\sum_{q=0}^{n-1}\left(Q^{(p)}Q^{(q)}\circ Q^{(m+n-p-q-3)}\right)_{ij,kl}\right)
\end{aligned} \tag{97}$$

or equivalently, denoting by $\{\,,\}_{WZ}$ the bracket for the Wess Zumino model and $\{\,,\}$ for previous brackets of the chiral model, we summarize the results by $(n \geq m)$

$$\{Q^{(m)}, Q^{(n)}\}_{WZ} = \{Q^{(m)}, Q^{(n)}\} + 4\alpha\{Q^{(m-1)}, Q^{(n)}\} \quad . \tag{98}$$

Some remarks are in order now. First, concerning the chain algebra, it clearly goes through to the Wess Zumino case. Therefore, the Jacobi identities are valid here as well. Using them, we can perform the proof of the redefinition of the charges in such a way that the Ansatz (97) is valid in the same way as we did before, except for the fact that now $a_{n+m} = 4\alpha$ (see Eq. (32)). With the argument that the linear term determines also the coefficient of the cubic term, we arrive at the result (97) for the complete algebra. The algebra for the right sector follows directly from (98) through $\alpha \to -\alpha$. Also the mixed brackets $\{Q^{L(m)}, Q^{R(n)}\}$ vanish since $\{(J_0^L)_{ij}(x), (J_0^R)_{kl}(y)\} = 0$.

. Conclusions

We have computed the Dirac algebra of conserved non-local charges. The result is characterized by the order n of the non-local charges $Q^{(n)}$, which in fact can be defined in terms of its genus [21], as computed from scattering theory. Therefore, classifying the genus, one verifies that in the right hand side of the Dirac algebra of charges, only the highest possible genus contributes with a non-vanishing coefficient.

We arrived at the results considering first the algebra of chains, in terms of which one is able to prove the Jacobi identity. Further on, we prove, still using the algebra of chains, that there is a simple relation between the linear and the cubic part of the algebra. Using the Jacobi identity, which at this time is known to hold, we prove that a redefinition of the charges is always possible, in such a way that the algebra is quite simple. Furthermore, we verified the results up to a very high order (see Eqs. (29)).

This result permits us to try to obtain constraints on the correlation functions of the theory, similarly to the massive perturbation of the $k=1$ WZW model [9]. Such problem evaded solution for several years, but with this approach, one should be able to accomplish such desired constraints, once one knows a realization of charges in terms of integro-differential operators. Indeed, for the asymptotic charges one finds such representations [5,10].

Further problems related to the role of monodromy matrices may also be obtained once one knows the expansion of this matrix in terms of non-local charge, a procedure in fact studied (although following the inverse way) in [4]. For sigma models with a simple gauge group the quantum non-local charge algebra must be the same as we have computed substituting Dirac brackets by $(-i)$ times commutators [15].

Finally, we remark that the WZNW theory presents an algebra which is analogous to the above one. In fact, the WZNW theory has been treated by the Bethe Ansatz [22] with results analogous in some sense to the purely bosonic case, and one expects many similarities.

Appendix

In this Appendix we list some useful formulae concerning the special product $A \circ B$ and the constraints involving the currents j and j_μ.

The product $A \circ B$ is defined as follows

$$(A \circ B)_{ij,kl} = A_{ik}B_{jl} - A_{il}B_{jk} + A_{jl}B_{ik} - A_{jk}B_{il} \tag{A.1}$$

and possesses the properties

$$\begin{aligned}
(A \circ B)_{ij,kl} &= (B \circ A)_{ij,kl} = (A^t \circ B^t)_{kl,ij} && (A.2)\\
(A \circ B)_{ij,ka}C_{al} - (k \leftrightarrow l) &= (A \circ BC)_{ij,kl} + (AC \circ B)_{ij,kl} && (A.3)\\
(A \circ B)_{ia,kl}C_{aj} - (i \leftrightarrow j) &= (A \circ C^t B)_{ij,kl} + (C^t A \circ B)_{ij,kl} && (A.4)\\
C_{ia}(A \circ B)_{aj,kl} - (i \leftrightarrow j) &= (CA \circ B)_{ij,kl} + (A \circ CB)_{ij,kl} && (A.5)\\
A_{ia}(B \circ C)_{ab,kl}D_{bj} - (i \leftrightarrow j) &= (AB \circ D^t C)_{ij,kl} + (D^t B \circ AC)_{ij,kl} && (A.6)\\
A_{ka}(B \circ C)_{ij,ab}D_{bl} - (k \leftrightarrow l) &= (BA^t \circ CD)_{ij,kl} + (BD \circ CA^t)_{ij,kl} && (A.7)
\end{aligned}$$

$$\frac{1}{4} t^a_{ji} t^b_{lk} (A \circ B)_{ij,kl} = \mathrm{tr}(t^a A t^b B). \tag{A.8}$$

Now we list the constraints among the currents:

$$\begin{aligned}
(j_\mu \circ j_\nu)_{ij,kl} &= (j_\mu)_{ij}(j_\nu)_{kl} + (j_\nu)_{ij}(j_\mu)_{kl} && (A.9)\\
(j_\mu \circ j)_{ij,kl} &= 0 && (A.10)\\
(j \circ j)_{ij,kl} &= 0 && (A.11)\\
[j_\mu, j]_+ &= j_\mu && (A.12)\\
[j, j]_+ &= 2j && (A.13)\\
[j_\mu, j] &= -\partial_\mu j && (A.14)\\
(j_1 j) &= \frac{1}{2} j_1 - \frac{1}{2}\partial_j \quad . && (A.15)
\end{aligned}$$

We cite also an useful relation containing the antiderivative operator ∂^{-1}

$$(\partial^{-1} A \circ A) = \frac{1}{2}\partial(\partial^{-1} A \circ \partial^{-1} A) \quad . \tag{A.16}$$

Acknowledgements

We would like to thank Prof. A. Salam for the hospitality at the International Centre for Theoretical Physics, where part of this work has been done.

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Figure Captions

Fig. 1: Pictorial representation of chains: empty and full circles represent the components $j_0(x)$ and $j_1(x)$ respectively; solid lines represent the sign function $\epsilon(x-y)$.

Fig. 2: Crossing in the algebra of chains: the figure represents the term obtained from the Dirac brackets of the factors $j_0(x_i)$ and $j_0(y_j)$ of two linear chains.

Fig. 3: Loops representing the four terms obtained in the trace projection of the crossing in Fig. 2: an a-labelled cross indicates the insertion of a matrix t^a; the broken line is the trace line of matrices; the curly line represents a delta function; the square indicates the omission of a component j_0.

Fig. 4: Four equivalent loops originated from different crossings.

Fig. 5: Vanishing surface terms corresponding to the sums (60) in the case $m = 2$, $n = 3$. Fig. 5a represents the first sum, where the terms join together in pairs (for instance, the first and the last loops) to produce a total derivative as in Eq. (61). In Fig. 5b, which corresponds to the second sum of (60), we have a similar pairing plus the single term (in the center of the figure) which generates a surface term of the form (62).

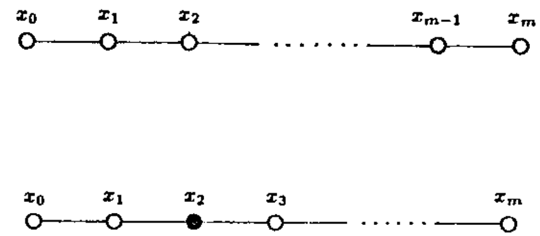

Fig. 1

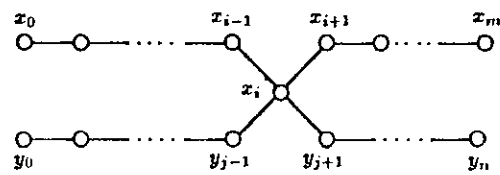

Fig. 2

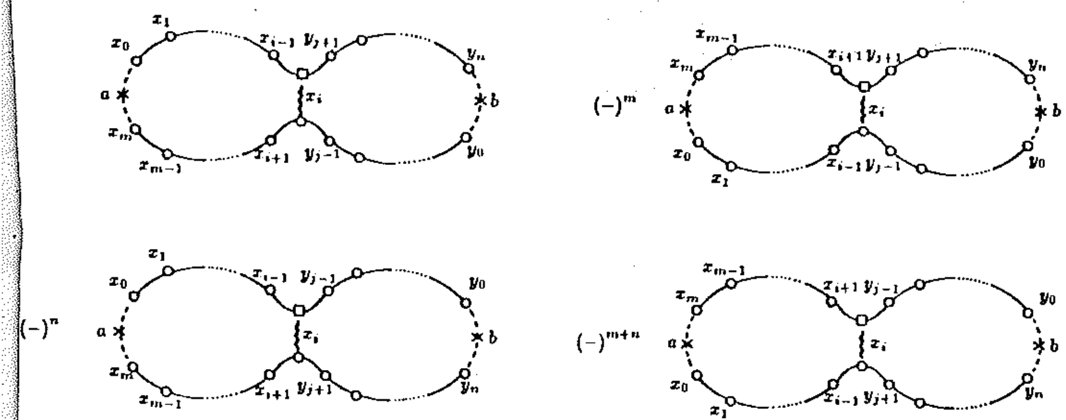

Fig. 3

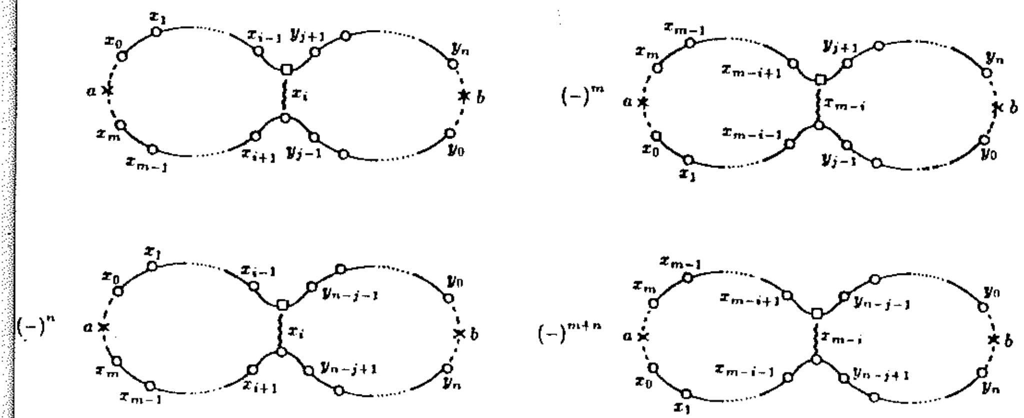

Fig. 4

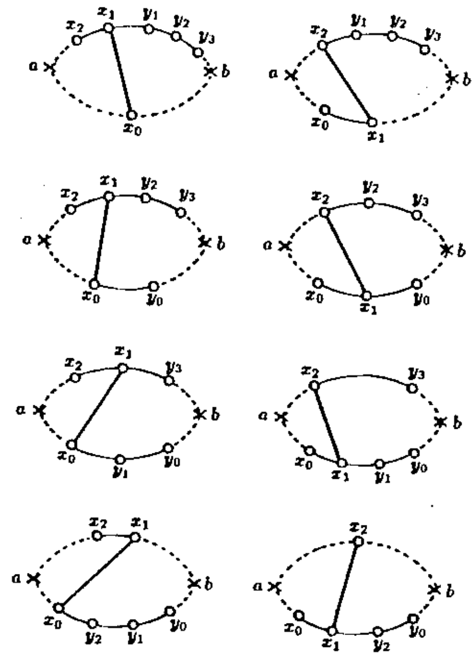

Fig. 5a

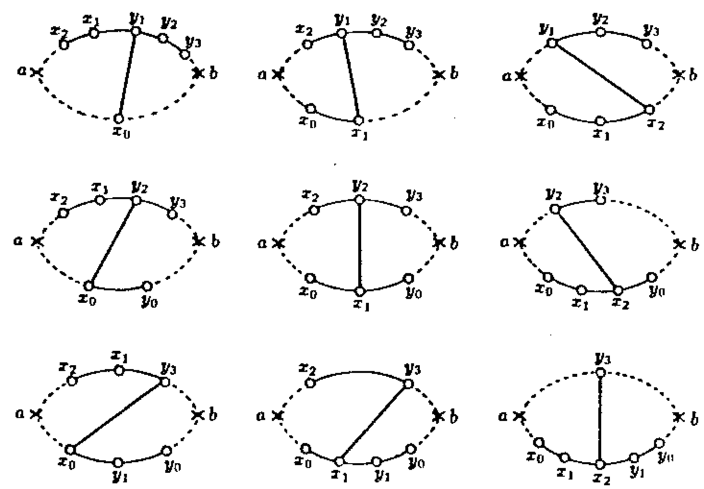

Fig. 5b