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STRINGS IN BACKGROUND FIELDS AND
EINSTEIN-CARTAN THEORY OF GRAVITY

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Abstract

Bosonic strings in graviton, dilaton, and antisymmetrical $B_{\mu\nu}$ background fields are studied. It is shown that the well known background fields equations gotten by the vanishing of one-loop beta functions, correspond to Einstein-Cartan gravity equations. This opens new possibilities of interpretation for the massless states of the closed bosonic string.

There is a great interest in the investigation of strings in curved backgrounds. The Einstein equation for the vacuum, $R_{\mu\nu} = 0$, appears naturally as a consequence of conformal invariance in a quantum analysis up to one-loop order for bosonic strings interacting with a graviton background field[1, 2]. The purpose of this work is to show that, the well known equations obtained as consequence of conformal invariance up to one-loop order for bosonic strings in interaction with graviton, dilaton, and antisymmetrical $B_{\mu\nu}$ background fields simultaneously[1, 2], correspond to Einstein-Cartan gravity equations for the vacuum, provided that one identifies the dilaton and the $B_{\mu\nu}$ background fields with certain non-riemannian quantities of the background manifold.

The classical action which describes a bosonic string on a curved N -dimensional background and that takes into account the relevant closed-string massless states (the graviton $g_{\mu\nu}$, the dilaton Φ , and the anti-symmetrical tensor $B_{\mu\nu}$) is given by [1, 2]

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left(\sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X) + \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X) - \alpha' \sqrt{h} \Phi(X) R^{(2)} \right), \quad (1)$$

where σ , $h_{\alpha\beta}$, $\epsilon^{\alpha\beta}$, and $R^{(2)}$ are respectively coordinates, the metric tensor, the totally anti-symmetrical symbol, and the scalar of curvature in the 2-dimensional world-sheet. The Einstein vacuum equation arises from an one-loop analyses of (1) with $B_{\mu\nu} = \Phi = 0$.

The necessary conditions to guarantee the conformal invariance up to

one-loop order of (1) are [1, 2]

$$\begin{aligned}\beta^\Phi &= \frac{N-26}{3\alpha'} + 4D_\mu\Phi D^\mu\Phi - 4D^2\Phi - R + \frac{1}{12}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma} = 0, \\ \beta_{\mu\nu}^g &= R_{\mu\nu} - \frac{1}{4}H_\mu^{\lambda\rho}H_{\nu\lambda\rho} + 2D_\mu D_\nu\Phi = 0, \\ \beta_{\mu\nu}^B &= D_\lambda H_{\mu\nu}^\lambda - 2(D_\lambda\Phi)H_{\mu\nu}^\lambda = 0,\end{aligned}\quad (2)$$

where $R_{\mu\nu}$, R , and D_μ are respectively the Ricci tensor, the scalar of curvature, and the covariant derivative in the background manifold \mathcal{M} , which usually is assumed to be Riemannian and so these quantities are calculated from the Christoffel symbols. The following conventions are adopted: $\text{sign}(g_{\mu\nu}) = (+, -, \dots)$, $R_{\alpha\nu\mu}^\beta = \partial_\alpha\Gamma_{\nu\mu}^\beta + \Gamma_{\alpha\rho}^\beta\Gamma_{\nu\mu}^\rho - (\alpha \leftrightarrow \nu)$, and $R_{\nu\mu} = R_{\alpha\nu\mu}^\alpha$. The new field $H_{\alpha\beta\gamma}$ in (2) is the third-rank anti-symmetrical strength tensor defined from $B_{\mu\nu}$,

$$H_{\alpha\beta\gamma} = \partial_\alpha B_{\beta\gamma} + \partial_\gamma B_{\alpha\beta} + \partial_\beta B_{\gamma\alpha}. \quad (3)$$

One can check that the equations (2) follow from the minimization of the action [1]-[4]:

$$S = - \int d^N x \sqrt{-g} e^{-2\Phi} \left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{12}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma} - \frac{N-26}{3\alpha'} \right). \quad (4)$$

The action (4) is the effective action for gravity with one-loop ‘‘stringy’’ corrections, and it is the starting point for the string cosmology discussions [5]. The aim is to show that (4) is in fact equivalent to the Hilbert-Einstein action for a N -dimensional Riemann-Cartan background space-time, and to do it, we introduce briefly some facts on Riemann-Cartan (RC) geometry.

The RC space-time U_N is a N -dimensional differentiable manifold endowed with a metric tensor $g_{\alpha\beta}(x)$ and with a metric-compatible connection

$\Gamma_{\alpha\beta}^\mu$, which is non-symmetrical in its lower indices. From the anti-symmetric part of the connection one can define the torsion tensor

$$S_{\alpha\beta}^\gamma = \frac{1}{2}(\Gamma_{\alpha\beta}^\gamma - \Gamma_{\beta\alpha}^\gamma). \quad (5)$$

The metric-compatible connection, that is used to define the covariant derivative \mathcal{D}_μ , can be written as

$$\Gamma_{\alpha\beta}^\gamma = \{\gamma_{\alpha\beta}\} - K_{\alpha\beta}^\gamma, \quad (6)$$

where $\{\gamma_{\alpha\beta}\}$ are the usual Christoffel symbols from Riemannian space-time V_N , and $K_{\alpha\beta}^\gamma$ is the contorsion tensor, which is given in terms of the torsion tensor by

$$K_{\alpha\beta}^\gamma = -S_{\alpha\beta}^\gamma + S_{\beta\alpha}^\gamma - S_{\alpha\gamma}^\beta. \quad (7)$$

The contorsion tensor (7) can be covariantly split in a traceless part and in a trace

$$K_{\alpha\beta\gamma} = \tilde{K}_{\alpha\beta\gamma} - \frac{2}{3}(g_{\alpha\gamma}S_\beta - g_{\alpha\beta}S_\gamma), \quad (8)$$

where $\tilde{K}_{\alpha\beta\gamma}$ is the traceless part and S_β is the trace of the torsion tensor, $S_\beta = S_{\alpha\beta}^\alpha$. The RC curvature tensor is defined by using the connection (6),

$$\mathcal{R}_{\alpha\nu\mu}^\beta = \partial_\alpha\Gamma_{\nu\mu}^\beta - \partial_\nu\Gamma_{\alpha\mu}^\beta + \Gamma_{\alpha\rho}^\beta\Gamma_{\nu\mu}^\rho - \Gamma_{\nu\rho}^\beta\Gamma_{\alpha\mu}^\rho, \quad (9)$$

and after some manipulations we get the following expression for the scalar of curvature

$$\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\alpha\mu\nu}^\alpha = R - 4\mathcal{D}_\mu S^\mu + 4S_\mu S^\mu - K_{\nu\rho\alpha}K^{\alpha\nu\rho}, \quad (10)$$

where R is the V_N scalar of curvature, calculated from the Christoffel symbols.

In V_N , the invariant volume element

$$dv = \sqrt{-g} d^4x, \quad (11)$$

is also covariantly constant, as one can check by calculating the V_N covariant derivative of the scalar density $\sqrt{-g}$, $D_\mu \sqrt{-g} = 0$. One can check that the V_N volume element (11) is not covariantly constant in U_N ,

$$\mathcal{D}_\mu \sqrt{-g} = \partial_\mu \sqrt{-g} - \Gamma_{\rho\mu}^\rho \sqrt{-g} = -2S_\mu \sqrt{-g}. \quad (12)$$

In order to construct an invariant and covariantly constant V_N -like volume element in U_N , one needs to find out a density $f(x)$ such that $\mathcal{D}_\mu f(x) = 0$. Such density exists only if the trace S_μ can be obtained from a scalar potential

$$S_\beta(x) = \partial_\beta \Theta(x), \quad (13)$$

and in this case we have

$$dv = e^{2\Theta} \sqrt{-g} d^4x, \quad (14)$$

that is the invariant and covariantly constant U_N volume element. With the volume element (14) we have the generalized Gauss formula

$$\int dv \mathcal{D}_\mu V^\mu = \int d^4x \partial_\mu (e^{2\Theta} \sqrt{-g} V^\mu) = \text{surf. term}, \quad (15)$$

where we used that $\Gamma_{\rho\mu}^\rho = \partial_\mu (\ln(e^{2\Theta} \sqrt{-g}))$ under the hypothesis (13).

Now one can construct the Hilbert-Einstein action, using the scalar of curvature (10), the condition (13), the volume element (14), and (15)

$$\begin{aligned} S_{\text{grav}} &= - \int dv (\mathcal{R} + \Lambda) \\ &= - \int e^{2\Theta} \sqrt{-g} d^N x (R + 4\partial_\mu \Theta \partial^\mu \Theta - K_{\nu\rho\alpha} K^{\alpha\nu\rho} + \Lambda) + \text{surf. terms.} \end{aligned} \quad (16)$$

where Λ is a cosmological constant. The similarity between (4) and (16) is surprising. They can be identified if one assumes that:

$$\begin{aligned} \Theta(x) &= -\Phi(x), \\ \Lambda &= -\frac{N-26}{3\alpha'}, \\ \frac{1}{12} H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} &= K_{\nu\rho\alpha} K^{\alpha\nu\rho} = \tilde{K}_{\nu\rho\alpha} \tilde{K}^{\alpha\nu\rho} - \frac{4}{3} \partial_\mu \Phi \partial^\mu \Phi. \end{aligned} \quad (17)$$

The expressions (17) deserve some explanation. The first equation, identifies the dilaton field with the scalar potential for the trace of the torsion tensor, i.e. $S_\mu(x) = -\partial_\mu \Phi(x)$. The last one, relates the totally anti-symmetrical tensor $H_{\alpha\beta\gamma}$, which is derived from the anti-symmetrical field $B_{\mu\nu}$, to the contorsion tensor, which is not anti-symmetrical in general and depends on the dilaton field. It admits as solution:

$$\tilde{K}_{\alpha\beta\gamma} = \frac{1}{2\sqrt{3}} \sqrt{H_{\mu\nu\omega} H^{\mu\nu\omega} + 16\partial_\nu \Phi \partial^\nu \Phi} \frac{H_{\alpha\beta\gamma}}{\sqrt{H_{\mu\nu\omega} H^{\mu\nu\omega}}}, \quad (18)$$

valid for $H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} \neq 0$. Let us explore (17) in more details with some particular cases, the 3 and 4-dimensional ones. In a 3-dimensional manifold, a totally anti-symmetrical third-rank tensor has the form

$$H_{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma} \Omega(x), \quad (19)$$

where $\Omega(x)$ is a scalar density. In this case $H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} = 0$ leads to $\Omega = 0$, and (18) does exist in this limit. For the 3-dimensional case (18) is valid for any $H_{\alpha\beta\gamma}$.

In an 4-dimensional manifold, a totally anti-symmetrical third-rank tensor has only 4 independent components, what allows us to write

$$H_{\alpha\beta\gamma}(x) = \epsilon_{\alpha\beta\gamma\delta} j^\delta(x) H^\delta(x), \quad (20)$$

where H^δ is a contravariant vector and $j(x)$ is a scalar density, in order to compensate de totally anti-symmetrical symbol. Defining

$$H^{\alpha\beta\gamma}(x) = \epsilon^{\alpha\beta\gamma\delta} j^{-1}(x) H_\delta(x), \quad (21)$$

and using that the traceless part of the contorsion tensor in (18) is also anti-symmetrical, and so derive from a vector \tilde{K}_α in the same way of (20), we have the following expression for the last equation of (17)

$$\frac{1}{2} H_\alpha H^\alpha = 6 \tilde{K}_\alpha \tilde{K}^\alpha - \frac{4}{3} S_\alpha S^\alpha. \quad (22)$$

The condition $H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} = 0$ implies that $H_\alpha H^\alpha = 0$, and in this case a solution of (22) is $\tilde{K}_\alpha = \frac{\sqrt{2}}{3} S_\alpha$. One needs to construct $\tilde{K}_{\alpha\beta\gamma}$ from S^α as in (20). A consistent choice for the density $j(x)$ is that allows us to identify (20) with the Hodge star (*) operation, and in order to get it one picks $j(x) = e^{2\Theta} \sqrt{-g}$ (see ref. [6]), which leads to the following equation

$$\tilde{K}_{\alpha\beta\gamma} = -\frac{\sqrt{2}}{3} \epsilon_{\alpha\beta\gamma\delta} e^{-2\Phi} \sqrt{-g} \partial^\delta \Phi, \quad (23)$$

valid for $H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} = 0$.

It is interesting to note that these solutions involve only the trace and the anti-symmetrical part of the contorsion tensor. The traceless part of the contorsion tensor is not anti-symmetrical in general, but only its anti-symmetrical part enters in the solution. One can think that, by (1), bosonic strings interact with the non-riemannian structure of the background manifold only by means of the trace and the anti-symmetrical part of the torsion tensor. The torsion tensor from (18) is given by:

$$S_{\alpha\beta\gamma} = -\frac{1}{2\sqrt{3}} \sqrt{H_{\mu\nu\omega} H^{\mu\nu\omega} + 16 \partial_\nu \Phi \partial^\nu \Phi} \frac{H_{\alpha\beta\gamma}}{\sqrt{H_{\mu\nu\omega} H^{\mu\nu\omega}}} - \frac{1}{3} (g_{\alpha\gamma} \partial_\beta \Phi - g_{\beta\gamma} \partial_\alpha \Phi). \quad (24)$$

where $H_{\alpha\beta\gamma} = \partial_\alpha B_{\beta\gamma} + \partial_\gamma B_{\alpha\beta} + \partial_\beta B_{\gamma\alpha}$.

The action (16) for the Einstein-Cartan theory of gravity was recently proposed, and one of its new predictions is that torsion propagates, what allows non-vanishing torsion solutions for the vacuum.

As the conclusion, we stress that these results allow us to say that (1) describes in fact a bosonic string moving in a Riemann-Cartan background manifold \mathcal{M} , where $g_{\mu\nu}$ is the metric of \mathcal{M} , and Φ and $B_{\mu\nu}$ are related to the non-riemannian structure of \mathcal{M} by (17). This is a new interpretation for the spin 0 and the anti-symmetrical massless states of the closed bosonic string. A two-loop analyses shall give "stringy" corrections to Einstein-Cartan gravity equations. These topics are now under investigation.

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