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PUBLICAÇÕES

IFUSP/P-1066

STRINGS IN BACKGROUND FIELDS AND EINSTEIN-CARTAN THEORY OF GRAVITY

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Julho/1993

hep-th/9307095

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Abstract

Bosonic strings in graviton, dilaton, and antisymmetrical $B_{\mu\nu}$ background fields are studied. It is shown that the well known background fields equations gotten by the vanishing of one-loop beta functions, correspond to Einstein-Cartan gravity equations. This opens new possibilities of interpretation for the massless states of the closed bosonic string.

There is a great interest in the investigation of strings in curved backgrounds. The Einstein equation for the vacuum, $R_{\mu\nu} = 0$, appears naturally as a consequence of conformal invariance in a quantum analysis up to one-loop order for bosonic strings interacting with a graviton background field[1, 2]. The purpose of this work is to show that, the well known equations obtained as consequence of conformal invariance up to one-loop order for bosonic strings in interaction with graviton, dilaton, and antisymmetrical $B_{\mu\nu}$ background fields simultaneously[1, 2], correspond to Einstein-Cartan gravity equations for the vacuum, provided that one identifies the dilaton and the $B_{\mu\nu}$ background fields with certain non-riemannian quantities of the background manifold.

The classical action which describes a bosonic string on a curved N-dimensional background and that takes into account the relevant closed-string massless states (the graviton $g_{\mu\nu}$, the dilaton Φ, and the anti-symmetrical tensor $B_{\mu\nu}$) is given by [1, 2]

$$\begin{aligned} S &= -\frac{1}{4\pi\alpha'}\int d^2\sigma \left(\sqrt{h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu g_{\mu\nu}(X) + \epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu B_{\mu\nu}(X)\right. \\ &\qquad \left. -\alpha'\sqrt{h}\Phi(X)R^{(2)}\right), \end{aligned} \tag{1}$$

where σ, $h_{\alpha\beta}$, $\epsilon^{\alpha\beta}$, and $R^{(2)}$ are respectively coordinates, the metric tensor, the totally anti-symmetrical symbol, and the scalar of curvature in the 2-dimensional world-sheet. The Einstein vacuum equation arises from an one-loop analyses of (1) with $B_{\mu\nu} = \Phi = 0$.

The necessary conditions to guarantee the conformal invariance up to one-loop order of (1) are[1, 2]

$$\begin{aligned}\beta^{\Phi} &= \frac{N-26}{3\alpha'} + 4D_\mu\Phi D^\mu\Phi - 4D^2\Phi - R + \frac{1}{12}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma} = 0,\\ \beta^{g}_{\mu\nu} &= R_{\mu\nu} - \frac{1}{4}H_{\mu}{}^{\lambda\rho}H_{\nu\lambda\rho} + 2D_\mu D_\nu\Phi = 0, \qquad (2)\\ \beta^{B}_{\mu\nu} &= D_\lambda H^{\lambda}{}_{\mu\nu} - 2\left(D_\lambda\Phi\right)H^{\lambda}{}_{\mu\nu} = 0,\end{aligned}$$

where $R_{\mu\nu}$, R, and D_μ are respectively the Ricci tensor, the scalar of curvature, and the covariant derivative in the background manifold $\mathcal{M}$, which usually is assumed to be Riemannian and so these quantities are calculated from the Christoffel symbols. The following conventions are adopted: $\mathrm{sign}(g_{\mu\nu}) = (+,-,-\ldots)$, $R_{\alpha\nu\mu}{}^{\beta} = \partial_\alpha\Gamma^{\beta}_{\nu\mu} + \Gamma^{\beta}_{\alpha\rho}\Gamma^{\rho}_{\nu\mu} - (\alpha \leftrightarrow \nu)$, and $R_{\nu\mu} = R_{\alpha\nu\mu}{}^{\alpha}$. The new field $H_{\alpha\beta\gamma}$ in (2) is the third-rank anti-symmetrical strength tensor defined from $B_{\mu\nu}$,

$$H_{\alpha\beta\gamma} = \partial_\alpha B_{\beta\gamma} + \partial_\gamma B_{\alpha\beta} + \partial_\beta B_{\gamma\alpha}. \qquad (3)$$

One can check that the equations (2) follow from the minimization of the action [1]-[4]:

$$S = -\int d^N x\sqrt{-g}e^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{12}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma} - \frac{N-26}{3\alpha'}\right). \qquad (4)$$

The action (4) is the effective action for gravity with one-loop "stringy" corrections, and it is the starting point for the string cosmology discussions [5]. The aim is to show that (4) is in fact equivalent to the Hilbert-Einstein action for a N-dimensional Riemann-Cartan background space-time, and to do it, we introduce briefly some facts on Riemann-Cartan (RC) geometry.

The RC space-time U_N is a N-dimensional differentiable manifold endowed with a metric tensor $g_{\alpha\beta}(x)$ and with a metric-compatible connection

$\Gamma^{\mu}_{\alpha\beta}$, which is non-symmetrical in its lower indices. From the anti-symmetric part of the connection one can define the torsion tensor

$$S_{\alpha\beta}{}^{\gamma} = \frac{1}{2}\left(\Gamma^{\gamma}_{\alpha\beta} - \Gamma^{\gamma}_{\beta\alpha}\right). \qquad (5)$$

The metric-compatible connection, that is used to define the covariant derivative $\mathcal{D}_\mu$, can be written as

$$\Gamma^{\gamma}_{\alpha\beta} = \left\{{}^{\gamma}_{\alpha\beta}\right\} - K_{\alpha\beta}{}^{\gamma}, \qquad (6)$$

where $\left\{{}^{\gamma}_{\alpha\beta}\right\}$ are the usual Christoffel symbols from Riemannian space-time V_N, and $K_{\alpha\beta}{}^{\gamma}$ is the contorsion tensor, which is given in terms of the torsion tensor by

$$K_{\alpha\beta}{}^{\gamma} = -S_{\alpha\beta}{}^{\gamma} + S_{\beta}{}^{\gamma}{}_{\alpha} - S^{\gamma}{}_{\alpha\beta}. \qquad (7)$$

The contorsion tensor (7) can be covariantly split in a traceless part and in a trace

$$K_{\alpha\beta\gamma} = \tilde{K}_{\alpha\beta\gamma} - \frac{2}{3}\left(g_{\alpha\gamma}S_\beta - g_{\alpha\beta}S_\gamma\right), \qquad (8)$$

where $\tilde{K}_{\alpha\beta\gamma}$ is the traceless part and S_β is the trace of the torsion tensor, $S_\beta = S_{\alpha\beta}{}^{\alpha}$. The RC curvature tensor is defined by using the connection (6),

$$\mathcal{R}_{\alpha\nu\mu}{}^{\beta} = \partial_\alpha\Gamma^{\beta}_{\nu\mu} - \partial_\nu\Gamma^{\beta}_{\alpha\mu} + \Gamma^{\beta}_{\alpha\rho}\Gamma^{\rho}_{\nu\mu} - \Gamma^{\beta}_{\nu\rho}\Gamma^{\rho}_{\alpha\mu}, \qquad (9)$$

and after some manipulations we get the following expression for the scalar of curvature

$$\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\alpha\mu\nu}{}^{\alpha} = R - 4\mathcal{D}_\mu S^\mu + 4S_\mu S^\mu - K_{\nu\rho\alpha}K^{\alpha\nu\rho}, \qquad (10)$$

where R is the V_N scalar of curvature, calculated from the Christoffel symbols.

In V_N, the invariant volume element

$$dv = \sqrt{-g}\, d^4x, \tag{11}$$

is also covariantly constant, as one can check by calculating the V_N covariant derivative of the scalar density $\sqrt{-g}$, $D_\mu\sqrt{-g} = 0$. One can check that the V_N volume element (11) is not covariantly constant in U_N,

$$\mathcal{D}_\mu\sqrt{-g} = \partial_\mu\sqrt{-g} - \Gamma^\rho_{\rho\mu}\sqrt{-g} = -2S_\mu\sqrt{-g}. \tag{12}$$

In order to construct an invariant and covariantly constant V_N-like volume element in U_N, one needs to find out a density $f(x)$ such that $\mathcal{D}_\mu f(x) = 0$. Such density exists only if the trace S_μ can be obtained from a scalar potential

$$S_\beta(x) = \partial_\beta\Theta(x), \tag{13}$$

and in this case we have

$$dv = e^{2\Theta}\sqrt{-g}\, d^4x, \tag{14}$$

that is the invariant and covariantly constant U_N volume element. With the volume element (14) we have the generalized Gauss formula

$$\int dv\mathcal{D}_\mu V^\mu = \int d^4x\partial_\mu\left(e^{2\Theta}\sqrt{-g}V^\mu\right) = \text{ surf. term}, \tag{15}$$

where we used that $\Gamma^\rho_{\rho\mu} = \partial_\mu\left(\ln(e^{2\Theta}\sqrt{-g})\right)$ under the hypothesis (13).

Now one can construct the Hilbert-Einstein action, using the scalar of curvature (10), the condition (13), the volume element (14), and (15)

$$\begin{aligned} S_{\text{grav}} &= -\int dv\,(\mathcal{R} + \Lambda) \qquad\qquad (16)\\ &= -\int e^{2\Theta}\sqrt{-g}\, d^Nx\,(R + 4\partial_\mu\Theta\partial^\mu\Theta - K_{\nu\rho\alpha}K^{\alpha\nu\rho} + \Lambda) + \text{surf. terms.} \end{aligned}$$

where Λ is a cosmological constant. The similarity between (4) and (16) is surprising. They can be identified if one assumes that:

$$\begin{aligned} \Theta(x) &= = -\Phi(x), \\ \Lambda &= = -\frac{N-26}{3\alpha'}, \\ \frac{1}{12}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma} &= K_{\nu\rho\alpha}K^{\alpha\nu\rho} = \tilde{K}_{\nu\rho\alpha}\tilde{K}^{\alpha\nu\rho} - \frac{4}{3}\partial_\mu\Phi\partial^\mu\Phi. \end{aligned} \tag{17}$$

The expressions (17) deserve some explanation. The first equation, identifies the dilaton field with the scalar potential for the trace of the torsion tensor, *i.e.* $S_\mu(x) = -\partial_\mu\Phi(x)$. The last one, relates the totally anti-symmetrical tensor $H_{\alpha\beta\gamma}$, which is derived from the anti-symmetrical field $B_{\mu\nu}$, to the contorsion tensor, which is not anti-symmetrical in general and depends on the dilaton field. It admits as solution:

$$\tilde{K}_{\alpha\beta\gamma} = \frac{1}{2\sqrt{3}}\sqrt{H_{\mu\nu\omega}H^{\mu\nu\omega} + 16\partial_\nu\Phi\partial^\nu\Phi}\,\frac{H_{\alpha\beta\gamma}}{\sqrt{H_{\mu\nu\omega}H^{\mu\nu\omega}}}, \tag{18}$$

valid for $H_{\alpha\beta\gamma}H^{\alpha\beta\gamma} \neq 0$. Let us explore (17) in more details with some particular cases, the 3 and 4-dimensional ones. In a 3-dimensional manifold, a totally anti-symmetrical third-rank tensor has the form

$$H_{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma}\Omega(x), \tag{19}$$

where $\Omega(x)$ is a scalar density. In this case $H_{\alpha\beta\gamma}H^{\alpha\beta\gamma} = 0$ leads to $\Omega = 0$, and (18) does exist in this limit. For the 3-dimensional case (18) is valid for any $H_{\alpha\beta\gamma}$.

In an 4-dimensional manifold, a totally anti-symmetrical third-rank tensor has only 4 independents components, what allows us to write

$$H_{\alpha\beta\gamma}(x) = \epsilon_{\alpha\beta\gamma\delta}\jmath(x)H^\delta(x), \tag{20}$$

where H^δ is a contravariant vector and $j(x)$ is a scalar density, in order to compensate de totally anti-symmetrical symbol. Defining

$$H^{\alpha\beta\gamma}(x) = \epsilon^{\alpha\beta\gamma\delta} j^{-1}(x) H_\delta(x), \tag{21}$$

and using that the traceless part of the contorsion tensor in (18) is also anti-symmetrical, and so derive from a vector $\tilde{K}_\alpha$ in the same way of (20), we have the following expression for the last equation of (17)

$$\frac{1}{2} H_\alpha H^\alpha = 6 \tilde{K}_\alpha \tilde{K}^\alpha - \frac{4}{3} S_\alpha S^\alpha. \tag{22}$$

The condition $H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} = 0$ implies that $H_\alpha H^\alpha = 0$, and in this case a solution of (22) is $\tilde{K}_\alpha = \frac{\sqrt{2}}{3} S_\alpha$. One needs to construct $\tilde{K}_{\alpha\beta\gamma}$ from S^α as in (20). A consistent choice for the density $j(x)$ is that allows us to identify (20) with the Hodge star (*) operation, and in order to get it one picks $j(x) = e^{2\Theta}\sqrt{-g}$ (see ref. [6]), which leads to the following equation

$$\tilde{K}_{\alpha\beta\gamma} = -\frac{\sqrt{2}}{3} \epsilon_{\alpha\beta\gamma\delta} e^{-2\Phi} \sqrt{-g} \partial^\delta \Phi, \tag{23}$$

valid for $H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} = 0$.

It is interesting to note that these solutions involve only the trace and the anti-symmetrical part of the contorsion tensor. The traceless part of the contorsion tensor is not anti-symmetrical in general, but only its anti-symmetrical part enters in the solution. One can think that, by (1), bosonic strings interact with the non-riemannian structure of the background manifold only by means of the trace and the anti-symmetrical part of the torsion tensor. The torsion tensor from (18) is given by:

$$S_{\alpha\beta\gamma} = -\frac{1}{2\sqrt{3}} \sqrt{H_{\mu\nu\omega} H^{\mu\nu\omega} + 16 \partial_\nu \Phi \partial^\nu \Phi} \frac{H_{\alpha\beta\gamma}}{\sqrt{H_{\mu\nu\omega} H^{\mu\nu\omega}}} - \frac{1}{3} \left(g_{\alpha\gamma} \partial_\beta \Phi - g_{\beta\gamma} \partial_\alpha \Phi \right). \tag{24}$$

where $H_{\alpha\beta\gamma} = \partial_\alpha B_{\beta\gamma} + \partial_\gamma B_{\alpha\beta} + \partial_\beta B_{\gamma\alpha}$.

The action (16) for the Einstein-Cartan theory of gravity was recently proposed, and one of its new predictions is that torsion propagates, what allows non-vanishing torsion solutions for the vacuum.

As the conclusion, we stress that these results allow us to say that (1) describes in fact a bosonic string moving in a Riemann-Cartan background manifold $\mathcal{M}$, where $g_{\mu\nu}$ is the metric of $\mathcal{M}$, and Φ and $B_{\mu\nu}$ are related to the non-riemannian structure of $\mathcal{M}$ by (17). This is a new interpretation for the spin 0 and the anti-symmetrical massless states of the closed bosonic string. A two-loop analyses shall give "stringy" corrections to Einstein-Cartan gravity equations. These topics are now under investigation.

The author wishes to thank Professors Élcio Abdalla and Josif Frenkel for valuable discussions. This work was supported by Fundação de Amparo à Pesquisa do Estado de São Paulo.

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