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FROM QUENCHED TO ANNEALED: A STUDY OF THE INTERMEDIATE DYNAMICS OF DISORDER

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Abstract

A model spin system with disorder is examined. The disorder is not necessarily quenched but it may evolve on a time scale that can be tuned. The annealed and quenched cases, obtained as limit cases are, respectively, the infinite range Ashkin-Teller (P=2 colors) model and the Hopfield neural net (finite number of patterns P=2). The intermediate-dynamics model behaves like a Hopfield associative memory in short time scales and like an Ashkin-Teller system in the long run. The time evolution of the order parameters is obtained from the master equations in the mean field approximation. *PACS : 05.50.+q, 87.10.e+10, 75.10 Hk, 64.60 Cn, 89.70.+c*

Competition between different interactions might lead to very interesting and complex behavior in models that arise in many different areas, from the physics of disordered systems and neural nets to e.g. economy, biology and cognitive psychology. A very useful approximation in the Statistical Mechanics study of disordered systems has been to consider the disorder completely static. The quenched disorder approximation sometimes simplifies the problems to a level amenable to analytical treatment. While the coupling "constants" in spin systems or synaptic couplings in neural nets are kept fixed the spins or neural activation evolve rapidly [1,2,3]. Another possibility, which receives the name of annealed approximation, is to let the disorder evolve in the same time scale as the spins or neural activation. The choice of the most suitable approximation in a given problem is based on physical or neurophysiological grounds. It is nevertheless quite obvious that sometimes it might be necessary to consider the case where the disorder evolves on a time scale intermediate between both limiting behaviors. The aim of this paper is to present a model system where the spins interact through couplings which themselves evolve with an intermediate characteristic dynamical time scale. This time scale can be tuned and the two limits, annealed and quenched can be recovered. Of course this type of problem is expected to be very hard in general. The simple nature of the model here presented can therefore be excused on the grounds that some simplifications are necessary if analytical results are expected.

A related problem has been recently studied in a quite different system and under different approximations by Coolen, Penney and Sherrington [4].

The basic observation behind the treatment here presented is the following. A system with at least two different classes of spins interacting through translationally invariant constant interactions resembles a disordered system if some of the classes evolve on a different time scale, much longer, say, than the others. Consider the master equation for the markovian time evolution of the probability distribution of the spin configurations (two classes). The Glauber transition probabilities can be thought of as the product of two terms, one the probabilty per unit time of choosing a given spin to be flipped, the other as the probability, once chosen, of being actually flipped. It is through this last term that the model equilibrium properties are determined, for example if they satisfy detailed balance (with the correct Gibbs distribution). To deal with intermediate dynamics the first factor has to be appropriately modifed so that one of the classes, in the average, is chosen with a different probabilty than the other. In this way, on a short time scale, the fast, or frequently chosen, spins evolve under a set of effective interactions that are almost quenched.

The method is better explained through a simple example but it is possible to extend it to more general settings. Consider the Hopfield model for an associative memory with two patterns. At each site $i = 1, \ldots, N$ there is an Ising variable $S_i = \pm 1$. The Hamiltonian is

$$H = \frac{1}{2N}\left(J_1(\sum_{i=1}^{N}\xi_i^1 S_i)^2 + J_1(\sum_{i=1}^{N}\xi_i^2 S_i)^2 + J_2(\sum_{i=1}^{N}\xi_i^1\xi_i^2)^2\right). \tag{1}$$

The two first terms are the usual hebbian contributions from each pattern while the third is just a constant if the disorder is taken as quenched. J_1 and J_2 are constant coupling constants. The introduction of the new sets of variables $\sigma_i = \pm 1$, $\tau_i \pm 1$ and $\mu_i \pm 1$, subject to the constraint $\sigma_i\mu_i = \tau_i$ at each site i and defined by

$$\sigma_i = \xi_i^1 S_i, \tag{2}$$

$$\mu_i = \xi_i^2 S_i, \tag{3}$$

$$\tau_i = \xi_i^1 \xi_i^2, \tag{4}$$

leads to the following form for the Hamiltonian

$$H = \frac{1}{2N}\left(J_1(\sum_{i=1}^{N}\sigma_i)^2 + J_1(\sum_{i=1}^{N}\mu_i)^2 + J_2(\sum_{i=1}^{N}\tau_i)^2\right). \tag{5}$$

Note that the form of this Hamiltonian is exactly the same as that of the infinite range symmetric Ashkin Teller model (ATM)[5]. But this is not the ATM if the $\{\tau_i\}$ are quenched. The difference is not in the form of the Hamiltonian but in the dynamics of the different degrees of freedom. In the ATM all the different classes of spins σ, μ, τ evolve under similar dynamics. Let $P(\{\sigma_i, \mu_i, \tau_i\}; t)$ be the probability of the ATM system being in a configuration $\{\sigma, \mu, \tau\}$ at time t. Its time evolution is given by the master equation:

$$P(\{\sigma,\mu,\tau\}; t+1) = P(\{\sigma,\mu,\tau\}; t) +$$

$$+\sum_i [P(\{f_i\sigma, f_i\mu, \tau\}; t)W(f_i\sigma, f_i\mu, \tau \to \sigma, \mu, \tau) - P(\{\sigma,\mu,\tau\}; t)W(\sigma,\mu,\tau \to f_i\sigma, f_i\mu, \tau)]$$
$$+\sum_i [P(\{f_i\sigma, \mu, f_i\tau\}; t)W(f_i\sigma, \mu, f_i\tau \to \sigma, \mu, \tau) - P(\{\sigma,\mu,\tau\}; t)W(\sigma,\mu,\tau \to f_i\sigma, \mu, f_i\tau)]$$
$$+\sum_i [P(\{\sigma, f_i\mu, f_i\tau\}; t)W(\sigma, f_i\mu, f_i\tau \to \sigma, \mu, \tau) - P(\{\sigma,\mu,\tau\}; t)W(\sigma,\mu,\tau \to \sigma, f_i\mu, f_i\tau)] \tag{6}$$

where f_i is the spin flip operator at site i. The W's are the transition rates and contain all the information about the system. The three terms in brackets, in the previous equation correspond to the increase or decrease of probability due to transitions into and out of a given state, from each of the possible spin flip types, that is flip of one of the three pairs: $\{\sigma_i, \mu_i\}$, $\{\sigma_i, \tau_i\}$ or $\{\mu_i, \tau_i\}$. Note that single spin flips are impossible since the constraint $\sigma_i\mu_i = \tau_i$ has to be satisfied at each site i. The choice of the transition probabilities is quite arbitrary, and the only requirement imposed is that the equilibrium distribution be the Gibbs distribution for the appropriate Hamiltonian. As usual, detailed balance ensures the correct equilibrium. For the ATM the Glauber-like probabilities are

$$W(f_i\sigma, f_i\mu, \tau \to \sigma, \mu, \tau)] = X(+++)/D,$$

$$W(\sigma, \mu, \tau \to f_i\sigma, f_i\mu, \tau)] = X(--+)/D,$$

$$W(\sigma, f_i\mu, f_i\tau \to \sigma, \mu, \tau)] = X(+++)/D,$$

$$W(\sigma, \mu, \tau \to \sigma, f_i\mu, f_i\tau)] = X(+--)/D,$$

$$W(f_i\sigma,\mu,f_i\tau \to \sigma,\mu,\tau)] = X(+++)/D,$$

$$W(\sigma,\mu,\tau \to f_i\sigma,\mu,f_i\tau)] = X(-+-)/D, \tag{7}$$

where the denominator is $D = X(+++) + X(--+) + X(+--) + X(-+-)$, and

$$X(\epsilon_1,\epsilon_2,\epsilon_3) = (e_\sigma)^{\epsilon_1\sigma_i}(e_\mu)^{\epsilon_2\mu_i}(e_\tau)^{\epsilon_3\tau_i}. \tag{8}$$

The ϵ_i are + or − and

$$e_\sigma = \exp(\frac{\beta J_1}{N}\sum_{j\neq i}\sigma_j)$$
$$e_\mu = \exp(\frac{\beta J_1}{N}\sum_{j\neq i}\mu_j)$$
$$e_\tau = \exp(\frac{\beta J_2}{N}\sum_{j\neq i}\tau_j) \tag{9}$$

Define the usual order parameters $m_\rho =< \frac{1}{N}\sum_i \rho_i >$, for $\rho = \sigma,\mu,\tau$ respectively where the angular brackets denote averages with respect to the probability distribution at time t. Multiplying by $\sum_{i=1}^N \sigma_i/N$ on both sides of the master equation, summing over all the configurations of $\{\sigma,\mu,\tau\}$, and making the usual single site mean field approximations, equations for the time evolution of the order parameters, are obtained:

$$\Delta m_\sigma(t+1) = -m_\sigma(t) + \frac{xyz + x/yz - y/zx - z/xy}{xyz + x/yz + y/zx + z/xy} \tag{10}$$

where $x = \exp(\beta J_1 m_\sigma)$, $y = \exp(\beta J_1 m_\mu)$ and $z = \exp(\beta J_2 m_\tau)$. The other two equations are obtained by cyclic permutations of x, y and z. In equilibrium the Δm_ρ's are zero and the fixed points are just the mean field equations . The phase diagram of this model (e.g.[5]) is shown in the $\beta J_1 > 0$, $\beta J_2 > 0$ space in fig(1)

A Monte Carlo simulation of the ATM model could proceed by choosing a random site i and then choosing with equal probability the pair of spins to be tentatively flipped with probabilities given by eq.(7). If the choice of the two pairs (σ,τ) and (μ,τ) is less probable than that of (σ,μ) then the system will evolve, in short time

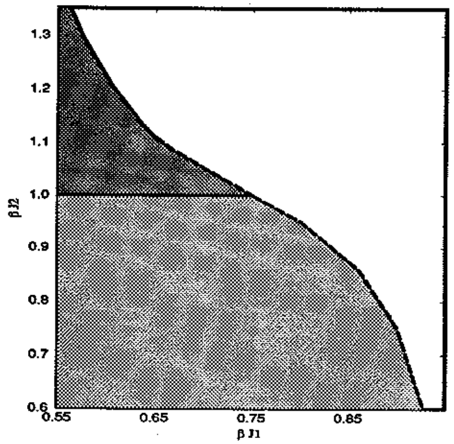

Figure 1: Phase Diagram of the ATM : Paramagnetic Phase in light grey. The dark region is the phase with $m_\tau \neq 0$ and the white region is ferromagnetic. The solid lines are second order transitions and the dashed line is a first order

scales, in an almost fixed configuration $\{\tau\}$. The asymptotic behavior of the system should nevertheless be characterized by the Gibbs distribution of the ATM, although it might take longer for the $\{\tau\}$ to equilibrate. A set of transition probabilities which: (i) can be interpreted as a probabilty per unit time w of choosing the pair (σ,μ) and u of choosing (μ,τ) or (σ,τ) times a transition probability, (ii) satisfy detailed balance for the ATM for $u, w \neq 0$, (iii) have the correct limits for $w = u$ (ATM) and $u = 0, w \neq 0$ (Hopfield model) and (iv) leads in a simple manner to mean field equations, is given below.

$$W(f_i\sigma, f_i\mu,\tau \to \sigma,\mu,\tau)] = wX(+++)/D_{\sigma\mu},$$

$$W(\sigma,\mu,\tau \to f_i\sigma, f_i\mu, \tau)] = wX(--+)/D_{\sigma\mu},$$
$$W(\sigma, f_i\mu, f_i\tau \to \sigma,\mu,\tau)] = uX(+++)/D_{\mu\tau},$$
$$W(\sigma,\mu,\tau \to \sigma, f_i\mu, f_i\tau)] = uX(+--)/D_{\mu\tau},$$
$$W(f_i\sigma, \mu, f_i\tau \to \sigma,\mu,\tau)] = uX(+++)/D_{\sigma\tau},$$
$$W(\sigma,\mu,\tau \to f_i\sigma, \mu, f_i\tau)] = uX(-+-)/D_{\sigma\tau}. \tag{11}$$

The X's are the same as before, while the denominators are contrived to simplify the algebra without spoiling the detailed balance. Notice that in each of the pairs of terms within the same brackets in eq(6) the denominators are the same. Defining

$$d(w,u) = w(e_\sigma e_\mu e_\tau + e_\sigma^{-1}e_\mu^{-1}e_\tau) + u(e_\sigma e_\mu^{-1}e_\tau^{-1} + e_\sigma^{-1}e_\mu e_\tau^{-1}) \tag{12}$$

then

$$\begin{aligned} D_{\sigma\mu}^{-1}(w,u) &= \frac{1}{2}\left[\frac{1+\tau_i}{d(w,u)} + \frac{1-\tau_i}{d(u,w)}\right] \\ D_{\mu\tau}^{-1}(w,u) &= \frac{1}{2}\left[\frac{1+\sigma_i}{d(w,u)} + \frac{1-\sigma_i}{d(u,w)}\right] \\ D_{\sigma\tau}^{-1}(w,u) &= \frac{1}{2}\left[\frac{1+\mu_i}{d(w,u)} + \frac{1-\mu_i}{d(u,w)}\right] \end{aligned} \tag{13}$$

The denominators associated with the transitions of a given pair of spins are invariant under the operator f_i acting on the pair of spins, thus ensuring conditon (iv) above. The same approximations that lead to the mean field equation (eq.(10)) of the ATM lead to the three evolution equations for the order parameters ($\rho = \sigma,\mu,\tau$)

$$\Delta m_\rho = -\sum_{\rho'=\sigma,\mu,\tau} m_{\rho'} F_{\rho'}^{\rho} + F_0^{\rho} \tag{14}$$

The F's are functions of $r = u/w$ and of (x,y,z) of eq.(10). For the σ equation, they are given below

$$\begin{aligned} F_0^\sigma &= \frac{1}{2}\frac{(xyz - z/xy) + r(xyz - y/zx)}{(xyz+z/xy)+r(x/yz+y/zx)} + \frac{1}{2}\frac{(x/yz - y/zx) + r(-z/xy + x/yz)}{r(xyz+z/xy)+(x/yz+y/zx)}, \\ F_\sigma^\sigma &= \frac{1}{2}\frac{(xyz + z/xy) + r(xyz + y/zx)}{(xyz+z/xy)+r(x/yz+y/zx)} + \frac{1}{2}\frac{(x/yz + y/zx) + r(z/xy + x/yz)}{r(xyz+z/xy)+(x/yz+y/zx)}, \\ F_\mu^\sigma &= \frac{1}{2}\frac{(xyz + z/xy) - r(xyz - y/zx)}{(xyz+z/xy)+r(x/yz+y/zx)} + \frac{1}{2}\frac{(x/yz + y/zx) + r(z/xy - x/yz)}{r(xyz+z/xy)+(x/yz+y/zx)}, \\ F_\tau^\sigma &= \frac{1}{2}\frac{(-xyz + z/xy) + r(xyz + y/zx)}{(xyz+z/xy)+r(x/yz+y/zx)} + \frac{1}{2}\frac{(x/yz - y/zx) - r(z/xy + x/yz)}{r(xyz+z/xy)+(x/yz+y/zx)}. \end{aligned} \tag{15}$$

The coeficients of the μ equation are obtained in the same way, and as expected from symmetry, satisfy the following relations

$$\begin{aligned} F_0^\mu(x,y,z) &= F_0^\sigma(y,x,z), \\ F_\sigma^\mu(x,y,z) &= F_\mu^\sigma(y,x,z), \\ F_\mu^\mu(x,y,z) &= F_\sigma^\sigma(y,x,z), \\ F_\tau^\mu(x,y,z) &= F_\tau^\sigma(y,x,z). \end{aligned} \tag{16}$$

And finally, for the τ equation

$$\begin{aligned} F_0^\tau &= \frac{r}{2}\left[\frac{(2xyz - x/yz - y/zx)}{(xyz+z/xy)+r(x/yz+y/zx)} + \frac{(2z/xy - x/yz - y/zx}{r(xyz+z/xy)+(x/yz+y/zx)}\right], \\ F_\sigma^\tau &= \frac{r}{2}\left[\frac{(x/yz+y/zx)}{(xyz+z/xy)+r(x/yz+y/zx)} - \frac{(x/yz+y/zx)}{r(xyz+z/xy)+(x/yz+y/zx)}\right], \\ F_\mu^\tau &= F_\sigma^\tau, \\ F_\tau^\sigma &= \frac{r}{2}\left[\frac{2xyz + x/yz + y/zx}{(xyz+z/xy)+r(x/yz+y/zx)} + \frac{2z/xy + x/yz + y/zx}{r(xyz+z/xy)+(x/yz+y/zx)}\right]. \end{aligned} \tag{17}$$

In the annealed limit, $r = 1$, when the two dynamic scales are the same, then eq.(10) together with its permutations are recovered, while for $r = 0$, the quenched limit, the evolution is given by

$$\Delta m_\sigma(t+1) = -m_\sigma(t) + \frac{1+m_\tau}{2}\tanh(m_\sigma + m_\mu) + \frac{1-m_\tau}{2}\tanh(m_\sigma - m_\mu), \tag{18}$$

$$\Delta m_\mu(t+1) = -m_\mu(t) + \frac{1+m_\tau}{2}\tanh(m_\sigma + m_\mu) - \frac{1-m_\tau}{2}\tanh(m_\sigma - m_\mu), \tag{19}$$

$\Delta m_\tau = 0.$

These are the correct evolution equations for the order parameters in the Hopfield model with two memory patterns and with correlation $< \xi^1 \xi^2 >= m_\tau$.

Results from the numerical iteration of the evolution equations (eqs.(14)) are shown in figs(2)-(3). These are the flows in the m_σ and m_μ plane, for different sets of initial conditions.

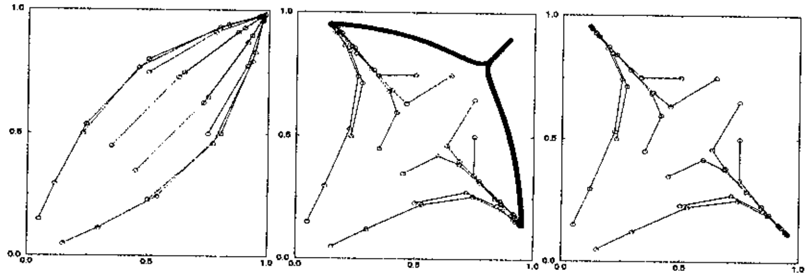

Figure 2: m_σ vs. m_μ initial value of $m_\tau = 0.1$ ($\beta J_1 = 2.0, \beta J_2 = 0.5$.) (a) The ATM, $r = 1$, (b)$r = 0.01$, the system evolves to the Hopfield fixed points in a short time scale and very slowly (thick line is due to crowding of symbols) to the ATM. (c) The Hopfield model, $r = 0$

In general for $r \neq 0$, the system behaves in short time scales as a Hopfield associative memory and the order parameters (m_σ, m_μ) flows are directed towards the Hopfield fixed points. These, however should be better called pseudo-fixed points, since they too are evolving, in the slower time scale, as the correlation of the patterns, $< \xi^1 \xi^2 >= m_\tau$, evolves to its equilibrium value $m_\tau^\infty(\beta J_1, \beta J_2)$. In figs(2a-c) the couplings are such that the annealed (ATM) system is in the ferromagnetic phase. It can be seen that the system flows to a fixed point where all the order parameters, including m_τ are large (2a). At this value of βJ_1, the Hopfield model can retrieve and distinguish both patterns

(2c). In the intermediate dynamics case (2b) the "memory"patterns are very slowly becoming more correlated and so the system, which behaves in the fast scale as an associative memory, eventually cannot separate the two patterns anymore, but it still remains in a ferromagnetic (mixture) phase.

The transition from the paramagnetic to ferromagnetic phase and part of the line separating the two ordered phases in the ATM are first order (dashed line fig.1), so there is a region where simple iteration of the mean field equations leads to results that depend on the initial values. An example in the region of spinodal decomposition, sensitive to the initial values, is shown in fig. 3.a.

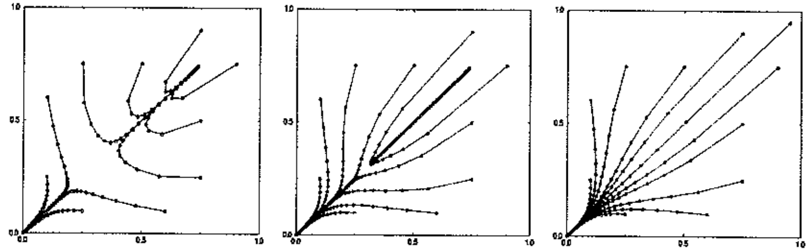

Figure 3: Same as figure 2. initial value of $m_\tau = 0.1$ ($\beta J_1 = 0.85, \beta J_2 = 0.85$.) (a) $r = 1$ ATM, (b)$r = 0.15$, (c) $r = 0$

The basin of attraction of the ferromagnetic fixed point is reduced when the quenching is increased. The flow is almost all towards the fixed point associated to the Hopfield model for the initial value of the patterns' correlation $m_\tau =< \xi^1 \xi^2 >$.

In fig.(4) the ATM is shown in the paramagnetic phase. For $m_\tau(0)$ sufficiently large the Hopfield model is in the ferromagnetic mixed phase. The intermediate dynamics model flows (4.b) toward the ferromagnetic pseudo fixed point at the begining of the

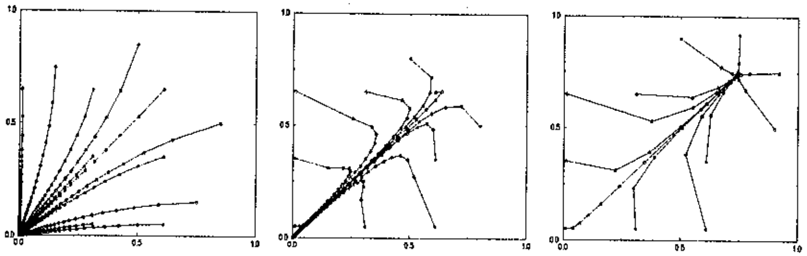

Figure 4: Same as figure 2. initial value of $m_r = 0.7$ ($\beta J_1 = 0.9, \beta J_2 = 0.001$. (a) $r = 1$ ATM, (b)$r = 0.2$, (c) $r = 0$

iterations and eventually turns towards the origin.

In conclusion a method for analysing the behavior of unquenched disorder, in a simple model has been presented. The system with intermediate dynamics flows rapidly to the pseudo equilibrium of the quenched model and follows the evolution of these pseudo fixed points as they slowly approach the fixed points of the annealed system. Other models with two different time scales can be treated using the same methods, the extension to the Hopfield model with a finite P is the next natural step. The diluted Ising model with unquenched disorder is now under investigation. The question of how to treat unquenched random disorder for more complex systems which in the quenched limit have a spinglass phase and thus require more sophisticated methods (e.g. replica or cavity methods) than the Hopfield model with finite P remains, however unanswered. It is possible that the confluence of pseudo fixed points, as in figure (2b), occurs in a hierarchy of steps, remanescent of the large number of relaxation times of the quenched complex system.

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