UNIVERSIDADE DE SÃO PAULO

INSTITUTO DE FÍSICA
CAIXA POSTAL 20516
01498-970 SÃO PAULO - SP
BRASIL

PUBLICAÇÕES

IFUSP/P-1072

EINSTEIN-CARTAN THEORY OF GRAVITY REVISITED

Alberto Saa
Instituto de Física, Universidade de São Paulo

Setembro/1993

IFUSP/P-1072
September, 1993

Einstein-Cartan theory of gravity revisited

Alberto Saa

Instituto de Física

Universidade de São Paulo, Caixa Postal 20516

01498-970 São Paulo, SP

Brazil

The role of space-time torsion in general relativity is reviewed in accordance with some recent results on the subject. It is shown that, according to the connection compatibility condition, the usual Riemannian volume element is not appropriate in the presence of torsion. A new volume element is proposed and used in the Lagrangian formulation for Einstein-Cartan theory of gravity. The dynamical equations for the space-time geometry and for matter fields are obtained, and some of their new predictions and features are discussed. In particular, one has that torsion propagates and that gauge fields can interact with torsion without the breaking of gauge invariance. It is shown also that the new Einstein-Hilbert action for Einstein-Cartan theory may provide a physical interpretation for dilaton gravity in terms of the non-riemannian structure of space-time.

04.50.+h, 04.90.+e

Typeset Using *REVTEX*

I. INTRODUCTION

The Einstein-Cartan (EC) theory of gravity is a simple and natural generalization of general relativity. It was proposed in the early twenties by Élie Cartan, who, before the introduction of the modern concept of spin, suggested the possibility of relating space-time torsion to an intrinsic angular momentum of matter. Nowadays, it is known that EC theory of gravity arises naturally from the local gauge theory for the Poincaré's group, and that it is in accordance with the available experimental data[1, 2]. In such theory, space-time is assumed to be a Riemann-Cartan (RC) manifold, and their dynamical equations are gotten via a minimal action principle of the Einstein-Hilbert type,

$$S_{\mathrm{grav}} = -\int d\mathrm{vol}\, R = -\int d^4x\sqrt{-g}\, R. \tag{1.1}$$

Although in the macroscopic world space-time is usually assumed to be torsionless, there are good reasons to believe that in the microscopic level space-time must have a non-vanishing torsion[3], and so, microscopic gravitational interactions should be described by EC theory. The interest in such theories of gravity has grown in recent years also due to their role in the semi-classical description of quantum fields on curved spaces, see for example [4] and references therein.

In this work it will be shown that the usual Riemannian volume element, which is used in the construction of the Einstein-Hilbert action, does not play in RC spacetimes the same role that it does in a Riemannian space-time. Endowed with an appropriate volume element, EC theory will predict new effects. The main new predictions are that torsion will propagate and that gauge fields can interact with torsion without breaking of gauge invariance. It is shown also that the proposed model can give a satisfactory physical interpretation for the dilaton gravity that comes from string theory, in terms of the non-riemannian structure of space-time.

The work is organized in four sections, where the first is this introduction. In the Sect. 2, RC geometry is briefly introduced and the problem of compatibility of volume elements is discussed. In the Sect. 3, the results of Sect. 2 are used to propose some modifications to EC theory of gravity, and the new equations for the vacuum is presented. Matter fields are also considered in this section, namely, scalar, gauge, and fermion fields. In the last section, it is shown that the proposed model requires some modifications in the traditional point of view that only fermion fields can be source to the non-riemannian structure of space-time, and further developments are discussed.

II. RC MANIFOLDS AND COMPATIBLE VOLUME ELEMENTS

The RC space-time (U_4) is a differentiable four dimensional manifold endowed with a metric tensor $g_{\alpha\beta}(x)$ and with a metric-compatible connection $\Gamma^{\mu}_{\alpha\beta}$, which is non-symmetrical in its lower indices. The following conventions will be adopted in this work: $\text{sign}(g_{\mu\nu}) = (+,-,-...)$, $R_{\alpha\nu\mu}{}^{\beta} = \partial_\alpha\Gamma^{\beta}_{\nu\mu} + \Gamma^{\beta}_{\alpha\rho}\Gamma^{\rho}_{\nu\mu} - (\alpha \leftrightarrow \nu)$, and $R_{\nu\mu} = R_{\alpha\nu\mu}{}^{\alpha}$. The anti-symmetric part of the connection defines a new tensor, the torsion tensor,

$$S_{\alpha\beta}{}^{\gamma} = \frac{1}{2}\left(\Gamma^{\gamma}_{\alpha\beta} - \Gamma^{\gamma}_{\beta\alpha}\right). \tag{2.1}$$

The metric-compatible connection can be written as

$$\Gamma^{\gamma}_{\alpha\beta} = \left\{ {\gamma \atop \alpha\beta} \right\} - K_{\alpha\beta}{}^{\gamma}, \tag{2.2}$$

where $\left\{ {\gamma \atop \alpha\beta} \right\}$ are the usual Christoffel symbols from Riemannian space-time (V_4), and $K_{\alpha\beta}{}^{\gamma}$ is the contorsion tensor, which is given in terms of the torsion tensor by

$$K_{\alpha\beta}{}^{\gamma} = -S_{\alpha\beta}{}^{\gamma} + S_{\beta}{}^{\gamma}{}_{\alpha} - S^{\gamma}{}_{\alpha\beta}. \tag{2.3}$$

The connection (2.2) is used to define the covariant derivative of a contravariant vector,

$$D_\nu A^\mu = \partial_\nu A^\mu + \Gamma^{\mu}_{\nu\rho}A^\rho, \tag{2.4}$$

and it is also important to our purposes to introduce the covariant derivative of a density $f(x)$,

$$D_\mu f(x) = \partial_\mu f(x) - \Gamma^{\rho}_{\rho\mu} f(x). \tag{2.5}$$

The contorsion tensor (2.3) can be covariantly split in a traceless part and in a trace,

$$K_{\alpha\beta\gamma} = \tilde{K}_{\alpha\beta\gamma} - \frac{2}{3}\left(g_{\alpha\gamma}S_\beta - g_{\alpha\beta}S_\gamma\right), \tag{2.6}$$

where $\tilde{K}_{\alpha\beta\gamma}$ is the traceless part and S_β is the trace of the torsion tensor, $S_\beta = S_{\alpha\beta}{}^{\alpha}$. The U_4 curvature tensor is defined by using the full connection (2.2), and it is given by:

$$R_{\alpha\nu\mu}{}^{\beta} = \partial_\alpha\Gamma^{\beta}_{\nu\mu} - \partial_\nu\Gamma^{\beta}_{\alpha\mu} + \Gamma^{\beta}_{\alpha\rho}\Gamma^{\rho}_{\nu\mu} - \Gamma^{\beta}_{\nu\rho}\Gamma^{\rho}_{\alpha\mu}. \tag{2.7}$$

After some algebraic manipulations we get the following expression for the scalar of curvature

$$R = g^{\mu\nu}R_{\alpha\mu\nu}{}^{\alpha} = R^{V_4} - 4D_\mu S^\mu + \frac{16}{3}S_\mu S^\mu - \tilde{K}_{\nu\rho\alpha}\tilde{K}^{\alpha\nu\rho}, \tag{2.8}$$

where R^{V_4} is the V_4 scalar of curvature, calculated from the Christoffel symbols.

In order to define a general covariant volume element in a manifold, it is necessary to introduce a density quantity $f(x)$,

$$d^4x \to f(x)d^4x = d\text{vol}. \tag{2.9}$$

This is done in order to compensate the Jacobian that arises from the transformation law of the usual volume element d^4x under a coordinate transformation, and usually, the density $f(x) = \sqrt{-g}$ is took to this purpose.

There are natural properties that a volume element shall exhibit. In V_4, the usual covariant volume element

$$\mathrm{dvol} = \sqrt{-g}\, d^4x, \tag{2.10}$$

is "parallel", in the sense that the scalar density $\sqrt{-g}$ obeys

$$D_\mu^{V_4}\sqrt{-g} = 0, \tag{2.11}$$

where $D_\mu^{V_4}$ is the V_4 covariant derivative, defined using the Christoffel symbols $\left\{ {\gamma \atop \alpha\beta} \right\}$. In mathematical precise language, the volume element (2.10) is said to be compatible with the connection in V_4 manifolds. One can infer that the volume element (2.10) is not "parallel" in U_4[5], since

$$D_\mu\sqrt{-g} = \partial_\mu\sqrt{-g} - \Gamma^\rho_{\rho\mu}\sqrt{-g} = -2S_\mu\sqrt{-g}, \tag{2.12}$$

as it can be checked using Christoffel symbols properties. This is the main point that we wish to stress, it is the basic argument to our claim that the usual volume element (2.10) is not appropriate in the presence of torsion.

The question that arises now is if it is possible to define a V_4-like, *i.e.* "parallel", volume element in U_4 manifolds. In order to do it, one needs to find out a density $f(x)$ such that $D_\mu f(x) = 0$. Such density exists only if the trace of the torsion tensor, S_μ, can be obtained from a scalar potential

$$S_\beta(x) = \partial_\beta\Theta(x), \tag{2.13}$$

and in this case we have $f(x) = e^{2\Theta}\sqrt{-g}$, and

$$\mathrm{dvol} = e^{2\Theta}\sqrt{-g}\, d^4x, \tag{2.14}$$

that is the "parallel" covariant U_4 volume element, or in another words, the volume element (2.14) is compatible with the connection in RC manifolds obeying (2.13).

With the volume element (2.14), we have the following generalized Gauss' formula

$$\int \mathrm{dvol}\, D_\mu V^\mu = \int d^4x \partial_\mu e^{2\Theta}\sqrt{-g}V^\mu = \text{ surface term}, \tag{2.15}$$

where we used that

$$\Gamma^\rho_{\rho\mu} = \partial_\mu \ln e^{2\Theta}\sqrt{-g} \tag{2.16}$$

under the hypothesis (2.13). It is easy to see that one cannot have a generalized Gauss' formula of the type (2.15) if the torsion does not obey (2.13). We will return to discuss the role of the condition (2.13) in the last section.

III. EINSTEIN-CARTAN THEORY OF GRAVITY

In this section, Einstein-Cartan theory of gravity will be reconstructed by using the results of the Sect. 2. Space-time will be assumed to be a Riemann-Cartan manifold with the "parallel" volume element (2.14), and of course, it is implicit the restriction that the trace of the torsion tensor is derived from a scalar potential, condition (2.13). With this hypothesis, EC theory of gravity will predict new effects, and they will be pointed out in the following subsections.

A. Vacuum equations

According to our hypothesis, in order to get the U_4 gravity equations we will assume that they can be obtained from an Einstein-Hilbert action using the scalar of curvature (2.8), the condition (2.13), and the volume element (2.14),

$$\begin{aligned} S_{\mathrm{grav}} &= -\int d^4x e^{2\Theta}\sqrt{-g}\, R \\ &= -\int d^4x e^{2\Theta}\sqrt{-g}\left(R^{V_4} + \frac{16}{3}\partial_\mu\Theta\partial^\mu\Theta - \tilde{K}_{\nu\rho\alpha}\tilde{K}^{\alpha\nu\rho}\right) + \text{surface terms.} \end{aligned} \tag{3.1}$$

where the generalized Gauss' formula (2.15) was used.

The equations for the $g^{\mu\nu}$, Θ, and $\tilde{K}_{\nu\rho\alpha}$ fields follow from the minimization of the action (3.1). The variations of $g^{\mu\nu}$ and $S_{\mu\nu}{}^\rho$ are assumed to vanish in the boundary. The equation $\frac{\delta S_{\mathrm{grav}}}{\delta \tilde{K}_{\nu\rho\alpha}} = 0$ implies that $\tilde{K}^{\nu\rho\alpha} = 0$, $\frac{\delta S_{\mathrm{grav}}}{\delta \tilde{K}_{\nu\rho\alpha}}$ standing for the Euler-Lagrange equations for $\delta\tilde{K}_{\nu\rho\alpha}$. For the other equations we have

$$-\frac{e^{-2\Theta}}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}S_{\rm grav}\Big|_{\tilde{K}=0} = R^{V_4}_{\mu\nu} - 2D_\mu\partial_\nu\Theta - \frac{1}{2}g_{\mu\nu}\left(R^{V_4} + \frac{8}{3}\partial_\rho\Theta\partial^\rho\Theta - 4\Box\Theta\right) = 0, \tag{3.2}$$
$$-\frac{e^{-2\Theta}}{2\sqrt{-g}}\frac{\delta}{\delta\Theta}S_{\rm grav}\Big|_{\tilde{K}=0} = R^{V_4} + \frac{16}{3}\left(\partial_\mu\Theta\partial^\mu\Theta - \Box\Theta\right) = 0,$$

where $R^{V_4}_{\mu\nu}$ is the V_4 Ricci tensor, calculated using the Christoffel symbols, and $\Box$ is the U_4 d'Alambertian operator, $\Box = D_\mu D^\mu$.

Taking the trace of the first equation of (3.2),

$$R^{V_4} + \frac{16}{3}\partial_\mu\Theta\partial^\mu\Theta = 6\Box\Theta, \tag{3.3}$$

and using it, one finally obtains the new U_4 gravity equations for the vacuum,

$$R^{V_4}_{\mu\nu} = 2D_\mu\partial_\nu\Theta - \frac{4}{3}g_{\mu\nu}\partial_\rho\Theta\partial^\rho\Theta = 2D_\mu S_\nu - \frac{4}{3}g_{\mu\nu}S_\rho S^\rho,$$
$$\Box\Theta = \frac{e^{-2\Theta}}{\sqrt{-g}}\partial_\mu e^{2\Theta}\sqrt{-g}\partial^\mu\Theta = D_\mu S^\mu = 0, \tag{3.4}$$
$$\tilde{K}_{\alpha\beta\gamma} = 0.$$

The vacuum equations (3.4) point out new features of EC theory. It is clear that torsion, described by the last two equations, propagates. The torsion mediated interactions are not of contact type anymore. The traceless tensor $\tilde{K}_{\alpha\beta\gamma}$ is zero for the vacuum, and only the trace S_μ can be non-vanishing outside matter distributions. As it is expected, the gravity field configuration for the vacuum is determined only by boundary conditions, and if due to such conditions we have that $S_\mu = 0$, our equations reduce to the usual vacuum equations, $S_{\alpha\gamma\beta} = 0$, and $R^{V_4}_{\alpha\beta} = 0$.

In a first sight, it seems that the equations (3.4) can be written without using the scalar potential $\Theta(x)$, so it can be questioned if they are valid for manifolds not obeying (2.13). The answer is clearly negative, since the first term in the right-handed side of the first equation is symmetrical under the change $(\mu \leftrightarrow \nu)$ only if the condition (2.13) holds.

Another remarkable consequence of the proposed model, is that the action (3.1) can provide a physical interpretation for the dilaton gravity in terms of the non-riemannian structure of space time[6]. Remember that dilaton gravity arises as consequence of conformal invariance in a quantum analysis up to one-loop order for string theory in background fields[7, 8], and its equations can be gotten from the following effective action:

$$S = -\int d^Nx\sqrt{-g}e^{-2\Phi}\left(R^{V_4} + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{12}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma} - \frac{N-26}{3\alpha'}\right), \tag{3.5}$$

where N is the space-time dimension, Φ is the dilaton field, and $H_{\alpha\beta\gamma}$ is the totally anti-symmetrical tensor, which is given by

$$H_{\alpha\beta\gamma} = \partial_\alpha B_{\beta\gamma} + \partial_\gamma B_{\alpha\beta} + \partial_\beta B_{\gamma\alpha}, \tag{3.6}$$

where $B_{\mu\nu}$ is the massless anti-symmetrical tensor background field. It is well known from sigma model context, that $B_{\mu\nu}$ is related to the torsion of space-time[9], but up to now there was no geometrical interpretation for the dilaton field Φ.

The similarity between (3.1) and (3.5) is clear. They can be identified if one assumes that:

$$\Theta(x) = -\Phi(x),$$
$$\frac{1}{12}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma} = \tilde{K}_{\nu\rho\alpha}\tilde{K}^{\alpha\nu\rho} - \frac{4}{3}\partial_\mu\Phi\partial^\mu\Phi = K_{\nu\rho\alpha}K^{\alpha\nu\rho} \tag{3.7}$$
$$\Lambda = -\frac{N-26}{3\alpha'},$$

where Λ stands for a cosmological constant, not present in the original action (3.1). The second equation of (3.7) can be used to write the contorsion tensor in terms of the tensor $H_{\alpha\beta\gamma}$, which is consistent with the result that $H_{\alpha\beta\gamma}$ is related to torsion[9]. The case $\Phi = 0$ is namely the prediction from sigma models. The interpretation of $K_{\alpha\gamma\beta}$ in the usual frame of EC gravity is problematical, since the totally anti-symmetrical tensor $H_{\alpha\beta\gamma}$ is invariant under the "gauge" transformation

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu, \tag{3.8}$$

that is a consequence of this definition (3.6), and there are no reasons *a priori* to expect such behavior of the contorsion tensor, or what is equivalent, it is not expected in the frame of EC theory that the contorsion tensor can be derived from an anti-symmetrical field like $B_{\mu\nu}$ in (3.6). However, it is clear the relation between the dilaton field Φ and the potential for the trace of the torsion tensor Θ, which allow us to interpret the dilaton field as being part of the non-riemannian structure of space-time.

B. Scalar fields

The first step to introduce matter fields in our discussion will be the description of scalar fields on RC manifolds. In order to do it, we will use minimal coupling procedure (MCP), which consists; given a Lorentz invariant action, to change the usual derivatives by covariant ones, the Lorentz metric tensor by the general one, and to introduce the covariant volume element, that according to our hypothesis will be given by (2.14). Using MCP for a massless scalar field one gets

$$\begin{aligned} S = S_{\rm grav} + S_{\rm scal} &= -\int d^4x e^{2\Theta}\sqrt{-g}\left(R - \frac{g^{\mu\nu}}{2}\partial_\mu\varphi\partial_\nu\varphi\right) \\ &= -\int d^4x e^{2\Theta}\sqrt{-g}\left(R^{V_4} + \frac{16}{3}\partial_\mu\Theta\partial^\mu\Theta - \tilde{K}_{\nu\rho\alpha}\tilde{K}^{\alpha\nu\rho} - \frac{g^{\mu\nu}}{2}\partial_\mu\varphi\partial_\nu\varphi\right), \end{aligned} \tag{3.9}$$

where surface terms were discarded. The equations for this case are obtained by varying (3.9) with respect to φ, $g^{\mu\nu}$, Θ, and $\tilde{K}_{\alpha\beta\gamma}$. As in the vacuum case, the equation $\frac{\delta S}{\delta \tilde{K}} = 0$ implies $\tilde{K} = 0$. Taking it into account we have

$$\begin{aligned} -\frac{e^{-2\Theta}}{\sqrt{-g}}\frac{\delta S}{\delta\varphi}\bigg|_{\tilde{K}=0} &= \frac{e^{-2\Theta}}{\sqrt{-g}}\partial_\mu e^{2\Theta}\sqrt{-g}\partial^\mu\varphi = \Box\varphi = 0, \\ -\frac{e^{-2\Theta}}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}\bigg|_{\tilde{K}=0} &= R^{V_4}_{\mu\nu} - 2D_\mu S_\nu - \frac{1}{2}g_{\mu\nu}\left(R^{V_4} + \frac{8}{3}S_\rho S^\rho - 4D_\rho S^\rho\right) \\ &\quad -\frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi + \frac{1}{4}g_{\mu\nu}\partial_\rho\varphi\partial^\rho\varphi = 0, \end{aligned} \tag{3.10}$$

$$-\frac{e^{-2\Theta}}{2\sqrt{-g}}\frac{\delta S}{\delta\Theta}\bigg|_{\tilde{K}=0} = R^{V_4} + \frac{16}{3}\left(S_\mu S^\mu - D_\mu S^\mu\right) - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi = 0.$$

Taking the trace of the second equation of (3.10),

$$R^{V_4} + \frac{16}{3}S_\mu S^\mu = 6D_\mu S^\mu + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi, \tag{3.11}$$

and using it, we get the following set of equations for the massless scalar case

$$\begin{aligned} \Box\varphi &= 0, \\ R^{V_4}_{\mu\nu} &= 2D_\mu S_\nu - \frac{4}{3}g_{\mu\nu}S_\rho S^\rho + \frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi, \\ D_\mu S^\mu &= 0, \\ \tilde{K}_{\alpha\beta\gamma} &= 0. \end{aligned} \tag{3.12}$$

As one can see, the torsion equations have the same form than the ones of the vacuum case, (3.4). Any contribution to the torsion will be due to boundary conditions, and not due to the scalar field itself. It means that if such boundary conditions imply that $S_\mu = 0$, the equations for the fields φ and $g_{\mu\nu}$ will be the same ones of the general relativity. One can interpret this by saying that, even feeling the torsion (see the second equation of (3.12)), massless scalar fields do not produce it. Such behavior is compatible with the idea that torsion must be governed by spin distributions.

However, considering massive scalar fields,

$$S_{\rm scal} = \int d^4x e^{2\Theta}\sqrt{-g}\left(\frac{g^{\mu\nu}}{2}\partial_\mu\varphi\partial_\nu\varphi - \frac{m^2}{2}\varphi^2\right), \tag{3.13}$$

we have the following set of equations instead of (3.12)

$$\begin{aligned} (\Box + m^2)\varphi &= 0, \\ R^{V_4}_{\mu\nu} &= 2D_\mu S_\nu - \frac{4}{3}g_{\mu\nu}S_\rho S^\rho + \frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}m^2\varphi^2, \\ D_\mu S^\mu &= \frac{3}{4}m^2\varphi^2, \\ \tilde{K}_{\alpha\beta\gamma} &= 0. \end{aligned} \tag{3.14}$$

The equation for the trace of the torsion tensor is different than the one of the vacuum case, we have that massive scalar fields can produce torsion. In contrast to the massless case, the equations (3.14) do not admit as solution $S_\mu = 0$ for non-vanishing φ. This is in disagreement with the traditional belief that torsion must be governed by spin distributions. We will return to this point in the last section.

C. Gauge fields

We need to be careful with the use of MCP to gauge fields. We will restrict ourselves to the abelian case in this work, non-abelian gauge fields will bring some technical difficulties that will not contribute to the understanding of the basic problems of gauge fields on Riemann-Cartan space-times.

It is well known that Maxwell field can be described by the differential 2-form

$$F = dA = d(A_\alpha dx^\alpha) = \frac{1}{2}F_{\alpha\beta}dx^\alpha \wedge dx^\beta, \tag{3.15}$$

where A is the (local) potential 1-form, and $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the usual electromagnetic tensor. It is important to stress that the forms F and A are covariant objects in any differentiable manifolds. Maxwell's equations can be written in Minkowski space-time in terms of exterior calculus as

$$\begin{aligned} dF &= 0, \\ d{}^*F &= 4\pi{}^*J, \end{aligned} \tag{3.16}$$

where * stands for the Hodge star operator and J is the current 1-form, $J = J_\alpha dx^\alpha$. The first equation in (3.16) is a consequence of the definition (3.15) and of Poincaré's lemma. In terms of components, one has the familiar homogeneous and non-homogeneous Maxwell's equations,

$$\begin{aligned} \partial_{[\gamma}F_{\alpha\beta]} &= 0, \\ \partial_\mu F^{\nu\mu} &= 4\pi J^\nu, \end{aligned} \tag{3.17}$$

where $_{[\ \ ]}$ means antisymmetrization. We know also that the non-homogenous equation follows from the minimization of the following action

$$S = -\int\left(4\pi{}^*J\wedge A + \frac{1}{2}F\wedge{}^*F\right) = \int d^4x\left(4\pi J^\alpha A_\alpha - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\right). \tag{3.18}$$

If one tries to cast (3.18) in a covariant way by using MCP in the tensorial quantities, we have that Maxwell tensor will be given by

$$F_{\alpha\beta} \to \tilde{F}_{\alpha\beta} = F_{\alpha\beta} - 2S_{\alpha\beta}{}^\rho A_\rho, \tag{3.19}$$

which explicitly breaks gauge invariance. With this analysis, it usually arises the conclusion that gauge fields cannot interact minimally with Einstein-Cartan gravity. We would like also to stress another undesired consequence, also related to the losing of gauge symmetry, of the use of MCP in the tensorial quantities. The homogeneous Maxwell's equation, the first of (3.17), does not come from a Lagrangian, and of course, if we choose to use MCP in the tensorial quantities we need also apply MCP to it. We get

$$\partial_{[\alpha}\tilde{F}_{\beta\gamma]} + 2S_{[\alpha\beta}{}^\rho\tilde{F}_{\gamma]\rho} = 0, \tag{3.20}$$

where $\tilde{F}_{\alpha\beta}$ is given by (3.19). One can see that (3.20) has no general solution for arbitrary $S_{\alpha\beta}{}^\rho$. Besides the losing of gauge symmetry, the use o MCP in the tensorial quantities also leads to a non consistent homogeneous equation. One has enough arguments to avoid the use of MCP in the tensorial quantities for gauge fields.

However, MCP can be successfully applied for general gauge fields (abelian or not) in the differential form quantities [10]. As its consequence, one has that the homogeneous equations is already in a covariant form in any differentiable manifold, and that the covariant non-homogeneous equations can be gotten from a Lagrangian obtained only by changing the metric tensor and by introducing the "parallel" volume element in the Minkowskian action (3.18). Considering the case where $J^\mu = 0$, we have the following action to describe the interaction of Maxwell fields and Einstein-Cartan gravity

$$S = S_{\rm grav} + S_{\rm Maxw} = -\int d^4x e^{2\Theta}\sqrt{-g}\left(R + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right). \tag{3.21}$$

As in the previous cases, the equation $\tilde{K}_{\alpha\beta\gamma} = 0$ follows from the minimization of (3.21). The other equations will be

$$\begin{aligned}
&\frac{e^{-2\Theta}}{\sqrt{-g}}\partial_\mu e^{2\Theta}\sqrt{-g}F^{\nu\mu} = 0,\\
&R_{\mu\nu}^{V_4} = 2D_\mu S_\nu - \frac{4}{3}g_{\mu\nu}S_\rho S^\rho - \frac{1}{2}\left(F_{\mu\alpha}F_\nu{}^\alpha + \frac{1}{2}g_{\mu\nu}F_{\omega\rho}F^{\omega\rho}\right),\\
&D_\mu S^\mu = -\frac{3}{8}F_{\mu\nu}F^{\mu\nu}.
\end{aligned} \tag{3.22}$$

One can see that the equations (3.22) are invariant under the usual $U(1)$ gauge transformations. It is also clear from the equations (3.22) that Maxwell fields interact with the non-Riemannian structure of space-time. We have that, as in the massive scalar case, the equations do not admit as solution $S_\mu = 0$ for arbitrary $F_{\alpha\beta}$, Maxwell fields are also sources to the space-time torsion. Similar results can be obtained also for non-abelian gauge fields[10].

D. Fermion fields

The Lagrangian for a (Dirac) fermion field with mass m in the Minkowski space-time is given by

$$\mathcal{L}_{\rm F} = \frac{i}{2}\left(\overline{\psi}\gamma^a\partial_a\psi - \left(\partial_a\overline{\psi}\right)\gamma^a\psi\right) - m\overline{\psi}\psi, \tag{3.23}$$

where γ^a are the Dirac matrices and $\overline{\psi} = \psi^\dagger\gamma^0$. Greek indices denote space-time coordinates (holonomic), and roman ones locally flat coordinates (non-holonomic). It is well known[1, 2] that in order to cast (3.23) in a covariant way, one needs to introduce the vierbein field, $e_a^\mu(x)$, and to generalize the Dirac matrices,

$\gamma^\mu(x) = e_a^\mu(x)\gamma^a$. The partial derivatives also must be generalized with the introduction of the spinorial connection ω_μ,

$$\begin{aligned}
\partial_\mu\psi &\rightarrow \nabla_\mu\psi = \partial_\mu\psi + \omega_\mu\psi,\\
\partial_\mu\overline{\psi} &\rightarrow \nabla_\mu\overline{\psi} = \partial_\mu\overline{\psi} - \overline{\psi}\omega_\mu,
\end{aligned} \tag{3.24}$$

where the spinorial connection is given by

$$\begin{aligned}
\omega_\mu &= \frac{1}{8}[\gamma^a,\gamma^b]e_a^\nu\left(\partial_\mu e_{\nu b} - \Gamma_{\mu\nu}^\rho e_{\rho b}\right)\\
&= \frac{1}{8}\left(\gamma^\nu\partial_\mu\gamma_\nu - \left(\partial_\mu\gamma_\nu\right)\gamma^\nu - [\gamma^\nu,\gamma_\rho]\Gamma_{\mu\nu}^\rho\right).
\end{aligned} \tag{3.25}$$

The last step, according to our hypothesis, shall be the introduction of the "parallel" volume element, and after that one gets the following action for fermion fields on RC manifolds

$$S_{\rm F} = \int d^4x e^{2\Theta}\sqrt{-g}\left\{\frac{i}{2}\left(\overline{\psi}\gamma^\mu(x)\nabla_\mu\psi - \left(\nabla_\mu\overline{\psi}\right)\gamma^\mu(x)\psi\right) - m\overline{\psi}\psi\right\}. \tag{3.26}$$

Varying the action (3.26) with respect to $\overline{\psi}$ one obtains:

$$\frac{e^{-2\Theta}}{\sqrt{-g}}\frac{\delta S_{\rm F}}{\delta\overline{\psi}} = \frac{i}{2}\left(\gamma^\mu\nabla_\mu\psi + \omega_\mu\gamma^\mu\psi\right) - m\psi + \frac{i}{2}\frac{e^{-2\Theta}}{\sqrt{-g}}\partial_\mu e^{2\Theta}\sqrt{-g}\gamma^\mu\psi = 0. \tag{3.27}$$

Using the result

$$[\omega_\mu,\gamma^\mu]\psi = -\left(\frac{e^{-2\Theta}}{\sqrt{-g}}\partial_\mu e^{2\Theta}\sqrt{-g}\gamma^\mu\right)\psi, \tag{3.28}$$

that can be check using (3.25), (2.16), and properties of ordinary Dirac matrices and of the vierbein field, we get the following equation for ψ on a RC space-time:

$$i\gamma^\mu(x)\nabla_\mu\psi - m\psi = 0. \tag{3.29}$$

The equation for $\overline{\psi}$ can be obtained in a similar way,

$$i\left(\nabla_\mu\overline{\psi}\right)\gamma^\mu(x) + m\overline{\psi} = 0. \tag{3.30}$$

We can see that the equations (3.29) and (3.30) are the same ones that arise from MCP used in the minkowskian equations of motion. In the usual EC theory, the equations obtained from the action principle do not coincide with the equations gotten by generalizing the minkowskian ones[2]. This is another new feature of the proposed model, it is more consistent in the sense that it makes no difference if one starts by the equations of motion or by the minimal action principle.

The Lagrangian that describes the interaction of fermion fields with the Einstein-Cartan gravity is

$$\begin{aligned} S &= S_{\text{grav}} + S_{\text{F}} \qquad (3.31) \\ &= -\int d^4x e^{2\Theta}\sqrt{-g}\left\{R - \frac{i}{2}\left(\overline{\psi}\gamma^\mu\partial_\mu\psi - \left(\partial_\mu\overline{\psi}\right)\gamma^\mu\psi + \overline{\psi}\left[\gamma^\mu,\omega_\mu\right]\psi\right) + m\overline{\psi}\psi\right\} \\ &= -\int d^4x e^{2\Theta}\sqrt{-g}\left\{R - \frac{i}{2}\left(\overline{\psi}\gamma^\mu\partial_\mu\psi - \left(\partial_\mu\overline{\psi}\right)\gamma^\mu\psi + \overline{\psi}\left[\gamma^\mu,\omega_\mu^{V_4}\right]\psi\right)\right. \\ &\qquad \left. -\frac{i}{8}\overline{\psi}\tilde{K}_{\mu\nu\omega}\gamma^{[\mu}\gamma^\nu\gamma^{\omega]}\psi + m\overline{\psi}\psi\right\}, \end{aligned}$$

where it was used that $\gamma^a\left[\gamma^b,\gamma^c\right] + \left[\gamma^b,\gamma^c\right]\gamma^a = 2\gamma^{[a}\gamma^b\gamma^{c]}$, and that

$$\omega_\mu = \omega_\mu^{V_4} + \frac{1}{8}K_{\mu\nu\rho}\left[\gamma^\nu,\gamma^\rho\right], \qquad (3.32)$$

where $\omega_\mu^{V_4}$ is the V_4 spinorial connection, calculated by using the Christoffel symbols instead of the full connection in (3.25).

The peculiarity of fermion fields is that one has no trivial equation for $\tilde{K}$ from (3.31). The Euler-Lagrange equations for $\tilde{K}$ is given by

$$\frac{e^{-2\Theta}}{\sqrt{-g}}\frac{\delta S}{\delta\tilde{K}} = \tilde{K}^{\mu\nu\omega} + \frac{i}{8}\overline{\psi}\gamma^{[\mu}\gamma^\nu\gamma^{\omega]}\psi = 0. \qquad (3.33)$$

Differently from the previous cases, we have that the traceless part of the contorsion tensor, $\tilde{K}_{\alpha\beta\gamma}$, is proportional to the spin distribution. It is still zero outside matter distribution, since its equation is an algebraic one, it does not allow propagation. The other equations follow from the minimization of (3.31). The main difference

between these equations and the usual ones obtained from standard EC gravity, is that in the present case one has non-trivial solution for the trace of the torsion tensor, that is derived from Θ. In the standard EC gravity, the torsion tensor is a totally anti-symmetrical tensor, its equation is identical to (3.33), and so it has vanishing trace.

IV. FINAL REMARKS

In this section, we are going to discuss the role of the condition (2.13), and to show that for the proposed model, the traditional point of view for the spin–torsion relationship must be modified. First of all, we would like to stress that, in contrast to the standard EC theory of gravity, due to the introduction of the new volume element, it makes no difference to use properly MCP in the equations of motion or in the action formulation. This is a strong difference between the standard EC theory and general relativity. We know that in general relativity, using MCP in the equations of motion or in the action, we get the same result.

It was already said the condition that the trace of the torsion tensor is derive from a scalar potential, (2.13), is the necessary condition in order to be possible the definition of a connection-compatible volume element on a manifold. Therefore, we have that our approach is restrict to space-times which admits such volume elements. However, assuming that one needs to use connection-compatible volume elements in action formulations, we automatic have this restriction if we wish to use minimal action principles.

In spite of it is not clear how to get EC gravity equations without using a minimal action principle, we can speculate about matter fields on space-times not obeying (2.13). Because of it is equivalent to use MCP in the equations of motion or in the action formulation, we can forget the last and to cast the equations of motion for matter fields in a covariant way directly. It can be done easily, as example, for scalar fields[5]. We get the following equation

$$\partial_\mu \partial^\mu \varphi = 0 \rightarrow \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} \partial^\mu \varphi + 2 S_\mu \partial^\mu \varphi = 0, \tag{4.1}$$

which is, apparently, a consistent equation. However, we need to define a inner product for the space of the solutions of (4.1) [11], and we are able to do it only if (2.13) holds. We have that the dynamics of matter fields requires some restrictions to the non-riemannian structure of space-time, namely, the condition (2.13). This is more evident for gauge fields, where (2.13) arises directly as an integrability condition for the equations of motion [10]. It seems that condition (2.13) cannot be avoided.

We could realize from the matter fields studied, that the trace of the torsion tensor is not directly related to spin distributions. This is a new feature of the proposed model, and we naturally arrive to the following question: What is the source of torsion? The situation for the traceless part of the torsion tensor is the same that one has in the standard EC theory, only fermion fields can be sources to it. For the trace, it is quite different. Take for example $\tilde{K}_{\alpha\beta\gamma} = 0$, that corresponds to scalar and gauge fields. In these cases, the equation for the trace of the torsion tensor is given by

$$D_\mu S^\mu = \frac{3}{2} \frac{e^{-2\Theta}}{\sqrt{-g}} \left(g^{\mu\nu} \frac{\delta S_{\text{mat}}}{\delta g^{\mu\nu}} + \frac{1}{2} \frac{\delta S_{\text{mat}}}{\delta \Theta} \right). \tag{4.2}$$

Using the definition of the energy-momentum tensor

$$\frac{e^{-2\Theta}}{\sqrt{-g}} \frac{\delta S_{\text{mat}}}{\delta g^{\mu\nu}} = -\frac{1}{2} T_{\mu\nu}, \tag{4.3}$$

and that for scalar and gauge fields we have

$$\frac{e^{-2\Theta}}{\sqrt{-g}} \frac{\delta S_{\text{mat}}}{\delta \Theta} = 2\mathcal{L}_{\text{mat}}, \tag{4.4}$$

equation (4.2) leads to

$$D_\mu S^\mu = \frac{3}{2} \left(\mathcal{L}_{\text{mat}} - \frac{1}{2} T \right), \tag{4.5}$$

where T is the trace of the energy-momentum tensor. The quantity between parenthesis, in general, has nothing to do with spin, and in spite of this, it is the source for a part of the torsion. For gauge fields, for which $T = 0$, the source for the trace of the torsion tensor is namely the Lagrangian. The presence of the scalar potential Θ in the "parallel" U_4 volume element seems to indicate that torsion is not directly related to spin distributions. These topics are now under investigation.

ACKNOWLEDGMENTS

The author is grateful to Professor Josif Frenkel. This work was supported by Fundação de Amparo à Pesquisa do Estado de São Paulo.

REFERENCES

[1] F.W. Hehl, P. von der Heyde, G.D. Kerlick, and J.M. Nester, Rev. Mod. Phys. **48**, 393 (1976).

[2] V. de Sabbata and M. Gasperini, *Introduction to Gravitation*, World Scientific, 1985.

[3] F.W. Hehl, Found. Phys. **15**, 451 (1985).

[4] I.L. Buchbinder, S.D, Odintsov, and I.L. Shapiro, *Effective Action in Quantum Gravity*, IOP Publishing, 1992.

[5] A. Saa, *On Minimal Coupling in Riemann-Cartan Space-Times*, to appear in Mod. Phys. Lett. A.

[6] A. Saa, *Strings in Background Fields and Einstein-Cartan Theory of Gravity*, to be published.

[7] C.G. Callan, D. Friedan, E.J. Martinec, and M.J. Perry, Nucl. Phys. **B262** 593 (1985).

[8] M.B. Green, J.H. Schwarz, and E. Witten, *Superstring Theory*, sect. 3.4, Cambridge University Press, 1987.

[9] E. Braaten, T.L. Curtright, and C.K. Zachos, Nucl. Phys. **B260**, 630 (1985).

[10] A. Saa, *Gauge Fields on Riemann-Cartan Space-Times*, to appear in Int. J. Mod. Phys. A.

[11] B.S. DeWitt, Phys. Rep. **19C**, 295 (1975).