UNIVERSIDADE DE SÃO PAULO

INSTITUTO DE FÍSICA
CAIXA POSTAL 20516
01452-990 SÃO PAULO - SP
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PATH INTEGRALS OVER VELOCITIES IN QUANTUM MECHANICS

Dmitri M. Gitman and **Shmaryu M. Shartsman**
Instituto de Física, Universidade de São Paulo
and
Department of Physics,
Case Western Reserve University
Cleveland, OH 44106, USA

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Dmitri M. Gitman and Shmaryu M. Shvartsman

Instituto de Física, Universidade de São Paulo

Caixa Postal 20516-CEP 01498-970-São Paulo, S.P., Brasil

Department of Physics, Case Western Reserve University

Cleveland, OH 44106, USA

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Abstract

Representations of propagators by means of path integrals over velocities are discussed both in nonrelativistic and relativistic quantum mechanics. It is shown that all the propagators can only be expressed through bosonic path integrals over velocities of space-time coordinates. For spinning and isospinning particles that is quite nontrivial statement, to prove which one needs to do all grassmannian integrations in conventional path integral representations. In the representations the integration over velocities is not restricted by any boundary conditions; matrices, which have to be inverted in course of doing Gaussian integrals, do not contain any derivatives in time, and spinor and isospinor structures of the propagators are given explicitly. One can define universal Gaussian and quasi-Gaussian integrals over velocities and rules of handling them. Such a technique allows one effectively calculate propagators in external fields. Thus, Klein-Gordon propagator is found in a constant homogeneous electromagnetic field and its combination with a plane wave field.

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I. INTRODUCTION

Propagators of relativistic particles in external fields (electromagnetic, non-Abelian or gravitational) contain important information about quantum behavior of these particles. Moreover, if such propagators are known in an arbitrary external field, one can find exact one-particle Green's functions in the corresponding quantum field theory, taking functional integrals over the external field. It is known that the propagators can be presented by means of path integrals over classical trajectories. Such representations were already discussed in the literature for a long time in different contexts [1–16]. Over recent years this activity got some additional motivation to learn on these simple examples how to quantize by means of path integrals more complicated theories such as string theory, gravity and so on. Path integral representations can be effectively used for calculations of propagators, for example, for concrete calculations of propagators in external electromagnetic or gravitational fields. However, in contrast with the field theory, where path integration rules are enough well defined, at least in the frame of perturbation theory [17,18], in relativistic and nonrelativistic quantum mechanics there are problems with uniqueness of definition of path integrals, with boundary conditions, and so on [1,19–23].

In this paper we discuss representations of relativistic and nonrelativistic propagators by means of path integrals over velocities. It is shown that all the propagators can only be expressed through bosonic path integrals over velocities of space-time coordinates. For spinning and isospinning particles that is not only a question of convenience, but a nontrivial statement, to prove which one needs, particularly, to do all grassmannian integrations in conventional path integral representations and find spinor and isospinor structure of the propagator explicitly. Conveniences of the representations are: the integration over velocities is not restricted by any boundary conditions, matrices, which have to be inverted in course of doing Gaussian integrals, do not contain any derivatives in time, and spinor and isospinor structures of the propagators are given explicitly by their decomposition in the independent γ-matrix structures or generators of a gauge group. One can define universal Gaussian and quasi-Gaussian integrals over velocities and rules of handling them. This approach is similar to one used in the field theory (in the frame of perturbation theory [17,18]). Using such a technique one can effectively calculate propagators in external fields. As examples, Klein-Gordon propagator is found in a constant homogeneous electromagnetic field and its combination with a plane wave field. One ought to say that path integration methods were already applied for such kind of calculations. For example, the causal propagators of relativistic particles in external electromagnetic field of a plane wave were found by means of path integrations in [2,4,24] and in crossed electric and magnetic fields in [5]. More complicated combination of electromagnetic field, consisting of parallel magnetic and electric field together with a plane wave, propagating along, was considered in [3,12]. In [8] they did particular functional integrations to prove a path integral representation for the causal propagator of spinning particle in an external electromagnetic field.

II. PATH INTEGRALS OVER VELOCITIES IN NONRELATIVISTIC QUANTUM MECHANICS

In nonrelativistic quantum mechanics they usually consider path integral representations for the propagation function (amplitude of the probability) $G(x,t;x't')$, $x=(x^i,\ i=\overline{1,3})$,

$$
\begin{aligned}
&G(x,t;x',t') = < x|e^{-i\hat{H}(t-t')}|x'> \,, \qquad (2.1)\\
&\left(i\frac{\partial}{\partial t}-\hat{H}\right)G(x,t;x',t')=0\,,\quad G(x,t;x',t)=\delta^3(x-x')\,.
\end{aligned}
$$

If we suppose, for example, that quantum Hamiltonian $\hat{H}$ is constructed from the classical one $H(x,p)$ by means of Weyl's ordering procedure, then the following path integral representation takes place:

$$
\begin{aligned}
&G=G(x_{out},t_{out};x_{in},t_{in})=\int_{x_{in}}^{x_{out}}D'x\int Dp\,\exp\left\{iS_H[x,p]\right\}\\
&=\lim_{N\to\infty}\int\frac{d^3p_N}{(2\pi)^3}\prod_{k=1}^{N-1}\frac{d^3x_kd^3p_k}{(2\pi)^3}\exp\left\{i\sum_{j=1}^{N}\left[p_j\frac{\Delta x_j}{\Delta t}-H(\bar{x}_j,p_j)\right]\Delta t\right\}\,, \qquad (2.2)\\
&\Delta x_j=x_j-x_{j-1},\ \Delta t=\frac{t_{out}-t_{in}}{N},\ \bar{x}_j=\frac{x_j+x_{j-1}}{2}\,,
\end{aligned}
$$

$$S_H[x,p]=\int_{t_{in}}^{t_{out}}\left[p\dot{x}-H(x,p)\right]dt\,,$$

where S_H is hamiltonian action and the integration in the right side of (2.2) is going over trajectories $x(t)$ with the boundary conditions $x(t_{in})=x_{in}$, $x(t_{out})=x_{out}$, and over trajectories $p(t)$ without any restrictions. We denoted the functional differential of x with prime to underline the number of integrations over x is less then one over p.

The expression (2.2) presents a hamiltonian form of the path integral for propagation function (2.1). To get a lagrangian form one can make a shift

$$p\to p+p_0\,, \qquad (2.3)$$

where $p_0=p_0(x,\dot{x})$ is a solution of the equation

$$\frac{\delta S_H}{\delta p}=0 \iff \dot{x}=\{x,H\}=\frac{\partial H}{\partial p}\,,$$

with respect to p. If H is constructed from a Lagrangian $L(x,\dot{x})$, then $p_0=\partial L/\partial\dot{x}$, so that

$$S_L[x]=S_H[x,p_0]=\int_{t_{in}}^{t_{out}}L(x,\dot{x})dt\,.$$

and

$$S_H[x,p+p_0]=S_L[x]+\Delta S_H,\quad \Delta S_H=-\int_{t_{in}}^{t_{out}}\sum_{n=2}\frac{p^n}{n!}\frac{\partial^nH}{\partial p^n}\bigg|_{p=p_0}dt\,.$$

Thus, one can write the path integral (2.2) in the following form

$$G=\int_{x_{in}}^{x_{out}}D'x\,\exp\left\{iS_L[x]\right\}\mathcal{M}[x]\,, \qquad (2.4)$$

with the measure

$$\mathcal{M}[x]=\int Dp\,\exp\left\{i\Delta S_H[x,p]\right\}\,. \qquad (2.5)$$

The expression (2.4) presents a lagrangian form of the path integral for the propagation function (2.1).

One can express the propagation function by means of a path integral over coordinates and velocities. To this end we make a change of variables in (2.2), $(x,p)\to(x,v)$, where v and p are connected by the equation

$$p = \left.\frac{\partial L}{\partial \dot{x}}\right|_{\dot{x}=v} = p_0(x,v) \ .$$

(We suppose for simplicity that Hessian is not zero in the case of consideration).

The Jacobian of the change of variables is

$$J(x,v) = \mathrm{Det}\frac{\partial^2 L(x,v)}{\partial v^i(t)\partial v^j(t)}\delta(t-t') \ ,$$

and

$$S_H[x,p_0(x,v)] = \int_{t_{in}}^{t_{out}}\left[L(x,v) + \frac{\partial L(x,v)}{\partial v}(\dot{x}-v)\right]dt \ ,$$

so, we get

$$G = \int_{x_{in}}^{x_{out}} D'x \int Dv \exp\left\{i\int_{t_{in}}^{t_{out}}\left[L(x,v) + \frac{\partial L(x,v)}{\partial v}(\dot{x}-v)\right]dt\right\} J(x,v) \ . \tag{2.6}$$

This formula presents the propagation function by means of a path integral over coordinates and velocities. A similar formula can be derived in the field theory for the generating functional of Green's function [25].

Making the shift of velocities, $v \to v + \dot{x}$, we get again the lagrangian form (2.4), but the expression for the measure $\mathcal{M}[x]$ is given now in terms of a path integral over velocities,

$$\mathcal{M}[x] = \int Dv \exp\{i\Delta S_L[x,v]\} J(x,v) \ , \tag{2.7}$$

$$\Delta S_L[x,v] = -\sum_{n=2}\frac{n-1}{n!}\int_{t_{in}}^{t_{out}}\frac{\partial^n L}{\partial \dot{x}^n}v^n dt \ .$$

In case if

$$L = L_0 + L_{int} \ , \quad L_0 = \frac{m\dot{x}^2}{2} \ , \quad L_{int} = -V(x) \ , \tag{2.8}$$

or

$$H = H_0 + H_{int} \ , \quad H_0 = \frac{p^2}{2m} \ , \quad H_{int} = V(x) \ .$$

we get simple Feynman's [1] answer (2.4) with x-independent measure (2.5) or (2.7).

Let us consider different kind of representations, containing path integrals over velocities. To this end it is useful first to make the number of integrations over x and p equal in the initial definition (2.2). Namely, one can write

$$G = \int_{x_{in}} Dx \int Dp \, \delta^3(x(t_{out}) - x_{out}) \exp\{iS_H[x,p]\}$$

$$= \lim_{N\to\infty}\int\prod_{k=1}^{N}\frac{\mathrm{d}^3x_k\mathrm{d}^3p_k}{(2\pi)^3}\delta^3(x_N - x_{out})\exp\left\{i\sum_{j=1}^{N}\left[p_j\frac{\Delta x_j}{\Delta t} - H(\bar{x}_j,p_j)\right]\Delta t\right\} \ , \tag{2.9}$$

where now only one boundary condition remains, $x(t_{in}) = x_{in}$. Making the shift of momenta (2.3) in the integral (2.9), a change of parameterization of trajectories, introducing instead of time t a parameter τ, $\tau \in [0,1]$,

$$\tau = \frac{t - t_{in}}{\Delta T} \ , \quad \Delta T = t_{out} - t_{in} \ ,$$

replacements

$$a(x - x_{in} - \tau\Delta x) \to x, \quad ap \to p, \quad \Delta x = x_{out} - x_{in}, \quad a = \sqrt{\frac{m}{\Delta T}} \ ,$$

and restricting ourselves for simplicity with the case (2.8), we get the expression[1]

$$G = a^3 \exp\left\{\frac{im\Delta x^2}{2\Delta T}\right\}\int_0 Dx \, \mathcal{M} \, \delta^3(x(1))$$

$$\times \exp\left\{i\int \mathrm{d}\tau\left[\frac{\dot{x}^2}{2} - V(ax + x_{in} + \tau\Delta x)\Delta T\right]\right\} \ , \tag{2.10}$$

where the integration over x is subjected the boundary condition $x(0) = 0$, and the measure $\mathcal{M}$ has the form

$$\mathcal{M} = \int \exp\left\{-\frac{i}{2}\int p^2 d\tau\right\} \ .$$

On this step we replace the integration over the trajectories $x(\tau)$ by one over velocities $v(\tau)$,

$$x(\tau) = \int \theta(\tau - \tau')v(\tau')d\tau' = \int_0^\tau v(\tau')d\tau' \ ,$$

$$v(\tau) = \dot{x}(\tau) \ , \quad x(1) = \int v d\tau \ . \tag{2.11}$$

The corresponding Jacobian can be written as

[1]Here and in what follow we use the notation $\int d\tau = \int_0^1 d\tau$.

$$J = \text{Det}\ \theta(\tau - \tau')$$

and regularized, for example, in the frame of discretization procedure. Thus, we get

$$G = a^3 \exp\left\{\frac{im\Delta x^2}{2\Delta T}\right\} \int Dv\ \mathcal{M}\ J\ \delta^3\left(\int v d\tau\right) \tag{2.12}$$
$$\times \exp\left\{i \int d\tau \left[\frac{v^2}{2} - V(a\int_0^\tau v(\tau')d\tau' + x_{in} + \tau\Delta x)\Delta T\right]\right\} ,$$

where integration over v as well as over p is already not subjected any boundary conditions. One can formally find the Jacobian J, switching off the potential $V(x)$ in (2.11) and using the expression for the free propagation function [1],

$$G_0 = \left(\frac{m}{2\pi i\Delta T}\right)^{\frac{3}{2}} \exp\left\{\frac{im\Delta x^2}{2\Delta T}\right\} .$$

So,

$$J = \left(\frac{1}{2\pi i}\right)^{\frac{3}{2}} \left[\int Dv\ \mathcal{M}\ \delta^3\left(\int v d\tau\right) \exp\left\{i\int d\tau \left(\frac{v^2}{2}\right)\right\}\right]^{-1} .$$

Gathering these results, we may write

$$G = G_0 \int \mathcal{D}v\ \delta^3\left(\int v d\tau\right) \exp\left\{i \int d\tau \left[\frac{v^2}{2} - V(a\int_0^\tau v(\tau')d\tau' + x_{in} + \tau\Delta x)\Delta T\right]\right\} , \tag{2.13}$$

where new measure $\mathcal{D}v$ has the form

$$\mathcal{D}v = Dv \left[\int Dv\ \ \delta^3\left(\int v d\tau\right) \exp\left\{i\int d\tau \left(\frac{v^2}{2}\right)\right\}\right]^{-1} . \tag{2.14}$$

Thus, we got a representation for the propagation function (2.1) by means of a special kind path integral over velocities. The conveniences of this representation are: the integration over velocities is not subjected any boundary conditions and no time derivatives appear in integrand, so, e.g. matrices, which have to be inverted in course of doing Gaussian integrals, do not contain any time derivatives. The same kind of path integrals arises in representations of relativistic particle propagators, which we present in the next section. One can formulate universal rules of handling such integrals in the frame of perturbation theory, what will be done in Sect.4.

III. PATH INTEGRALS OVER VELOCITIES IN RELATIVISTIC QUANTUM MECHANICS

A. Scalar particle propagator in an external electromagnetic field

As known, the propagator of a scalar particle in an external electromagnetic field $A_\mu(x)$ is the causal Green's function $D^c(x,y)$ of the Klein-Gordon equation in this field,

$$\left[(i\partial - gA)^2 - m^2 + i\epsilon\right] D^c(x,y) = -\delta^4(x-y) , \tag{3.1}$$

where $x = \left(x^\mu,\ \mu = \overline{0,3}\right)$, Minkowski tensor $\eta_{\mu\nu} = \text{diag}(1 - 1 - 1 - 1)$, and infinitesimal term $i\epsilon$ selects the causal solution.

Consider a lagrangian form of the path integral representation for $D^c(x,y)$ [13], modified by inserting of a δ- function, similar to the nonrelativistic case,

$$D^c = D^c(x_{out}, x_{in}) = \frac{i}{2}\int_0^\infty de_0 \int_{e_0} De \int D\pi \int_{x_{in}} Dx\ M(e)\ \delta^4(x(1) - x_{out})$$
$$\times \exp\left\{i\int d\tau \left[-\frac{\dot{x}^2}{2e} - \frac{e}{2}m^2 - g\dot{x}A(x) + \pi\dot{e}\right]\right\} , \tag{3.2}$$

where $x^\mu(\tau)$, $e(\tau)$, $\pi(\tau)$ are trajectories of integration, parameterized by some parameter $\tau \in [0,1]$, and subjected the boundary conditions $x(0) = x_{in}$, $e(0) = e_0$; the measure $M(e)$ can formally be written as

$$M(e) = \int Dp\ \exp\left\{\frac{i}{2}\int ep^2 d\tau\right\} . \tag{3.3}$$

The propagator D^c can be only presented via a path integral over velocities, in the form similar to (2.13). First we integrate over π and then use the arisen δ-function $\delta(\dot{e})$ to remove the functional integration over e,

$$D^c = \frac{i}{2}\int_0^\infty \frac{de_0}{e_0^2} \int_{x_{in}} Dx\ M(e_0)\ \delta^4(x(1) - x_{out})$$
$$\times \exp\left\{i\int d\tau \left[-\frac{\dot{x}^2}{2e_0} - \frac{e_0}{2}m^2 - g\dot{x}A(x)\right]\right\}.$$

Then, after the replacement

$$\frac{x - x_{in} - \tau\Delta x}{\sqrt{e_0}} \to x\,, \quad \Delta x = x_{out} - x_{in}\,, \tag{3.4}$$

we get the expression

$$D^c = \frac{i}{2}\int_0^\infty \frac{de_0}{e_0^2}\exp\left[-\frac{i}{2}\left(e_0 m^2 + \frac{\Delta x^2}{e_0}\right)\right]\int_0 Dx\; M(1)\,\delta^4(x(1))$$
$$\times\exp\left\{i\int d\tau\left[-\frac{\dot{x}^2}{2} - g(\sqrt{e_0}\dot{x} + \Delta x)A(\sqrt{e_0}x + x_{in} + \tau\Delta x)\right]\right\}, \tag{3.5}$$

where the trajectories x obey already zero boundary conditions, $x(0) = 0$.

As in nonrelativistic case, we replace the integration over the trajectories x by one over velocities v, according to eq. (2.11). Thus,

$$D^c = \frac{i}{2}\int_0^\infty \frac{de_0}{e_0^2}\exp\left[-\frac{i}{2}\left(e_0 m^2 + \frac{\Delta x^2}{e_0}\right)\right]\int Dv\; M(1)\; J\;\delta^4\left(\int v d\tau\right) \tag{3.6}$$
$$\times\exp\left\{i\int d\tau\left[-\frac{v^2}{2} - g\,(\sqrt{e_0}v + \Delta x)\,A\left(\sqrt{e_0}\int_0^\tau v(\tau')d\tau' + x_{in} + \tau\Delta x\right)\right]\right\}.$$

One can formally find the Jacobian J, switching off the potential $A_\mu(x)$ in (3.6) and using the expression for the free causal Green's function D_0^c,

$$D_0^c = D_0^c(x_{out}, x_{in}) = \frac{1}{2(2\pi)^2}\int_0^\infty \frac{de_0}{e_0^2}\exp\left[-\frac{i}{2}\left(e_0 m^2 + \frac{\Delta x^2}{e_0}\right)\right].$$

So, we get formally

$$J = \frac{1}{i\,(2\pi)^2}\left[\int Dv\; M(1)\,\delta^4\left(\int v d\tau\right)\exp\left\{i\int d\tau\left(-\frac{v^2}{2}\right)\right\}\right]^{-1}. \tag{3.7}$$

Gathering these results, we may write

$$D^c = \frac{1}{2(2\pi)^2}\int_0^\infty \frac{de_0}{e_0^2}\exp\left[-\frac{i}{2}\left(e_0 m^2 + \frac{\Delta x^2}{e_0}\right)\right]\Delta_1(e_0)\,, \tag{3.8}$$

$$\Delta_1(e_0) = \int \mathcal{D}v\;\delta^4\left(\int v d\tau\right)\exp\left\{i\int d\tau\left[-\frac{v^2}{2} - g\,(\sqrt{e_0}v + \Delta x)\right.\right.$$
$$\left.\left.\times A\left(\sqrt{e_0}\int_0^\tau v(\tau')d\tau' + x_{in} + \tau\Delta x\right)\right]\right\}, \tag{3.9}$$

where the new measure $\mathcal{D}v$ has the form

$$\mathcal{D}v = Dv\left[\int Dv\;\;\delta^4\left(\int v d\tau\right)\exp\left\{i\int d\tau\left(-\frac{v^2}{2}\right)\right\}\right]^{-1}. \tag{3.10}$$

It is clear that $\Delta_1(e_0) = 1$ at $A = 0$.

Thus, we got a representation for the propagator (3.1) by means of a path integral over velocities of similar kind as in nonrelativistic case.

B. Spinning particle propagator in external electromagnetic field

The propagator of a spinning particle in an external electromagnetic field $A_\mu(x)$ is the causal Green's function $S^c(x,y)$ of the Dirac equation in this field,

$$[\gamma^\mu(i\partial_\mu - gA_\mu) - m]\,S^c(x,y) = -\delta^4(x-y)\,, \tag{3.11}$$
$$[\gamma^\mu\,,\,\gamma^\nu]_+ = 2\eta^{\mu\nu}\,.$$

Consider a lagrangian form of the path integral representation for, transformed by $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$ function $\tilde{S}^c(x,y) = S^c(x,y)\gamma^5$, (see [13]), modified as in the two previous cases by inserting of a δ-function,

$$\tilde{S}^c = \tilde{S}^c(x_{out}, x_{in}) = \exp\left\{i\gamma^n\frac{\partial_\ell}{\partial\theta^n}\right\}\int_0^\infty de_0\int d\chi_0\int_{e_0} De\int_{\chi_0} D\chi\int_{x_{in}} Dx\int D\pi_e\int D\pi_\chi$$
$$\times\int_{\psi(1)+\psi(0)=\theta}\mathcal{D}\psi\; M(e)\,\delta^4(x(1) - x_{out})\exp\left\{i\int\left[-\frac{\dot{x}^2}{2e} - \frac{e}{2}m^2 - g\dot{x}A(x)\right.\right.$$
$$\left.\left.+iegF_{\mu\nu}(x)\psi^\mu\psi^\nu\; + i\left(\frac{\dot{x}_\mu\psi^\mu}{e} - m\psi^5\right)\chi - i\psi_n\dot{\psi}^n + \pi_e\dot{e} + \pi_\chi\dot{\chi}\right]d\tau + \psi_n(1)\psi^n(0)\right\}\Big|_{\theta=0}, \tag{3.12}$$

where

$$[\gamma^m\,,\,\gamma^n]_+ = 2\eta^{mn},\; m,n = \overline{0,3},5;\;\; \eta^{mn} = \mathrm{diag}(1\,-1\,-1\,-1\,-1)\,;$$

θ^n are auxiliary grassmannian (odd) variables, anticommuting by definition with the γ-matrices; $x^\mu(\tau)$, $e(\tau)$, $\pi_e(\tau)$ are bosonic trajectories of integration; $\psi^n(\tau)$, $\chi(\tau)$, $\pi_\chi(\tau)$ are odd trajectories of integration; and boundary conditions $x(0) = x_{in}$, $e(0) = e_0$, $\psi^n(0) + \psi^n(1) = \theta^n$, $\chi(0) = \chi_0$ take place; the measure $M(e)$ is defined in (3.3) and

$$\mathcal{D}\psi = D\psi\left[\int_{\psi(0)+\psi(1)=0} D\psi\exp\left\{\int_0^1 \psi_n\dot{\psi}^n d\tau\right\}\right]^{-1}, \tag{3.13}$$

is the measure in the integration over ψ.

We are going to demonstrate that the propagator of a spinning particle can also be expressed through a bosonic path integral over velocities of coordinates x. To this end one needs to fulfil several functional integrations. First, one can integrate over π_e and π_χ, and then use arisen δ-functions to remove the functional integration over e and χ,

$$\tilde{S}^c = -\exp\left\{i\gamma^n\frac{\partial_\ell}{\partial\theta^n}\right\}\int_0^\infty de_0\int_{x_{in}}Dx\int_{\psi(0)+\psi(1)=\theta}\mathcal{D}\psi\; M(e_0)\,\delta^4(x(1)-x_{out})$$
$$\times\int\left(\frac{\dot{x}_\mu\psi^\mu}{e_0}-m\psi^5\right)d\tau\exp\left\{i\int\left[-\frac{\dot{x}^2}{2e_0}-\frac{e_0}{2}m^2-g\dot{x}A(x)\right.\right.$$
$$\left.\left.+ige_0F_{\mu\nu}(x)\psi^\mu\psi^\nu-i\psi_n\dot{\psi}^n\right]d\tau+\psi_n(1)\psi^n(0)\right\}\Big|_{\theta=0}\,, \tag{3.14}$$

Then, it is convenient to replace the integration over ψ by one over related odd velocities ω,

$$\psi(\tau)=\frac{1}{2}\int\varepsilon(\tau-\tau')\omega(\tau')d\tau'+\frac{1}{2}\theta\,,\quad \omega(\tau)=\dot{\psi}(\tau)\,,\ \varepsilon(\tau)=\text{sign}\,\tau\,. \tag{3.15}$$

There are not more any restrictions on ω; because of (3.15) the boundary conditions for ψ are obeyed automatically. The corresponding Jacobian does not depend on variables and cancels with the same one from the measure (3.13). Thus[2],

$$\tilde{S}^c = -\frac{1}{2}\exp\left\{i\gamma^n\frac{\partial_\ell}{\partial\theta^n}\right\}\int_0^\infty de_0\int_{x_{in}}Dx\int\mathcal{D}\omega\; M(e_0)\,\delta^4(x(1)-x_{out})$$
$$\times\left[\frac{\dot{x}_\mu}{e_0}\left(\varepsilon\omega^\mu+\theta^\mu\right)-m\left(\varepsilon\omega^5+\theta^5\right)\right]\exp\left\{i\left[-\frac{\dot{x}^2}{2e_0}-\frac{e_0}{2}m^2-g\dot{x}A(x)\right.\right.$$
$$\left.\left.-\frac{ie_0g}{4}\left(\omega^\mu\varepsilon-\theta^\mu\right)F_{\mu\nu}(x)\left(\varepsilon\omega^\nu+\theta^\nu\right)+\frac{i}{2}\omega_n\varepsilon\omega^n\right]\right\}\Big|_{\theta=0}\,, \tag{3.16}$$

where the measure $\mathcal{D}\omega$ is

$$\mathcal{D}\omega = D\omega\left[\int D\omega\exp\left\{-\frac{1}{2}\omega^n\varepsilon\omega_n\right\}\right]^{-1}\,. \tag{3.17}$$

One can prove, that for a function $f(\theta)$ in the Grassmann algebra, the following identity holds

$$\exp\left\{i\gamma^n\frac{\partial_\ell}{\partial\theta^n}\right\}f(\theta)\Big|_{\theta=0}=f\left(\frac{\partial_\ell}{\partial\zeta}\right)\exp\left\{i\zeta_n\gamma^n\right\}\Big|_{\zeta=0}$$
$$=\sum_{k=0}^{4}\sum_{n_1\cdots n_k}f_{n_1\cdots n_k}\frac{\partial_\ell}{\partial\zeta_{n_1}}\cdots\frac{\partial_\ell}{\partial\zeta_{n_k}}\sum_{l=0}^{4}\frac{i^l}{l!}\left(\zeta_n\gamma^n\right)^l\Big|_{\zeta=0}\,, \tag{3.18}$$

where ζ_n are some odd variables. Taking (3.18) into account in (3.16), we get

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[2]Here and further we are using condensed notations, $\omega\varepsilon\omega=\int d\tau d\tau'\omega(\tau)\varepsilon(\tau-\tau')\omega(\tau')$, $\dot{x}A(x)=\int d\tau\dot{x}A(x)$ and so on.

$$\tilde{S}^c = -\frac{1}{2}\int_0^\infty de_0\int_{x_{in}}Dx\;M(e_0)\,\delta^4(x(1)-x_{out}) \tag{3.19}$$
$$\times\left[\frac{\dot{x}_\mu}{e_0}\left(\varepsilon\frac{\delta_\ell}{\delta\rho_\mu}+\frac{\partial_\ell}{\partial\zeta_\mu}\right)-m\left(\varepsilon\frac{\delta}{\delta\rho_5}+i\gamma^5\right)\right]\exp\left\{i\left[-\frac{\dot{x}^2}{2e_0}-\frac{e_0}{2}m^2\right.\right.$$
$$\left.\left.-g\dot{x}A(x)+\frac{ie_0g}{4}F_{\mu\nu}(x)\frac{\partial_\ell}{\partial\zeta_\mu}\frac{\partial_\ell}{\partial\zeta_\nu}\right]\right\}R\left[x,\rho,\frac{\partial_\ell}{\partial\zeta}\right]\exp\left\{i\zeta_\mu\gamma^\mu\right\}\Big|_{\rho=0\,,\,\zeta=0}\,,$$

where $\rho_n(\tau)$ are odd sources for $\omega^n(\tau)$ and

$$R\left[x,\rho,\frac{\partial_\ell}{\partial\zeta}\right]=\int\mathcal{D}\omega\;\exp\left\{-\frac{1}{2}\omega^nT_{nk}(x|g)\omega^k+I_n\omega^n\right\}\,, \tag{3.20}$$
$$I_\mu=\rho_\mu-\frac{e_0g}{2}\frac{\partial_\ell}{\partial\zeta_\nu}F_{\nu\mu}(x)\varepsilon\,,\quad I_5=\rho_5\,,$$

with

$$T_{nk}(x|g)=\begin{pmatrix}\Lambda_{\mu\nu}(x|g) & 0\\ 0 & -\varepsilon\end{pmatrix}\,,\quad \Lambda_{\mu\nu}(x|g)=\eta_{\mu\nu}\varepsilon-\frac{e_0}{2}\varepsilon gF_{\mu\nu}(x)\varepsilon\,. \tag{3.21}$$

Integral in (3.20) is gaussian one. It can be easily done [26], taking into account its original definition [13],

$$R\left[x,\rho,\frac{\partial_\ell}{\partial\zeta}\right]=\left[\frac{\text{Det}T(x|g)}{\text{Det}T(x|0)}\right]^{1/2}\exp\left\{-\frac{1}{2}I_n\left[T^{-1}(x|g)\right]^{nk}I_k\right\}\,, \tag{3.22}$$

$$\left[T^{-1}(x|g)\right]^{nk}=\begin{pmatrix}\left(\Lambda^{-1}(x|g)\right)^{\mu\nu} & 0\\ 0 & -\varepsilon^{-1}\end{pmatrix}\,. \tag{3.23}$$

The ratio $\text{Det}T(x|g)/\text{Det}T(x|0)$ in (3.22) can be replaced by $\text{Det}\Lambda(x|g)/\text{Det}\Lambda(x|0)$ due to the structure (3.21) of the matrix $T(x|g)$, and the latter can be presented in a convenient form, which allows one to avoid problems with calculations of determinants of matrices with continuous indices. Namely, let us differentiate the well known formula

$$\text{Det}\Lambda(x|g)=\exp\left\{\text{Tr}\,\ln\Lambda(x|g)\right\}$$

with respect to g. So we get the equation

$$\frac{d}{dg}\text{Det}\Lambda(x|g)=\text{Det}\Lambda(x|g)\;\text{Tr}\;\Lambda^{-1}(x|g)\frac{d\Lambda(x|g)}{dg}=-e_0g\text{Det}\Lambda(x|g)\text{Tr}\;\mathcal{G}(x|g)F(x)\,,$$

with

$$\mathcal{G}^{\mu\nu}(x|g) = \frac{1}{2}\varepsilon\left[\Lambda^{-1}(x|g)\right]^{\mu\nu}\varepsilon\,. \tag{3.24}$$

This equation can be solved in the form

$$\frac{\mathrm{Det}\Lambda(x|g)}{\mathrm{Det}\Lambda(x|0)} = \exp\left\{-e_0\int_0^g dg'\ \mathrm{Tr}\ \mathcal{G}(x|g')F(x)\right\}\,. \tag{3.25}$$

Besides, the representation (3.19) contains only first derivatives with respect to $\rho_n(\tau)$, acting on R at $\rho_n = 0$. This circumstance allows one to replace in (3.19) the expression (3.22) by

$$\tilde{R}\left[x,\rho,\frac{\partial_\ell}{\partial\zeta}\right] = \left[\frac{\mathrm{Det}\Lambda(x|g)}{\mathrm{Det}\Lambda(x|0)}\right]^{1/2}$$
$$\times\exp\left\{e_0 g\rho_\mu\varepsilon^{-1}\mathcal{G}^{\mu\alpha}(x|g)F_{\alpha\nu}(x)\frac{\partial_\ell}{\partial\zeta_\nu} - \frac{e_0^2g^2}{4}\left(F(x)\mathcal{G}(x|g)F(x)\right)_{\mu\nu}\frac{\partial_\ell}{\partial\zeta_\mu}\frac{\partial_\ell}{\partial\zeta_\nu}\right\}\,. \tag{3.26}$$

Substituting (3.25,3.26) into (3.19), and performing functional differentiation with respect to ρ_μ, we get

$$\tilde{S}^c = -\frac{1}{2}\int_0^\infty de_0\int_{x_{in}} Dx\ M(e_0)\,\delta^4\left(x(1)-x_{out}\right)\left[\frac{\dot{x}^\mu}{e_0}K_{\mu\nu}(x)\frac{\partial_\ell}{\partial\zeta_\nu} - im\gamma^5\right]$$
$$\times\exp\left\{i\left[-\frac{\dot{x}^2}{2e_0} - \frac{e_0}{2}m^2 - g\dot{x}A(x) + \frac{ie_0}{2}\int_0^g dg'\ \mathrm{Tr}\ \mathcal{G}(x|g')F(x)\right.\right.$$
$$\left.\left.+\frac{ie_0g}{4}\left(F(x)K(x)\right)_{\mu\nu}\frac{\partial_\ell}{\partial\zeta_\mu}\frac{\partial_\ell}{\partial\zeta_\nu}\right]\right\}\exp\left\{i\zeta_\mu\gamma^\mu\right\}\Big|_{\zeta=0}\,, \tag{3.27}$$

where

$$K_{\mu\nu} = \eta_{\mu\nu} + e_0 g\left(\mathcal{G}(x|g)F(x)\right)_{\mu\nu}\,. \tag{3.28}$$

The differentiation over ζ in (3.27) can be fulfilled explicitly, using eq.(3.18),

$$S^c = \frac{i}{2}\int_0^\infty de_0\int_{x_{in}} Dx\ M(e_0)\,\delta^4\left(x(1)-x_{out}\right)\Phi(x,e_0)$$
$$\times\exp\left\{i\left[-\frac{\dot{x}^2}{2e_0} - \frac{e_0}{2}m^2 - g\dot{x}A(x)\right]\right\}\,, \tag{3.29}$$

$$\Phi(x,e_0) = \left[m + (2e_0)^{-1}\dot{x}K(x)\left(1-2gF(x)K(x)\right)\gamma + im\frac{e_0g}{4}\left(F(x)K(x)\right)_{\mu\nu}\sigma^{\mu\nu}\right.$$
$$\left.+\,i\frac{g}{4}\left(\dot{x}K(x)\gamma\right)\left(F(x)K(x)\right)_{\mu\nu}\sigma^{\mu\nu} + m\frac{e_0^2g^2}{16}\left(F(x)K(x)\right)_{\mu\nu}\left(F(x)K(x)\right)^{*\mu\nu}\gamma^5\right]$$
$$\times\exp\left\{-\frac{e_0}{2}\int_0^g dg'\ \mathrm{Tr}\ \mathcal{G}(x|g')F(x)\right\}\,, \tag{3.30}$$

where

$$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu\,,\,\gamma^\nu\right]\,,\quad \left(F(x)K(x)\right)^{*\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\left(F(x)K(x)\right)_{\alpha\beta}\,; \tag{3.31}$$

and $\varepsilon^{\mu\nu\alpha\beta}$ is Levi-Civita symbol.

The eq.(3.30) gives a representation for Dirac propagator as a path integral over bosonic trajectories of a functional, which spinor structure is found explicitly, namely, its decomposition in all independent γ-structures is given. The functional $\Phi(x,e_0)$ can be called spin factor, and namely it distinguishes Dirac propagator from the scalar one. One needs to stress that spin factor is gauge invariant, because of its dependence of $F_{\mu\nu}(x)$ only.

Making the replacement (3.4) in (3.29) and going over to the integration over velocities v, according (2.11), and using the expression (3.7) for the corresponding Jacobian, one can present Dirac propagator via path integral over velocities only,

$$S^c = \frac{1}{2(2\pi)^2}\int_0^\infty\frac{de_0}{e_0^2}\exp\left[-\frac{i}{2}\left(e_0m^2 + \frac{\Delta x^2}{e_0}\right)\right]\Delta_2(e_0)\,, \tag{3.32}$$

$$\Delta_2(e_0) = \int\mathcal{D}v\ \delta^4\left(\int v\,d\tau\right)\Phi\left(\sqrt{e_0}\int_0^\tau v(\tau')d\tau' + x_{in} + \tau\Delta x, e_0\right)$$
$$\exp\left\{i\int d\tau\left[-\frac{v^2}{2} - g(\sqrt{e_0}v + \Delta x)A\left(\sqrt{e_0}\int_0^\tau v(\tau')d\tau' + x_{in} + \tau\Delta x\right)\right]\right\}\,, \tag{3.33}$$

where the new measure $\mathcal{D}v$ has the form (3.10)

C. Scalar particle propagator in non-Abelian external field

In the same manner one can present relativistic particle propagators in non-Abelian external field. Here we restrict ourselves with a consideration of a scalar particle propagator in an external electromagnetic $A_\mu(x)$ and non-Abelian $B_\mu(x)$ fields. Such a propagator is the causal Green's function $D^c(x,y)$ of the Klein-Gordon equation in the fields,

$$\left[\left(i\partial - gA(x) - B^a(x)T_a\right)^2 - m^2\right]D^c(x,y) = -\delta^4(x-y)\,,$$

where T^a are generators of a corresponding group. Choosing for simplicity $SU(2)$ as the group, we have $T_a = \frac{1}{2}\sigma_a$, where σ_a are Pauli matrices. The propagator D^c can be presented via bosonic and grassmannian path integrals [9,12],

$$D^c = D^c(x_{out}, x_{in}) = \frac{i}{2}\exp\left\{i\sigma_a\frac{\partial_\ell}{\partial\theta_a}\right\}\int_0^\infty de_0 \int_{e_0} De \int_{x_{in}}^{x_{out}} Dx \int D\pi_e \int_{\phi(0)+\phi(1)=\theta} \mathcal{D}\phi$$
$$\times M(e)\exp\left\{i\left[-\frac{\dot{x}^2}{2e} - \frac{e}{2}m^2 - g\dot{x}A(x) - \dot{x}B^a(x)\mathcal{T}_a - i\phi_a\dot{\phi}_a + \pi_e\dot{e}\right] + \phi_a(1)\phi_a(0)\right\}\Bigg|_{\theta=0}, \quad (3.34)$$
$$\mathcal{D}\phi = D\phi\left[\int_{\phi(0)+\phi(1)=0} D\phi\exp\left\{\phi_a\dot{\phi}_a\right\}\right]^{-1},$$

where θ_a are auxiliary odd variables, anticommuting by definition with the σ-matrices; $\phi_a(\tau)$ are odd trajectories of integration and $\mathcal{T}_a = -i\epsilon_{abc}\phi_b\phi_c$. All grassmannian integrals can be done similar to the spinning particle case and final result presented in the form

$$D^c = \frac{i}{2}\int_0^\infty de_0 \int_{x_{in}}^{x_{out}} Dx\ M(e_0)\ \Phi(x)\exp\left\{i\left[-\frac{\dot{x}^2}{2e_0} - \frac{e_0}{2}m^2 - g\dot{x}A(x)\right]\right\}, \quad (3.35)$$
$$\Phi(x) = \left[1 + \mathrm{Tr}\ R(x)\mathcal{G}(x|1)R(x) - \frac{i}{2}L_{ab}(x)\epsilon_{abc}T_c\right]\exp\left\{-\frac{1}{2}\int_0^1 d\lambda\ \mathrm{Tr}\ \mathcal{G}(x|\lambda)R(x)\right\}, \quad (3.36)$$
$$\mathcal{G}(x|\lambda) = \frac{1}{2}\varepsilon Q^{-1}(x|\lambda)\varepsilon\ ,\ Q(x|\lambda) = \varepsilon I - \frac{\lambda}{2}\varepsilon R(x)\varepsilon\ ,\ R_{ab}(x) = \dot{x}B^c(x)\epsilon_{cab}\ ,$$
$$L_{ab}(x) = R_{ab}(x) - \left[R(x)\mathcal{G}(x|1)R(x)\right]_{ab}\ ,$$

where I is unit matrix in the group space. The isospinor factor (3.36) in (3.35) is presented by its decomposition in the generators T_a of the $SU(2)$ group. Explicit description of the spinor and isospinor structure of Dirac propagator in both Abelian and non-Abelian external fields is more complicated problem which, nevertheless, can be solved in the frame of the same approach.

Propagator D^c can be written in terms of path integral over velocities as in spinning particle case. The result has the form (3.32,3.33), where spinor factor $\Phi(x, e_0)$ has to be replaced by the isospinor factor $\Phi(x)$ (3.36).

IV. GAUSSIAN AND QUASI-GAUSSIAN PATH INTEGRALS OVER VELOCITIES

In the previous sections we demonstrated that all the propagators both in nonrelativistic and relativistic quantum mechanics can be only presented by means of bosonic path integrals over velocities of space-time coordinates. All these integrals have the following structure

$$\int \mathcal{D}v\ \delta^n\left(\int v d\tau\right) F[v], \quad (4.1)$$

where

$$\mathcal{D}v = Dv\left[\int Dv\ \ \delta^n\left(\int v d\tau\right)\exp\left\{i\int d\tau\left(-\frac{v^2}{2}\right)\right\}\right]^{-1}, \quad (4.2)$$
$$v = (v^\mu),\ v^2 = \eta_{\mu\nu}v^\mu v^\nu,\ \mu,\nu = \overline{0, n-1},$$

with some functional $F[v]$. In relativistic case $n = 4$, and $\eta_{\mu\nu}$ is Minkowski tensor; in nonrelativistic case $n = 3$, and $\eta_{\mu\nu}$ is reducing to $-\delta_{ij}$.

Ways of doing path integrals of general form are unknown at present time, only Gaussian path integrals, treated in certain sense, can be taken directly. That is also valid with regards to the integrals in question (4.1). However, if we restrict ourselves with a limited class of functionals $F[v]$, which are called quasi-Gaussian and are defined below, then one can formulate some universal rules of their calculation and handling them. Similar idea has been realized in the field theory [17,18]. The restriction with quasi-Gaussian functionals corresponds, in fact, to a perturbation theory, with Gaussian path integral as a zero order approximation.

Introduce Gaussian functional $F_G[v, I]$,

$$F_G[v, I] = \exp\left\{-\frac{i}{2}\int d\tau d\tau' v^\mu(\tau)L_{\mu\nu}(g,\tau,\tau')v^\nu(\tau') - i\int d\tau I_\mu(\tau)v^\mu(\tau)\right\}, \quad (4.3)$$

where $v^\mu(\tau)$ are the velocities and $I_\mu(\tau)$ are corresponding sources to them. A functional $F_{qG}[v, I]$ we call quasi-Gaussian if

$$F_{qG}[v, I] = F[v]F_G[v, I]\ , \quad (4.4)$$

where $F[v]$ is a functional, which can be expanded in the functional series of v,

$$F[v] = \sum_{n=0} \int d\tau_1 \dots d\tau_n F_{\mu_1 \dots \mu_n}(\tau_1 \dots \tau_n)\, v^{\mu_1}(\tau_1) \dots v^{\mu_n}(\tau_n)\ . \quad (4.5)$$

In (4.3) the matrix $L_{\mu\nu}(g, \tau, \tau')$ supposes to have the following form

$$L_{\mu\nu}(q, \tau, \tau') = \eta_{\mu\nu}\delta(\tau - \tau') + gM_{\mu\nu}(\tau, \tau')\ . \quad (4.6)$$

Define the path integral over velocities v of the Gaussian functional as

$$\int \mathcal{D}v\, \delta^n\left(\int v d\tau\right) F_G[v,I] \tag{4.7}$$
$$= \left[\frac{\text{Det } L(g) \ \det\ l(g)}{\text{Det } L(0) \ \det\ l(0)}\right]^{-1/2} \exp\left\{\frac{i}{2}\int d\tau d\tau' I(\tau)K(\tau,\tau')I(\tau')\right\} ,$$

where

$$K(\tau,\tau') = L^{-1}(g,\tau,\tau') - Q^T(\tau)l^{-1}(g)Q(\tau') ,$$
$$l(g) = \int d\tau d\tau' L^{-1}(g,\tau,\tau'), \quad Q(\tau) = \int d\tau' L^{-1}(g,\tau',\tau) . \tag{4.8}$$

The formula (4.7) can be considered as infinitedimensional generalization of the straightforward calculations result in the frame of the discretization procedure, connected with the original definition of path integrals for propagators discussed in the previous sections. In course of doing of finitedimensional integrals it is implied a supplementary definition of arisen improper Gaussian integrals by means of the analytical continuation in the matrix elements of the nonsingular matrix L.

To avoid problems with calculations of determinants of matrices with continuous indices we can use the formula

$$\frac{\text{Det } L(g)}{\text{Det } L(0)} = \exp\left\{\int_0^g dg' \text{Tr } L^{-1}(g')M\right\} , \tag{4.9}$$

which may be derived similar to one (3.25). Taking into account that det $l(0) = -1$, we can rewrite the path integral of the Gaussian functional in the following form

$$\int \mathcal{D}v\, \delta^n\left(\int v d\tau\right) F_G[v,I] \tag{4.10}$$
$$= [-\det\ l(g)]^{-1/2} \exp\left\{\frac{i}{2}\int d\tau d\tau' I(\tau)K(\tau,\tau')I(\tau') - \frac{1}{2}\int_0^g dg' \text{Tr } L^{-1}(g')M\right\} .$$

The path integral of the quasi-Gaussian functional we define through one of the Gaussian functional

$$\int \mathcal{D}v\, \delta^n\left(\int v d\tau\right) F_{qG}[v,I] = F\left(i\frac{\delta}{\delta I}\right)\int \mathcal{D}v\, \delta^n\left(\int v d\tau\right) F_G[v,I]$$
$$= [-\det\ l(g)]^{-1/2} F\left(i\frac{\delta}{\delta I}\right) \exp\left\{\frac{i}{2}\int d\tau d\tau' I(\tau)K(\tau,\tau')I(\tau') - \frac{1}{2}\int_0^g dg' \text{Tr } L^{-1}(g')M\right\} . \tag{4.11}$$

One can formulate rules of handling integrals from quasi-Gaussian functionals, using the formula (4.11). For example, such integrals are invariant under shifts of integration variables,

$$\int \mathcal{D}v\, \delta^n\left(\int (v+u) d\tau\right) F_{qG}[v+u,I] = \int \mathcal{D}v\, \delta^n\left(\int v d\tau\right) F_{qG}[v,I] . \tag{4.12}$$

The validity of this assertion for the Gaussian integral can be verified by a direct calculation. Then the general formula (4.12) follows from the (4.11). Using the property (4.12), one can derive an useful generalization of the formula (4.11),

$$\int \mathcal{D}v\, \delta^n\left(\int v d\tau - a\right) F_{qG}[v,I]$$
$$= [-\det\ l(g)]^{-1/2} F\left(i\frac{\delta}{\delta I}\right) \exp\left\{\frac{i}{2}\int d\tau d\tau' I(\tau)K(\tau,\tau')I(\tau')\right.$$
$$\left. -\frac{i}{2} a l^{-1}(g) a - i a l^{-1}(g)\int Q(\tau)I(\tau)d\tau - \frac{1}{2}\int_0^g dg' \text{Tr } L^{-1}(g')M\right\} , \tag{4.13}$$

where a ia a constant vector. The integral of the total functional derivative over $v^\mu(\tau)$ is equal to zero,

$$\int \mathcal{D}v\, \frac{\delta}{\delta v^\mu(\tau)}\, \delta^n\left(\int v d\tau\right) F_{qG}[v,I] = 0 . \tag{4.14}$$

This property may be obtained as a consequence of the functional integral invariance under the shift of variables, as well as by direct calculations of integral (4.14). Using the latter, one can derive formulas of integration by parts, which we do not present here. If a quasi-Gaussian functional depends on a parameter α , then the derivative with respect to this parameter is commutative with the integral sign,

$$\frac{\partial}{\partial\alpha}\int \mathcal{D}v\, \delta^n\left(\int v d\tau\right) F_{qG}[v,I,\alpha] = \int \mathcal{D}v\, \delta^n\left(\int v d\tau\right) \frac{\partial}{\partial\alpha} F_{qG}[v,I,\alpha] . \tag{4.15}$$

Finally, the formula for the change of the variables holds:

$$\int \mathcal{D}v\, \delta^n\left(\int v d\tau\right) F_{qG}[v,I] = \int \mathcal{D}v\, \delta^n\left(\int \phi d\tau\right) F_{qG}[\phi,I]\ \text{Det}\ \frac{\delta\phi_\tau(v)}{\delta v(\tau')} , \tag{4.16}$$

where $\phi_\tau(v)$ is a set of analytical functionals in v , parameterized by τ . One can prove formulas (4.15,4.16) in the same manner it was done in [18] for the case of the field theory.

Thus, in quantum mechanics, in the frame of perturbation theory, one can define quasi-Gaussian path integrals over velocities and rules of handling them. This definition is close to one in field theory [17,18], the analogy is stressed by the circumstance that, as in the field theory, the integrals over velocities do not contain explicitly any boundary condition for trajectories of the integration. After the rules of integration are formulated, one can forget about the origin of the integrals over velocities and fulfil integrations, using the rules only. In the next Section we demonstrate this technique on some examples.

V. EXAMPLE

Here we are going to calculate the propagator of a scalar particle in an external electromagnetic field, using representation (3.8) and rules of integrations, presented in the previous sections. We consider a combination of a constant homogeneous field and a plane wave field. The potentials for this field may be taken in the form

$$A_\mu(x) = -\frac{1}{2}F_{\mu\nu}x^\nu + f_\mu(nx) \ , \tag{5.1}$$

where $F_{\mu\nu}$ is the field strength tensor of the constant homogeneous field with nonzero invariants

$$\mathcal{F} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \neq 0 \ , \ \ \mathcal{G} = -\frac{1}{4}F^*_{\mu\nu}F^{\mu\nu} \neq 0 \ ,$$

($F^*_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$, $\epsilon_{\mu\nu\alpha\beta}$ is totally antisymmetric tensor), in terms of which its eigenvalues $\mathcal{E}$ and $\mathcal{H}$ are expressed

$$\begin{aligned} &F_{\mu\nu}n^\nu = -\mathcal{E}n_\mu \ , \ \ F_{\mu\nu}\bar{n}^\nu = \mathcal{E}\bar{n}_\mu \ , \ \ F_{\mu\nu}\ell^\nu = i\mathcal{H}\ell_\mu \ , \ \ F_{\mu\nu}\bar{\ell}^\nu = -i\mathcal{H}\bar{\ell}_\mu \ , \\ &\mathcal{E} = \left[(\mathcal{F}^2+\mathcal{G}^2)^{\frac{1}{2}} - \mathcal{F}\right]^{\frac{1}{2}} \ , \ \mathcal{H} = \left[(\mathcal{F}^2+\mathcal{G}^2)^{\frac{1}{2}} + \mathcal{F}\right)]^{\frac{1}{2}} \ . \end{aligned} \tag{5.2}$$

The eigenvectors n, $\bar{n}$, ℓ, $\bar{\ell}$ are isotropic and obey the conditions

$$n^2 = \bar{n}^2 = \ell^2 = \bar{\ell}^2 = 0 \ , \ \ n\bar{n} = 2 \ , \ \ell\bar{\ell} = -2 \ , \ n\ell = \bar{n}\ell = n\bar{\ell} = \bar{n}\bar{\ell} = 0 \ . \tag{5.3}$$

The functions $f_\mu(nx)$ are arbitrary, except for the fact that they are subject to the conditions

$$f_\mu(nx)n^\mu = f_\mu(nx)\bar{n}^\mu = 0 \ . \tag{5.4}$$

The total field strength tensor for the potential (5.1) is

$$F_{\mu\nu}(x) = F_{\mu\nu} + \Psi_{\mu\nu}(nx), \ \ \Psi_{\mu\nu}(nx) = n_\mu f'_\nu(nx) - n_\nu f'_\mu(nx) \ . \tag{5.5}$$

Since the invariants $\mathcal{F}$, $\mathcal{G}$ of the tensor $F_{\mu\nu}$ are nonzero, there exists a special reference frame, where the electric and magnetic fields, corresponding to this tensor, are collinear with respect to one another and to the spatial part $\mathbf{n}$ of the four-vector n. In this reference frame, the total field $F_{\mu\nu}(x)$ corresponds to a constant homogeneous and collinear electric and magnetic fields together with a plane wave, propagating along them; $\mathcal{E}$, $\mathcal{H}$, being equal to the strengths of a constant homogeneous electric and magnetic fields, respectively. In terms of the defined eigenvectors the tensor $F_{\mu\nu}$ can be written as

$$F_{\mu\nu} = \frac{\mathcal{E}}{2}(\bar{n}_\mu n_\nu - n_\mu\bar{n}_\nu) + \frac{i\mathcal{H}}{2}\left(\bar{\ell}_\mu\ell_\nu - \ell_\mu\bar{\ell}_\nu\right) \ , \tag{5.6}$$

and the completeness relation holds

$$\eta_{\mu\nu} = \frac{1}{2}\left(\bar{n}_\mu n_\nu + n_\mu\bar{n}_\nu - \bar{\ell}_\mu\ell_\nu - \ell_\mu\bar{\ell}_\nu\right) \ . \tag{5.7}$$

The latter allows one to express any four-vector u in terms of the eigenvectors (5.2),

$$\begin{aligned} &u^\mu = n^\mu u^{(1)} + \bar{n}^\mu u^{(2)} + \ell^\mu u^{(3)} + \bar{\ell}^\mu u^{(4)} \ , \\ &u^{(1)} = \frac{1}{2}\bar{n}u \ , \ u^{(2)} = \frac{1}{2}nu \ , \ u^{(3)} = -\frac{1}{2}\bar{\ell}u \ , \ u^{(4)} = -\frac{1}{2}\ell u \ . \end{aligned} \tag{5.8}$$

In these concrete calculations it is convenient for us to make a shift of variables in the formula (3.9), to rewrite it in the following form

$$\begin{aligned} \Delta(e_0) = &\exp\left(i\frac{\Delta x^2}{2e_0}\right)\int \mathcal{D}v \ \delta^4\left(\int v d\tau - \frac{\Delta x}{\sqrt{e_0}}\right) \\ &\times \exp\left\{i\int d\tau\left[-\frac{v^2}{2} - g\sqrt{e_0}vA\left(\sqrt{e_0}\int_0^\tau v(\tau')d\tau' + x_{in}\right)\right]\right\} \ . \end{aligned} \tag{5.9}$$

The calculations will be made in two steps: first in a constant homogeneous field only, and then in the total combination (5.1), using some results of the first problem. Thus, on the first step the potentials of the electromagnetic field are

$$A_\mu(x) = -\frac{1}{2}F_{\mu\nu}x^\nu \; . \tag{5.10}$$

Substituting the external field (5.10) into (5.9), one can find

$$\Delta(e_0) = \exp\left(i\frac{\Delta x^2}{2e_0}\right)\int \mathcal{D}v\; \delta^4\left(\int v d\tau - \frac{\Delta x}{\sqrt{e_0}}\right) \tag{5.11}$$
$$\times \exp\left\{-\frac{i}{2}\int d\tau d\tau' v(\tau) L(g,\tau,\tau') v(\tau') - i\int \frac{g\sqrt{e_0}}{2}\, x_{in}F\, v d\tau\right\} ,$$

where

$$L_{\mu\nu}(g,\tau,\tau') = \eta_{\mu\nu}\delta(\tau-\tau') - \frac{ge_0}{2}F_{\mu\nu}\epsilon(\tau-\tau') \; . \tag{5.12}$$

The path integral (5.11) is the Gaussian one (see (4.13)). To get an answer, one needs to find the inverse matrix $L^{-1}(g,\tau,\tau')$, which satisfies the equation

$$\int L(g,\tau,\tau'')L^{-1}(g,\tau'',\tau')d\tau'' = \delta(\tau-\tau') \; .$$

One can demonstrate, that this equation is equivalent to a differential one,

$$\frac{\partial}{\partial\tau}L^{-1}(g,\tau,\tau') - ge_0F\; L^{-1}(g,\tau,\tau') = \delta'(\tau-\tau') \; , \tag{5.13}$$

with initial condition

$$L^{-1}(g,0,\tau') + \frac{ge_0F}{2}\int L^{-1}(g,\tau'',\tau')d\tau'' = \delta(\tau') \; .$$

Its solution has the form

$$L^{-1}(g,\tau,\tau') = \delta(\tau-\tau') + \frac{ge_0F}{2}\exp\{ge_0(\tau-\tau')F\}\left[\epsilon(\tau-\tau') - \tanh\left(\frac{ge_0F}{2}\right)\right] \; . \tag{5.14}$$

Using (5.14), one can find all ingredients of the general formula (4.13), taking into account that $a = -\Delta x/\sqrt{e}_0$, $I(\tau) = g\sqrt{e_0}x_{in}F/2$.

Thus,

$$K(\tau,\tau') = \delta(\tau-\tau') + \frac{ge_0F}{2}\exp\{ge_0(\tau-\tau')F\}\left[\epsilon(\tau-\tau') - \coth\left(\frac{ge_0F}{2}\right)\right] , \tag{5.15}$$
$$\int d\tau d\tau' K(\tau,\tau') = 0 \; , \; \int d\tau Q(\tau) = l(g) \; , \; l(g) = \frac{\tanh ge_0F/2}{ge_0F/2} \; ,$$
$$M(\tau,\tau') = -\frac{e_0}{2}F\epsilon(\tau-\tau') \; , \; \int_0^g dg' \mathrm{Tr} L^{-1}(g')M = \mathrm{tr}\ln(\cosh ge_0F/2) \; ,$$

where the symbol "tr" is being taken over four dimensional indices only. Then

$$\Delta(e_0) = \left[-\det\left(\frac{\sinh ge_0F/2}{gF/2}\right)\right]^{-1/2} \tag{5.16}$$
$$\times \exp\left\{\frac{i}{2}\left[\frac{\Delta x^2}{e_0} + g\; x_{out}Fx_{in} - \frac{1}{2}\Delta x\; gF\coth\left(\frac{ge_0F}{2}\right)\Delta x\right]\right\} \; .$$

Substituting (5.16) into (3.8), we get a final expression for the causal propagator of a scalar particle in a constant homogeneous electromagnetic field

$$D^c(x_{out},x_{in}) = \frac{1}{2(2\pi)^2}\int_0^\infty de_o\left[-\det\left(\frac{\sinh ge_0F/2}{gF/2}\right)\right]^{-1/2} \tag{5.17}$$
$$\times \exp\left\{\frac{i}{2}\left[gx_{out}Fx_{in} - e_0m^2 - \frac{1}{2}\Delta x\; gF\coth(\frac{ge_0F}{2})\Delta x\right]\right\} \; .$$

This result was first derived by Schwinger, using his proper time method [27].

Now we return to the total electromagnetic field (5.1). Let us substitute the potential (5.1) into (5.9),

$$\Delta(e_0) = \exp\left(i\frac{\Delta x^2}{2e_0}\right)\int \mathcal{D}v\; \delta^4\left(\int v d\tau - \frac{\Delta x}{\sqrt{e_0}}\right) \tag{5.18}$$
$$\times \exp\left\{-\frac{i}{2}\int d\tau d\tau' v(\tau)L(g,\tau,\tau')v(\tau') - i\int\frac{g\sqrt{e_0}}{2}\, x_{in}F\, v d\tau\right.$$
$$\left. -ig\sqrt{e_0}\int d\tau v(\tau) f\left(nx_{in} + \sqrt{e_0}\int_0^\tau nv(\tau')d\tau'\right)\right\} ,$$

with $L(g,\tau,\tau')$ defined in (5.12). One can take the integral (5.18) as quasi-Gaussian, in accordance with the formula (4.13). So, one can write

$$\Delta(e_0) \tag{5.19}$$
$$= \exp\left\{g\sqrt{e_0}\int d\tau f\left(nx_{in} + i\sqrt{e_0}\int_0^\tau n\frac{\delta}{\delta I(\tau')}d\tau'\right)\frac{\delta}{\delta I(\tau)}\right\} B(I)|_{I=0} \; ,$$

where

$$B(I) = \exp\left(i\frac{\Delta x^2}{2e_0}\right)\int \mathcal{D}v\; \delta^4\left(\int v d\tau - \frac{\Delta x}{\sqrt{e_0}}\right) \tag{5.20}$$
$$\times \exp\left\{-\frac{i}{2}\int d\tau d\tau' v(\tau)L(g,\tau,\tau')v(\tau') - i\int\left(\frac{g\sqrt{e_0}}{2}x_{in}F + I(\tau)\right)v(\tau)d\tau\right\} \; .$$

The integral can be found similar to (5.11). As a result we get

$$B(I)=\exp\left\{\frac{i}{2}\int d\tau d\tau' I(\tau)K(\tau,\tau')\,I(\tau')-i\int I(\tau)a(\tau)d\tau\right\}\Delta(e_0)|_{\Psi=0}\,, \tag{5.21}$$

where $\Delta(e_0)|_{\Psi=0}$ is the expression given by (5.16), $K(\tau,\tau')$ is defined in (5.15), and

$$a(\tau)=\frac{\Delta x}{2\sqrt{e_0}}\left(1+\coth\left(ge_0F/2\right)\right)ge_0F\exp\left(-ge_0F\tau\right)\,.$$

To obtain the action of the operator, involved in (5.19), on the functional $B(I)$, we decompose the sources $I^\mu(\tau)$ in the eigenvectors (5.2), using (5.8)

$$I^\mu(\tau)=\frac{1}{2}\left(n^\mu\,\bar{n}I(\tau)+\bar{n}^\mu\,nI(\tau)-\ell^\mu\,\bar{\ell}I(\tau)-\bar{\ell}^\mu\,\ell I(\tau)\right)\,.$$

Then, it is possible to write

$$n\frac{\delta}{\delta I(\tau)}=2\frac{\delta}{\delta\bar{n}I(\tau)}\,,\quad f\frac{\delta}{\delta I(\tau)}=\bar{\ell}f\frac{\delta}{\delta\bar{\ell}I(\tau)}+\ell f\frac{\delta}{\delta\ell I(\tau)}\,.$$

Using this, we get

$$\begin{aligned}
&f\left(nx_{in}+i\sqrt{e_0}\int_0^\tau n\frac{\delta}{\delta I(\tau')}d\tau'\right)\frac{\delta}{\delta I(\tau)}\\
&=\bar{\ell}f\left(nx_{in}+i\sqrt{e_0}\int_0^\tau\frac{\delta}{\delta\bar{n}I(\tau')}d\tau'\right)\frac{\delta}{\delta\bar{\ell}I(\tau)}+\ell f\left(nx_{in}+i\sqrt{e_0}\int_0^\tau\frac{\delta}{\delta\bar{n}I(\tau')}d\tau'\right)\frac{\delta}{\delta\ell I(\tau)}\,,\\
&\int d\tau d\tau' I(\tau)K(\tau,\tau')\,I(\tau')\\
&=\int d\tau d\tau'\left[\bar{n}I(\tau)\,nI(\tau')K(\tau,\tau',\mathcal{E})-\bar{\ell}I(\tau)\,\ell I(\tau')K(\tau,\tau',i\mathcal{H})\right]\,,\\
&\int I(\tau)a(\tau)d\tau=\frac{1}{2}\int\left[\bar{n}I(\tau)\,na(\tau)+nI(\tau)\,\bar{n}a(\tau)-\bar{\ell}I(\tau)\,\ell a(\tau)-\ell I(\tau)\,\bar{\ell}a(\tau)\right]d\tau\,,
\end{aligned}$$

where

$$\begin{aligned}
K(\tau,\tau',\mathcal{E})&=\delta(\tau-\tau')+\frac{ge_0\mathcal{E}}{2}\exp\{ge_0(\tau-\tau')\mathcal{E}\}\left[\epsilon(\tau-\tau')-\coth\left(\frac{ge_0\mathcal{E}}{2}\right)\right]\,,\\
K(\tau,\tau',i\mathcal{H})&=\delta(\tau-\tau')+\frac{ige_0\mathcal{H}}{2}\exp\{ige_0(\tau-\tau')\mathcal{H}\}\left[\epsilon(\tau-\tau')+i\cot\left(\frac{ge_0\mathcal{H}}{2}\right)\right]\,.
\end{aligned}$$

Now the exponent of the functional $B[I]$ is linear in $nI(\tau)$, $\bar{n}I(\tau)$, $\ell I(\tau)$, $\bar{\ell}I(\tau)$. Thus, one can easy to get a result

$$\begin{aligned}
\Delta(e_0)&=\exp\left\{\frac{i}{2}g^2e_0\int d\tau d\tau' f\left(nx_{cl}(\tau)\right)K(\tau,\tau')f\left(nx_{cl}(\tau')\right)\right.\\
&\left.+ig\sqrt{e_0}\int d\tau a(\tau)f\left(nx_{cl}(\tau)\right)\right\}\Delta(e_0)|_{\Psi=0}\,,
\end{aligned} \tag{5.22}$$

$$nx_{cl}(\tau)=nx_{in}+\frac{1-\exp\left(ge_0\mathcal{E}\tau\right)}{1-\exp\left(ge_0\mathcal{E}\right)}n\Delta x\,,\quad nx_{cl}(0)=nx_{in}\,,\quad nx_{cl}(1)=nx_{out}\,,$$

where $x_{cl}(\tau)$ is the solution [12] of the Lorentz equation in the external electromagnetic field (5.1). Gathering (5.22) and (5.16), we get

$$\begin{aligned}
\Delta(e_0)&=\left[-\det\frac{\sinh\left(ge_0F/2\right)}{ge_0F/2}\right]^{-1/2}\exp\left\{\frac{i}{2}\left[gx_{out}Fx_{in}\right.\right.\\
&-\frac{1}{2}\left(\Delta x+l(e_0,1)\right)\,gF\coth\left(ge_0F/2\right)\,\left(\Delta x+l(e_0,1)\right)+2\Phi(e_0)\\
&\left.\left.+\Delta x\,gFl(e_0,1)+\frac{\Delta x^2}{e_0}\right]\right\}\,,
\end{aligned} \tag{5.23}$$

where

$$\begin{aligned}
\Phi(e_0)&=e_0\int gf\left(nx_{cl}(\tau)\right)\left[gf\left(nx_{cl}(\tau)\right)+gFl(e_0,\tau)\right]d\tau\,,\\
l(e_0,\tau)&=e_0\int_0^\tau\exp\left\{ge_0(\tau-\tau')F\right)gf\left(nx_{cl}(\tau')\right\}d\tau'\,.
\end{aligned} \tag{5.24}$$

Substituting (5.23) into (3.8), we arrive to the final expression for the causal propagator of a scalar particle in the external electromagnetic field (5.1):

$$\begin{aligned}
D^c\left(x_{out},x_{in}\right)&=\frac{1}{2\left(2\pi\right)^2}\int_0^\infty de_0\left[-\det\left(\frac{\sinh ge_0F/2}{gF/2}\right)\right]^{-1/2}\\
&\times\exp\left\{\frac{i}{2}\left[gx_{out}Fx_{in}-e_0m^2+\Delta x\,gFl(e_0,1)\right.\right.\\
&\left.\left.-\frac{1}{2}\left(\Delta x+l(e_0,1)\right)\,gF\coth\left(ge_0F/2\right)\,\left(\Delta x+l(e_0,1)\right)+2\Phi(e_0)\right]\right\}\,.
\end{aligned} \tag{5.25}$$

This expression coincides with the one, obtained in [28], by means of the method of summation over exact solutions of Klein-Gordon equation in the external field (5.1). A detailed description of quantum electrodynamical processes in such a field one can find in [12,29].

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