UNIVERSIDADE DE SÃO PAULO

INSTITUTO DE FÍSICA
CAIXA POSTAL 20516
01452-990 SÃO PAULO - SP
BRASIL

PUBLICAÇÕES

IFUSP/P-1088

PERTURBATION THEORY ON A FINITE EUCLIDEAN LATTICE
I. PROPAGATORS OF POLYNOMIAL SCALAR MODELS

J.L. deLyra
Instituto de Física, Universidade de São Paulo

Novembro/1993

Perturbation Theory on a Finite Euclidean Lattice
I. Propagators of Polynomial Scalar Models

J. L. deLyra*
Departamento de Física Matemática
Instituto de Física,
Universidade de São Paulo,
Caixa Postal 20516,
01498-970 - São Paulo,
São Paulo, Brazil

In this paper we examine, in a new perspective, the perturbative expansion of the $SO(\mathcal{N})$-symmetric $\lambda\phi^4$ scalar field models. The lattice definition of the models is used, to allow for direct comparison with their known non-perturbative behavior, which has been determined mostly by means of computer simulations. Perturbation theory is developed in a way appropriate for the d-dimensional Euclidean treatment on a finite lattice. The propagators of the models are calculated perturbatively to order one-loop, and used to probe into their critical behavior. The mechanism which leads to the divergences in the continuum limit of the lattice is explicitly displayed. We show that lowest-order perturbation theory predicts correctly the qualitative critical behavior of the models in the continuum limit. While the interpretation of the perturbative results directly on the finite lattice is more involved, perturbation theory can also be used for the calculation of physical quantities on finite lattices.
PACS:03.70.+k;11.10.-z

I. INTRODUCTION

In this paper we examine the perturbative treatment of the $SO(\mathcal{N})$-symmetric $\lambda\phi^4$ polynomial models defined on the d-dimensional Euclidean lattice. The non-perturbative behavior of these models, as defined by means of the lattice, is reasonably well-known[1], specially due to computer simulations, and therefore these models are a good laboratory for the study of the perturbative expansion. To facilitate the comparison with the results of computer simulations, it is interesting to develop the perturbative treatment from the beginning on finite Euclidean lattices.

It is not easy to pinpoint, in the usual continuum formalism, the basic mathematical origin of the infinities that plague the perturbative expansion. Using the lattice definition we will be able to display clearly the mechanism responsible for the divergences, and therefore to understand how the infinities come about when one takes the limit. This mechanism is related to some very simple but fundamental properties of the free field theory, namely, the discontinuous character of the dominant field configurations[2]. We will also present some results related to the structure of the phase diagrams of the $\lambda\phi^4$ models.

We start in Sec. II by presenting the definition of the models, and discussing qualitatively some aspects of their non-perturbative behavior. In Sec. III we develop the formalism of the perturbative expansion in a way appropriate for the lattice theory, and discuss some important consequences of the basic properties of free scalar fields. In Sec. IV we present the results of one-loop calculations of the two-point functions of the models. Our conclusions are contained in Sec. V.

II. DEFINITION OF THE MODELS

The dynamical variables of the models are a set of scalar fields ϕ_i, $i = 1\ldots\mathcal{N}$. The action defining the classical Euclidean theories in the continuum limit, in d dimensions, is

$$S[\phi] = \int_{L^d} \mathrm{d}^d x \left[\frac{1}{2}\sum_{\nu=1}^{d}\left(\partial_\nu\vec{\phi}\cdot\partial_\nu\vec{\phi}\right) + \frac{m^2}{2}\left(\vec{\phi}\cdot\vec{\phi}\right) + \frac{\Lambda}{4}\left(\vec{\phi}\cdot\vec{\phi}\right)^2\right],$$

where the scalar-product "dot" notation is defined by $\vec{\tau}\cdot\vec{\omega} = \sum_{i=1}^{\mathcal{N}} \tau_i\,\omega_i$, m is the bare mass, Λ the bare coupling constant, and we will restrict the discussion to dimensions $d \geq 3$. We will be considering the models defined in a finite cubic box of side L, and the integral is over the volume of this box. This finite box is needed to allow for direct comparison with the results of computer simulations, which are by necessity performed in a finite box. The boundary conditions are chosen to be periodic, so that the box becomes a d-dimensional flat torus.

In order to define the model on a lattice with N sites in each direction and spacing $a = L/N$, we introduce the dimensionless fields $\varphi_i = a^{(d-2)/2}\phi_i$, and the dimensionless parameters $\alpha = m^2a^2$ and $\lambda = a^{4-d}\Lambda$. A lattice realization of the action is obtained by expressing derivatives in terms of finite forward differences $\Delta_\nu\varphi_i$, and replacing the integral by a sum over the N^d sites, denoted symbolically by x,

$$S_N[\varphi] = \sum_x^{N^d} \left[\frac{1}{2} \sum_{\nu=1}^{d} (\Delta_\nu\vec{\varphi} \cdot \Delta_\nu\vec{\varphi}) + \frac{\alpha}{2}(\vec{\varphi}\cdot\vec{\varphi}) + \frac{\lambda}{4}(\vec{\varphi}\cdot\vec{\varphi})^2 \right]. \tag{1}$$

This lattice action is the one usually considered in the literature and employed in most computer simulations. Using it we can define a quantum theory for these models. The typical expectation value of an observable $\mathcal{O}[\phi]$ is given on the N-lattice by

$$\langle \mathcal{O} \rangle_N = \frac{\int [\mathrm{d}\phi]\, \mathcal{O}[\phi]\, e^{-S_N[\phi]}}{\int [\mathrm{d}\phi]\, e^{-S_N[\phi]}},$$

where the notation for the functional-integral measure is

$$[\mathrm{d}\phi] = \prod_i^{\mathcal{N}} \prod_x^{N^d} \mathrm{d}\phi_i(x).$$

The continuum-limit Euclidean expectation value is obtained by taking the $N \to \infty$ limit of the lattice expression,

$$\langle \mathcal{O} \rangle = \lim_{N\to\infty} \langle \mathcal{O} \rangle_N .$$

The final step in obtaining the expectation values of the quantum theory is to extend the Euclidean results back into the Minkowski sector by analytic continuation.

In order to put the perturbative calculations in proper perspective, it is necessary to consider the critical behavior of the models[4]. There are two free parameters, α and λ, but there are also conditions which must be satisfied in the continuum limit in order that the models exist as field theoretical models. These conditions are essentially that all physical observables be finite, and they imply that one must choose some definite functions $\alpha(N)$ and $\lambda(N)$, as $N \to \infty$. A fundamental condition is that, in the continuum limit, the renormalized mass be finite, or that the correlation length be different from zero. This condition implies a relation between the limiting values $\alpha(\infty) = \alpha_c$ and $\lambda(\infty) = \lambda_c$, which defines the so-called *critical curves* of the models, $\lambda_c = f_c(\alpha_c)$.

A relevant non-perturbative aspect of the models is that while λ must be positive to ensure the existence of the ground states and therefore the stability of the models, α may be negative if λ is not zero. In fact, in order that continuum limits to points other than the Gaussian point ($\alpha = 0, \lambda = 0$) may exist, α *must* be strictly negative in the limit, a fact which will play an important role in the perturbative analysis.

The relation between α and λ in the limit determines whether the symmetry breaks or not, defining two separate regions or *phases* in the (α, λ) parameter plane. The curve that separates the two phases in the (α, λ) plane is the critical curve $\lambda_c = f_c(\alpha_c)$, which is roughly a straight line starting at the Gaussian point and extending to the $(\alpha < 0, \lambda > 0)$ quadrant. It is possible to estimate the position of this critical curve from a qualitative analysis of the symmetry-breaking mechanism[5], which gives a linear relation between λ and α, and is qualitatively consistent with both the mean-field and the perturbative results. A typical such (α, λ) *phase diagram* of the models can be found in Fig. 3 of an earlier paper[5].

In the region which is to become the symmetric phase in the continuum limit, one expects that the expectation values of the field components be small and become zero in the limit. Also, the renormalized mass parameters of the propagators for each field component should all be equal, and different from zero. In the region which is to become the broken-symmetric phase, it is to be expected that the expectation value of at least one of the field components be different from zero, and that the renormalized mass parameters of all but one field component be small, and become zero in the limit. We will verify the extent to which these expectations are realized by perturbation theory on the lattice.

III. THE PERTURBATIVE EXPANSION

The main idea of the perturbative theory is to develop an expansion for the complete models around solvable Gaussian models. Since any Gaussian model is exactly solvable, one is free to choose to expand the theory around any particular Gaussian ensemble. Presumably, for small values of the coupling constant the exact results are not too different from the corresponding results of the free theory, and the expansion can be used to obtain useful approximations to the complete theory for small values of the coupling constant.

The first step in the development of the perturbative theory is the separation of the action in two parts,

$$S = S_0 + S_V,$$

where S_0 is a Gaussian action, and we assume from now on that all quantities are written on a finite lattice. We have for an observable $\mathcal{O}$

$$\langle \mathcal{O} \rangle = \frac{\int [\mathrm{d}\varphi] \mathcal{O}[\varphi] e^{-S_0} e^{-S_V}}{\int [\mathrm{d}\varphi] e^{-S_0} e^{-S_V}}.$$

We now write this in terms of the measure of the Gaussian theory defined by S_0, dividing both the denominator and the numerator by $\int [\mathrm{d}\varphi] e^{-S_0}$, and thus obtaining

$$\langle \mathcal{O} \rangle = \frac{\left\langle \mathcal{O}\, e^{-S_V} \right\rangle_0}{\left\langle e^{-S_V} \right\rangle_0}, \tag{2}$$

where the subscript 0 denotes expectation values in the theory defined by S_0,

$$\langle \mathcal{O} \rangle_0 = \frac{\int [\mathrm{d}\varphi] \mathcal{O}[\varphi] e^{-S_0}}{\int [\mathrm{d}\varphi] e^{-S_0}}.$$

The term S_V of the action is the one containing λ, which is presumed small. But S_V may also contain other parameters, and in order to develop the perturbative expansion, rather than using λ directly as the expansion parameter, it is convenient to introduce a new parameter ϵ in the following way:

$$f(\epsilon) = \frac{\left\langle \mathcal{O}\, e^{-\epsilon S_V} \right\rangle_0}{\left\langle e^{-\epsilon S_V} \right\rangle_0}.$$

We have therefore $f(0) = \langle \mathcal{O} \rangle_0$ and $f(1) = \langle \mathcal{O} \rangle$. The perturbative expansion consists of the expansion of $f(\epsilon)$ around $\epsilon = 0$, to a certain desired order, and the application of the resulting expressions at $\epsilon = 1$. Of course, this can only be a good approximation if S_V is a small quantity. Classically, one can make S_V small by decreasing λ and any other parameters that it might contain, but, as we will see shortly, this is not possible in the continuum limit of the quantum theory, and this fact is at the root of all difficulties with the perturbative expansion.

In order to understand the origin of the difficulties, one must recall some important properties of the free theory[3]. Assuming for this discussion that S_0 has the generic form

$$S_0 = \sum_x^{N^d} \left[\frac{1}{2} \sum_{\nu=1}^{d} (\Delta_\nu \vec{\varphi} \cdot \Delta_\nu \vec{\varphi}) + \frac{\alpha_0}{2} (\vec{\varphi} \cdot \vec{\varphi}) \right], \tag{3}$$

where α_0 is non-negative, to ensure the stability of the measure, one can calculate[3], for an arbitrary field component, the quantity

$$\langle \varphi_i^2 \rangle_0 = \sigma_0^2(N, d, \alpha_0) = \frac{1}{N^d} \sum_k^{N^d} \frac{1}{\rho_k^2 + \alpha_0},$$

where the quantities ρ_k^2 and the integers k are defined in the Appendix, no sum over i is implied, and symmetry requires that σ_0 be independent of i. The summation convention will *not* be used in this paper, and, except for the scalar-product "dot" notation, all sums will be indicated explicitly.

This σ_0 is the width of the local distribution of values of the i-th component of the field. It has a singular behavior on finite lattices for $\alpha_0 = 0$, due the the existence of a zero-mode on the torus. We may write it as

$$\sigma_0^2(N, d, \alpha_0) = \frac{1}{N^d \alpha_0} + \sigma_0'^2(N, d, \alpha_0),$$
$$\sigma_0'^2(N, d, \alpha_0) = \frac{1}{N^d} \sum_k^{N^d}{}' \frac{1}{\rho_k^2 + \alpha_0},$$

where $\sum'$ denotes the sum without the zero-mode $\rho_k = 0$. One can see here that the $\alpha_0 \to 0$ limit of the first term diverges for finite N. But if we make $\alpha_0 = m_0^2 a^2 = m_0^2/N^2$ where m_0 is some finite and non-zero real number, then the $N \to \infty$ limit of the first term is zero as long as $d \geq 3$. It can be verified that in this case $\sigma_0'^2$ converges, in each dimension, to finite values of order one in the continuum limit. Numerical evaluations of the sums give the following large-N asymptotic results:

$$\sigma_0'^2(d=3) \approx 0.25274,$$
$$\sigma_0'^2(d=4) \approx 0.15493,$$
$$\sigma_0'^2(d=5) \approx 0.11563,$$

where the last digit in each result is uncertain due to the numerical errors. The results above were calculated for the case $m_0 = 0$ but, remarkably, it can be verified that the limiting values are independent of m_0. Because σ_0^2 will play an important role in the perturbative calculations, here it becomes clear that we cannot make $\alpha_0 = 0$ on finite lattices, although it is possible to make $\alpha_0 \to 0$ in the continuum limit without introducing divergencies into σ_0^2.

One can also show that in the free measure, for a given field component,

$$\langle \varphi_i^4 \rangle_0 = 3 \langle \varphi_i^2 \rangle_0^2,$$

and from this it follows that, assuming the general form for S_V

$$S_V = \sum_x^{N^d} \left[\frac{\alpha}{2} (\vec{\varphi} \cdot \vec{\varphi}) + \frac{\lambda}{4} (\vec{\varphi} \cdot \vec{\varphi})^2 \right],$$

we have for its expectation value

$$\langle S_V\rangle_0 = \frac{\mathcal{N}\sigma_0^2(N,d,\alpha_0)}{2}\left[\alpha + \frac{\mathcal{N}+2}{2}\sigma_0^2(N,d,\alpha_0)\lambda\right] N^d.$$

This means that, as long as the factor in brackets is not zero in the limit[6], $\langle S_V\rangle_0$ diverges as N^d in the continuum limit, a result which is directly related to the issue of the discontinuity of the fields discussed in a previous paper[3]. Note that this divergence of S_V is *not* due to an integration over an infinite volume, since we are considering here the theory defined in a finite box. This makes it clear that we cannot make S_V small by just changing the values of α and λ, except if we make them equal to zero in the limit, converging to the Gaussian point.

If we consider the denominator of Eq. (2), it is now clear that we will have, in the continuum limit,

$$\left\langle e^{-S_V}\right\rangle_0 \to 0,$$

and, for $\epsilon \neq 0$, the perturbative expansion of the exponential will contain terms that diverge in the continuum limit,

$$\left\langle e^{-\epsilon S_V}\right\rangle_0 \approx 1 - \epsilon\langle S_V\rangle_0 + \ldots, \quad \langle S_V\rangle_0 \to \infty.$$

We can see now that the limit of Eq. (2) for large N is of the form $0/0$. It nevertheless exists, as long as the theory is non-perturbatively well-defined. The denominator of Eq. (2) can be understood as the ratio of the measures of the free and interacting theories,

$$\left\langle e^{-S_V}\right\rangle_0 = \frac{\int[\mathrm{d}\varphi]e^{-S_0}e^{-S_V}}{\int[\mathrm{d}\varphi]e^{-S_0}}, \tag{4}$$

and the conclusion is, therefore, that in the continuum limit these two measures are related in a singular way. For any finite N-lattice, $\langle S_V\rangle_0$ is finite, one can make the parameters sufficiently small, and thus improve the approximation of the full theory by its perturbative expansion. But in the continuum limit the only way to avoid the divergences is to make $\alpha \to 0$, $\lambda \to 0$.

We argue that this behavior of $\langle S_V\rangle_0$ is the fundamental cause of all infinities that appear in the perturbative expansion. However, since in Eq. (2) we have the ratio of two quantities involving S_V, it is possible that some or all of these N^d divergences will cancel out. Also, the singularity in the relation between the two measures in Eq. (4) in the continuum limit indicates that even if all such strongly divergent term do cancel

out, there may still be other divergences left over when one writes observables of the full theory in terms of observables of the free theory. These are probably not strong power-law divergences, but weaker ones, possibly logarithmic.

If the theory is to be perturbatively renormalizable, all the strongly divergent terms must cancel out order by order in the perturbative expansion. The theory will be perturbatively finite, or perturbatively renormalizable in a strong sense, if and only if *all* divergent terms, strong and weak, cancel out order by order in the expansion. Even if this is not the case, the theory may still be perturbatively renormalizable in a weaker sense if one is able to reparametrize the theory so that the remaining weak divergences are absorbed into its bare constants. We will see that the models considered here are in fact one-loop finite in what concerns the propagators.

Notwithstanding all this, it is reasonable to think that in each phase the observables $\langle\mathcal{O}\rangle$ of the full theory are some smooth functions of its parameters, and therefore that $f(\epsilon)$ is a smooth function of ϵ, so that there should be a convergent expansion for it around $\epsilon = 0$. At least, if the function is differentiable to a certain order, there should be an approximation up to that order. In this paper we will examine only the λ^0 and λ^1 terms, for which we get

$$\begin{aligned} f(\epsilon) &= f(0) + \epsilon f'(0) + \ldots, \\ f(0) &= \langle\mathcal{O}\rangle_0, \\ f'(0) &= -\left[\langle\mathcal{O}S_V\rangle_0 - \langle\mathcal{O}\rangle_0\langle S_V\rangle_0\right]. \end{aligned}$$

Making $\epsilon = 1$ we get

$$\langle\mathcal{O}\rangle \approx \langle\mathcal{O}\rangle_0 - \left[\langle\mathcal{O}S_V\rangle_0 - \langle\mathcal{O}\rangle_0\langle S_V\rangle_0\right]. \tag{5}$$

This is the approximation for $\langle\mathcal{O}\rangle$ up to order λ. We will use it in Sec. IV to calculate perturbatively some particular observables. Here we see that it is indeed possible that the divergences in $\langle\mathcal{O}S_V\rangle_0$ will cancel those in $\langle S_V\rangle_0$.

We must now return to the issue of the separation of the action into S_0 and S_V. This will depend on whether one is in the symmetric or broken-symmetric phases or, more precisely, on whether or not $\langle\vec{\varphi}\rangle = 0$. In any case S_0 must satisfy two conditions: it must be quadratic on the fields, and it must be stable, meaning that it must correspond to a well-behaved free theory with a finite mass, whose Euclidean action is bounded from below.

The stability issue must be examined carefully here. It can be shown that, in any continuum limit of the theory that does not approach the Gaussian point, α will always become negative[5]. We cannot, therefore, include the α term in S_0, because this quadratic action would become unbounded from below, and the corresponding measure would be ill-defined. The alternative of leaving only the derivative term in S_0 and simply including the α term in S_V is also unsuitable, because the massless free theory thus produced has a zero-mode that may lead to spurious infrared divergences in the expansion.

Let us consider first the symmetric phase. In order to avoid infrared trouble, we introduce a new parameter $\alpha_0 > 0$, and choose the form of Eq. (3) for the free part of the action. The potential term S_V will be the remainder of the original action. In this phase the corresponding S_V is therefore

$$S_V = \sum_x^{N^d} \left[\frac{\alpha - \alpha_0}{2} (\vec{\varphi} \cdot \vec{\varphi}) + \frac{\lambda}{4} (\vec{\varphi} \cdot \vec{\varphi})^2 \right].$$

As long as it is positive, we are free to choose α_0 at will. It is reasonable to choose $\alpha_0 = m_0^2/N^2$, with a fixed and finite m_0, so that S_0 will correspond to a well-behaved free theory in the continuum limit. In particular, making $m_0 = m_R$ seems to be a natural and convenient choice. Since we can choose at will the Gaussian ensemble we are to expand about, it is clearly convenient to make it as similar as possible to the ensemble of the complete theory. The choice $\alpha_0 = \alpha_R$ ensures that the second moment of the two ensembles are identical. We will see later that this choice of α_0 is in fact needed in order to make the perturbative propagator well-behaved.

In the broken-symmetric phase we have $\langle \vec{\varphi} \rangle \neq 0$, and in order to develop the perturbative expansion in a simple way we rewrite the model in terms of a shifted field $\vec{\varphi}'$, with

$$\vec{\varphi} = \vec{\varphi}' + \vec{v}, \quad \langle \vec{\varphi}' \rangle = \vec{0}, \quad \langle \vec{\varphi} \rangle = \vec{v} = (0, \ldots, 0, v),$$

where we chose the coordinate system in the internal manifold so that the direction of symmetry breakdown is the direction of the component $\varphi_{\mathcal{N}}$. In computer simulations this condition can be satisfied exactly, since one may eliminate the drifting of the direction of symmetry breakdown by making global rotations of the fields at periodic intervals. This amounts, in fact, to the complete elimination from the ensemble of the theory of the zero-modes of all but the $\mathcal{N}^{th}$ field component. One can also consider excluding these zero modes from the Gaussian ensemble to be used in the perturbative theory, but it turns out that doing so does not significantly change the results.

Under this change of variables from φ to φ' the derivative term of the action in Eq. (1) remains unaltered and, dropping field-independent terms, which cancel off in the ratios of functional integrals defining the observables, we have,

$$S_N[\varphi] = \sum_x^{N^d} \left[\frac{1}{2} \sum_{\nu=1}^{d} \Delta_\nu \vec{\varphi}' \cdot \Delta_\nu \vec{\varphi}' + \frac{\alpha + \lambda v^2}{2} \left(\vec{\varphi}' \cdot \vec{\varphi}' \right) + \lambda v^2 {\varphi'_{\mathcal{N}}}^2 \right.$$
$$\left. + v \left(\alpha + \lambda v^2 \right) \varphi'_{\mathcal{N}} + \lambda v \left(\vec{\varphi}' \cdot \vec{\varphi}' \right) \varphi'_{\mathcal{N}} + \frac{\lambda}{4} \left(\vec{\varphi}' \cdot \vec{\varphi}' \right)^2 \right].$$

We introduce now the constant $\alpha_0 > 0$, and separate the action in the free part

$$S_0 = \sum_x^{N^d} \left[\frac{1}{2} \sum_{\nu=1}^{d} \left(\Delta_\nu \vec{\varphi}' \cdot \Delta_\nu \vec{\varphi}' \right) + \frac{\alpha_0}{2} \left(\vec{\varphi}' \cdot \vec{\varphi}' \right) \right],$$

and the potential part

$$S_V = \sum_x^{N^d} \left[v \left(\alpha + \lambda v^2 \right) \varphi'_{\mathcal{N}} + \frac{\alpha - \alpha_0 + \lambda v^2}{2} \left(\vec{\varphi}' \cdot \vec{\varphi}' \right) \right.$$
$$\left. + \lambda v^2 {\varphi'_{\mathcal{N}}}^2 + \lambda v \left(\vec{\varphi}' \cdot \vec{\varphi}' \right) \varphi'_{\mathcal{N}} + \frac{\lambda}{4} \left(\vec{\varphi}' \cdot \vec{\varphi}' \right)^2 \right].$$

This is the separation of the action to be used in the broken-symmetric phase. Note that, since in this phase one component of the field will be singled out by the symmetry breakdown mechanism, it is likely that α_0 will have to be chosen differently for different components of the field, so that we may have the condition $\alpha_0 = \alpha_R$ for all field components.

IV. ONE-LOOP CALCULATIONS

We examine first the propagator in the symmetric phase, that is, we calculate the propagator under the assumption that $\langle \vec{\varphi} \rangle = 0$. This should be a good approximation to the complete result in the region which becomes the symmetric phase in the continuum limit, where $\langle \vec{\varphi} \rangle$ is small. To order zero-loop we have for the dimensionless propagator

$$\langle \varphi_i(x) \varphi_i(y) \rangle \approx \frac{1}{N^d} \sum_k^{N^d} f_p^N(x-y) \frac{R}{\rho_k^2 + \alpha_R},$$

where the lattice mode functions $f_p^N(x-y)$ are defined in the Appendix, the renormalized mass parameter is $\alpha_R = \alpha$, and the residue of the pole of the corresponding dimensional propagator is $R = 1$. We calculate now the propagator to the first non-classical order, that is, to order one loop. Due to the symmetry, it is enough to pick an arbitrary field component for the calculation. We have now, using Eq. (5),

$$\langle\varphi_i(x)\varphi_i(y)\rangle \approx g_1(x,y),$$

$$g_1(x,y) = \langle\varphi_i(x)\varphi_i(y)\rangle_0 - \left[\langle\varphi_i(x)\varphi_i(y)S_V\rangle_0 - \langle\varphi_i(x)\varphi_i(y)\rangle_0\langle S_V\rangle_0\right].$$

The expectation values appearing here involve only Gaussian integrals, and are given by

$$g_0(x,y) = \frac{1}{N^d}\sum_k^{N^d} f_p^N(x-y)\frac{1}{\rho_k^2+\alpha_0},$$

$$\langle\varphi_i(x)\varphi_i(y)\rangle_0 = g_0(x,y),$$

$$\langle S_V\rangle_0 = \mathcal{N}\frac{\alpha-\alpha_0}{2}N^d\sigma_0^2(N,d,\alpha_0) + \mathcal{N}(\mathcal{N}+2)\frac{\lambda}{4}N^d\sigma_0^4(N,d,\alpha_0),$$

$$\begin{aligned}\langle\varphi_i(x)\varphi_i(y)S_V\rangle_0 = &\mathcal{N}\frac{\alpha-\alpha_0}{2}N^d\sigma_0^2(N,d,\alpha_0)g_0(x,y)\\ &+(\alpha-\alpha_0)\sum_z^{N^d} g_0(x,z)g_0(z,y)\\ &+\mathcal{N}(\mathcal{N}+2)\frac{\lambda}{4}N^d\sigma_0^4(N,d,\alpha_0)g_0(x,y)\\ &+(\mathcal{N}+2)\lambda\sigma_0^2(N,d,\alpha_0)\sum_z^{N^d} g_0(x,z)g_0(z,y).\end{aligned}$$

In these calculations all the strong divergences, consisting of terms proportional to N^d, cancel out, a fact that corresponds to the usual cancellation of vacuum bubbles. Given all this we can write for the propagator

$$g_1(x,y) = \frac{1}{N^d}\sum_k^{N^d} f_p^N(x-y)\left[\frac{1}{\rho_k^2+\alpha_0} + \frac{\alpha_0-\alpha-(\mathcal{N}+2)\lambda\sigma_0^2(N,d,\alpha_0)}{\left(\rho_k^2+\alpha_0\right)^2}\right].$$

If we now choose α_0 such that this propagator has a simple pole at $\rho_k^2 = -\alpha_R$, we discover that we must have

$$\alpha_0 = \alpha + (\mathcal{N}+2)\lambda\sigma_0^2(N,d,\alpha_0),$$

and that the propagator can then be written as

$$g_1(x,y) = \frac{1}{N^d}\sum_k^{N^d} f_p^N(x-y)\frac{1}{\rho_k^2+\alpha_R},$$

which corresponds to the momentum-space propagator

$$\tilde{g}_1(p) = \frac{1}{N^d}\frac{1}{\rho_p^2+\alpha_R},$$

where the renormalized mass m_R is given by $\alpha_R = \alpha_0 = m_R^2L^2/N^2$.

Superficially, this expression for the renormalized mass can be understood as the sum of an order-zero term α with an order-λ correction, but in truth this is misleading, since when one takes into account the quantum fluctuations by including the one-loop term, α becomes in fact negative, and the first-order term cannot be understood as a small correction to the classical result, where α must be positive. Note also that the residue of the pole of the dimensional propagator corresponding to the expression above is equal to one, showing that to this order there is no need for field renormalization. This is qualitatively consistent with the situation observed non-perturbatively in the few preliminary computer simulations tried so far. Note also that there are no divergences left in the results, showing that the propagators are one-loop finite.

We have to this order, therefore, the propagator of a free theory, with a renormalized mass. A closer examination of the renormalized mass parameter will give us some insight into the symmetry breaking mechanism of the models. Note that we do not really have an explicit expression for α_R, but rather a self-consistent equation, due to the facts that we were led the choice $\alpha_0 = \alpha_R$, and that σ_0^2 depends on α_0,

$$\alpha_R = \alpha + (\mathcal{N}+2)\lambda\sigma_0^2(N,d,\alpha_R).$$

In order to examine the behavior of this α_R in both the finite lattices and the continuum limit, we first rewrite it in terms of $\sigma_0'^2$,

$$\alpha_R = \alpha + (\mathcal{N}+2)\lambda\sigma_0'^2(N,d,\alpha_R) + \frac{(\mathcal{N}+2)\lambda}{N^d\alpha_R}.$$

In this expression the main dependencies on α_R are made explicit, since, whether or not α_R is zero, $\sigma_0'^2$ is a finite non-zero number, either on the finite lattices or in the continuum limit. We may now write this as

$$\alpha_R^2 - \alpha_R\left[\alpha + (\mathcal{N}+2)\lambda\sigma_0'^2(N,d,\alpha_R)\right] - \frac{(\mathcal{N}+2)\lambda}{N^d} = 0. \tag{6}$$

In the continuum limit the third term vanishes and we get

$$\alpha_R = \alpha + (\mathcal{N}+2)\lambda\sigma_0'^2(\infty, d),$$

where $\sigma_0'^2(\infty, d)$ is no longer dependent on α_R or m_R. Since in a second-order phase transition α_R is zero at the transition, this gives at once an equation for the critical curve of the models,

$$\alpha + (\mathcal{N}+2)\lambda\sigma_0'^2(\infty, d) = 0.$$

The fact that we must also have $\alpha_R \geq 0$ establishes the range of validity of the calculation to be that region of the parameter plane where

$$\alpha + (\mathcal{N}+2)\lambda\sigma_0'^2(\infty, d) \geq 0,$$

which is therefore the symmetric phase.

In order to further analyze the situation on the finite lattices, we write the roots of Eq. (6), getting

$$\alpha_R = \frac{\alpha + (\mathcal{N}+2)\lambda\sigma_0'^2(N, d, \alpha_R)}{2} \pm \sqrt{\left[\frac{\alpha + (\mathcal{N}+2)\lambda\sigma_0'^2(N, d, \alpha_R)}{2}\right]^2 + \frac{(\mathcal{N}+2)\lambda}{N^d}}.$$

Although $\sigma_0'^2$ is still a function of α_R in this expression, for the analysis it is enough to know that it is a finite and positive number, for any values of α_R and N. As long as N is finite, one can see here that the negative sign for the square root leads to a negative α_R, and must therefore be discarded, leaving as the only possibility

$$\alpha_R = \frac{\alpha + (\mathcal{N}+2)\lambda\sigma_0'^2(N, d, \alpha_R)}{2} + \sqrt{\left[\frac{\alpha + (\mathcal{N}+2)\lambda\sigma_0'^2(N, d, \alpha_R)}{2}\right]^2 + \frac{(\mathcal{N}+2)\lambda}{N^d}}. \qquad (7)$$

Here we can see that as long as N is finite we cannot have $\alpha_R = 0$, because the value of the square root is positive and larger than the other term. This result is in accordance with the well-known fact that there is no phase transition on finite lattices with periodic boundary conditions. Note that for finite N the expression above for α_R can be calculated at any point (α, λ) of the parameter plane where the theory is stable.

In the region where $[\alpha + (\mathcal{N}+2)\lambda\sigma_0'^2(N, d, \alpha_R)]$ is positive α_R approaches this quantity in the continuum limit, while where $[\alpha + (\mathcal{N}+2)\lambda\sigma_0'^2(N, d, \alpha_R)]$ is negative, if this is at all possible, α_R approaches zero. Whether or not Eq. (7) is a good approximation on finite lattices will depend on whether or not $\langle\vec{\varphi}\rangle$ is close to zero for the given values of (α, λ). Of course, a precise determination of α_R for given values of (α, λ) on a finite lattice will have to rely on a numerical solution of Eq. (6), taking into account the dependence of $\sigma_0'^2$ of α_R.

Since on the one hand there is no true phase transition on finite lattices, and on the other hand there is some amount of spontaneous magnetization at least in some regions of the parameter plane, we expect that at any point (α, λ) of the plane we will in fact have $\langle\vec{\varphi}\rangle \neq 0$ on finite lattices, whether it is small or not. Therefore, it is to be expected that a calculation of the propagator assuming that $\langle\vec{\varphi}\rangle \neq 0$ will be a better representation of the exact results on finite lattices, specially in the region which becomes the broken-symmetric phase in the continuum limit.

We calculate next, therefore, the propagator in the broken-symmetric phase, assuming now that $\langle\varphi_i\rangle = 0$ for $i \neq \mathcal{N}$, but that $\langle\varphi_{\mathcal{N}}\rangle \neq 0$. In this case we must calculate the propagator of the $\mathcal{N}^{th}$ component separately from the others, which are all equivalent due to the remaining $SO(\mathcal{N}-1)$ symmetry. The calculations are otherwise similar to the ones in the symmetric phase. Again the N^d divergences cancel off, and we end up, for all but the $\mathcal{N}^{th}$ component, with

$$\langle\varphi_i(x)\varphi_i(y)\rangle \approx g_{1,i\neq\mathcal{N}}(x, y),$$

$$g_{1,i\neq\mathcal{N}}(x, y) = \frac{1}{N^d}\sum_k^{N^d} f_p^N(x-y)\left[\frac{1}{\rho_k^2+\alpha_0} + \frac{\alpha_0}{(\rho_k^2+\alpha_0)^2}\right].$$

Here we encounter a difficulty, because the choice of α_0 which causes this to display a simple pole is $\alpha_0 = 0$, which we cannot impose on a finite lattice. We may, however, make α_0 approach zero in the continuum limit, and then we get $(\mathcal{N}-1)$ massless Goldstone bosons, as the Goldstone theorem requires for a symmetry breakdown from $SO(\mathcal{N})$ to $SO(\mathcal{N}-1)$. Presumably, on a finite lattice the Goldstone bosons are not completely massless, but have a small mass that tends to zero in the continuum limit. On such a finite lattice, however, we are unable to draw any definite conclusions from the result above.

If we calculate now the propagator for the $\mathcal{N}^{th}$ field component we get

$$\langle\varphi_{\mathcal{N}}(x)\varphi_{\mathcal{N}}(y)\rangle - \langle\varphi_{\mathcal{N}}(x)\rangle\langle\varphi_{\mathcal{N}}(y)\rangle \approx g_{1,\mathcal{N}}(x, y),$$

$$g_{1,\mathcal{N}}(x,y) = \frac{1}{N^d}\sum_k^{N^d} f_p^N(x-y)\left[\frac{1}{\rho_k^2+\alpha_R} + \frac{\alpha_0+\alpha+(\mathcal{N}+2)\lambda\sigma_0'^2(N,d,\alpha_R)}{(\rho_k^2+\alpha_R)^2}\right],$$

which has a simple pole as long as we choose

$$\alpha_0 = -2\left[\alpha+(\mathcal{N}+2)\lambda\sigma_0^2(N,d,\alpha_0)\right],$$

so that the momentum-space propagator reduces to

$$\tilde{g}_{1,\mathcal{N}}(p) = \frac{1}{N^d}\frac{1}{\rho_p^2+\alpha_R},$$

where the renormalized mass m_R is again given by $\alpha_R = \alpha_0 = m_R^2L^2/N^2$. In this case we have for the self-consistent equation for α_R

$$\alpha_R^2 + 2\alpha_R\left[\alpha+(\mathcal{N}+2)\lambda\sigma_0'^2(N,d,\alpha_R)\right] + \frac{2(\mathcal{N}+2)\lambda}{N^d} = 0. \tag{8}$$

In the continuum limit we now have

$$\alpha_R = -2\left[\alpha+(\mathcal{N}+2)\lambda\sigma_0'^2(\infty,d)\right], \tag{9}$$

and hence the condition $\alpha_R = 0$ gives the same equation as before for the critical curve. In this broken-symmetric calculation, the range of validity of the results is the one defined by

$$\alpha+(\mathcal{N}+2)\lambda\sigma_0'^2(\infty,d) \leq 0,$$

the complement of the one encountered in the symmetric-phase calculation.

In order to analyze the situation on finite lattices, we write the roots of the self-consistent Eq. (8),

$$\alpha_R = -\left[\alpha+(\mathcal{N}+2)\lambda\sigma_0'^2(N,d,\alpha_R)\right] \pm\sqrt{\left[\alpha+(\mathcal{N}+2)\lambda\sigma_0'^2(N,d,\alpha_R)\right]^2 - \frac{2(\mathcal{N}+2)\lambda}{N^d}}.$$

In this case if $[\alpha+(\mathcal{N}+2)\lambda\sigma_0'^2(N,d,\alpha_R)]$ is positive both signs for the square root give a negative α_R and must be discarded. Therefore, the result only applies if

$$\alpha+(\mathcal{N}+2)\lambda\sigma_0'^2(N,d,\alpha_R) < 0.$$

Besides, in order that the argument of the square root be non-negative, we must also have

$$\left[\alpha+(\mathcal{N}+2)\lambda\sigma_0'^2(N,d,\alpha_R)\right]^2 \geq \frac{2(\mathcal{N}+2)\lambda}{N^d}.$$

Whenever these conditions are satisfied, one has two possible values for α_R, neither one of which can be zero on finite lattices,

$$\alpha_R = -\left[\alpha+(\mathcal{N}+2)\lambda\sigma_0'^2(N,d,\alpha_R)\right] +\sqrt{\left[\alpha+(\mathcal{N}+2)\lambda\sigma_0'^2(N,d,\alpha_R)\right]^2 - \frac{2(\mathcal{N}+2)\lambda}{N^d}},$$

and

$$\alpha_R = -\left[\alpha+(\mathcal{N}+2)\lambda\sigma_0'^2(N,d,\alpha_R)\right] -\sqrt{\left[\alpha+(\mathcal{N}+2)\lambda\sigma_0'^2(N,d,\alpha_R)\right]^2 - \frac{2(\mathcal{N}+2)\lambda}{N^d}}. \tag{10}$$

We can see now that in the continuum limit the first possibility approaches the finite, non-zero result in Eq. (9), while the second possibility approaches zero. Therefore, we must conclude that the first possibility represents the mass of the $\mathcal{N}^{th}$ field component. The interpretation of the second possibility is more delicate. Since it has the expected qualitative behavior for the mass of the Goldstone bosons in the continuum limit, it is tempting to interpret it as the mass parameter for these bosons on the finite lattice, but since it comes out of the calculation of the massive mode, while the calculation of the massless modes was inconclusive for finite lattices, this interpretation cannot be correct. The alternative interpretation is that this second solution corresponds to the unstable symmetric solution which exists even in the broken-symmetric phase. This corresponds, classically, to the unstable situation where the fields are at the local maximum of the potential at $\vec{\varphi} = 0$.

V. CONCLUSIONS

In the process of developing perturbation theory on a finite Euclidean lattice, we have discovered that the emergence of divergences in the continuum limit is closely related to the discontinuous character of the dominant field configurations in the functional integral[3]. This establishes a clear distinction between true divergences in the theory, such as the divergence of the expectation value of the potential part of the action, and artifacts of the perturbative expansion, which are not truly relevant for the definition of the theory.

We have seen that in the Euclidean perturbative formalism one must take the stability of the ensembles involved carefully into consideration, and that the results of the perturbative calculations of the propagators of the models improve greatly by the imposition of the self-consistence condition $\alpha_0 = \alpha_R$. The zero-modes of the models play then a very important role in the mechanism of the $SO(\mathcal{N}) \to SO(\mathcal{N}-1)$ symmetry breakdown. In particular the singular behavior of the quantity $\sigma_0^2 = \langle\varphi_i^2\rangle$ is important for the analysis of the phase transitions on the lattice.

For a phase transition to be possible it is necessary that σ_0^2 be finite even for $\alpha_R = 0$. The fact that this is not possible on finite lattices, but only in the continuum limit, is related to the well-known fact that there is no phase transition on finite periodic lattices. It is also interesting to note that the fact that σ_0^2 is never finite for $d = 1$ and $d = 2$ is related in a similar way to another well-known fact, namely that there can be no phase transition in these dimensions.

Perturbative calculations performed on finite lattices may be very useful as tools for the confirmation and control of computer simulations. They may also be useful to extend the results of these simulations. For example, the $\mathcal{N}$-dependence encountered in the results seems to be either exact or a very good approximation, a fact also confirmed by mean-field calculations[5]. Therefore, simulations performed with one symmetry group may give relevant information for all symmetry groups.

In the calculations presented here we have shown that the propagators are finite to order one-loop, and it would be interesting to try the corresponding two-loop calculations, which could be important, for example, for the determination of the critical exponents of the transitions. In particular, improvements on the detailed knowledge of α_R on finite lattices would give us further insight into the mechanism of the approach to the continuum, and of the nature of the phase transition. These calculations, which are considerably more complex than the ones presented here, are in progress.

It would be very interesting to compare some of the perturbative results presented here with corresponding non-perturbative results obtained from precise Monte Carlo simulations of the models. This would determine, in a quantitative way, how well perturbation theory works in these models, and may give us hints about the circumstances under which we may trust the perturbative results. In particular it would be interesting to examine the slopes of the constant-α_R curves near the α-axis, which are important because they can guide us to the understanding of the possible continuum-limit flows of the theory. Also, one may be able to clarify the interpretation of the result in Eq. (10).

The few preliminary simulations which were tried so far were qualitatively consistent with the results presented here, but lacked sufficient statistics and fast enough convergence to thermal equilibrium to allow for precise comparison. In particular, it turns out that it is quite difficult to run precise Monte Carlo simulations near the Gaussian point, due to the shallowness of the potential there, which causes very slow relaxation of the initial configuration. Further work on the technical issues involved, aiming at better simulations, is in progress.

ACKNOWLEDGMENTS

The author would like to thank Dr. T. E. Gallivan and Mr. A. Kapulkin for many interesting discussions and suggestions on the matters discussed here. He would also like to thank Prof. H. Fleming for his continued support and encouragement, his interest in this work, and for many fruitful discussions about it.

The computer work involved in this paper was carried out entirely on the cluster of Unix workstations of the "Departamento de Física Matemática", which were acquired mainly with several grants from FAPESP, and also with a grant from CNPq.

APPENDIX: FOURIER TRANSFORMS

We discuss here the issue of transformation to momentum space for a finite lattice with periodic boundary conditions. In the continuum, the appropriate mode functions for transformation to momentum space in a finite cubic box are the usual complex phases,

$$f_p(x) = \exp\left(\imath\frac{2\pi k_1 x_1}{L} + \ldots + \imath\frac{2\pi k_d x_d}{L}\right),$$

where $p(k) = 2\pi\,(k_1, \ldots, k_d)\,/L$, and $k_\nu = 0, \pm1, \pm2, \pm3, \ldots$ with $\nu = 1, \ldots, d$ labels the discrete modes of the Laplacian. They satisfy the orthonormality and completeness conditions

$$\int_0^L \mathrm{d}x_1 \ldots \int_0^L \mathrm{d}x_d\, f_p^*(x) f_{p'}(x) = L^d \delta_{k_1 k_1'} \ldots \delta_{k_d k_d'},$$
$$\sum_{k_1=-\infty}^{\infty} \ldots \sum_{k_d=-\infty}^{\infty} f_p^*(x) f_p(x') = L^d \delta^d(x, x').$$

With the use of these mode functions, the expression for the mode-transformed continuum free-field Green's function in a box is

$$\widetilde{G}(p) = \left\langle \widetilde{\phi}_i^*(p)\widetilde{\phi}_i(p)\right\rangle = \frac{1}{p^2 + m^2},$$

where the discrete squared momenta are given in terms of the modes by

$$p^2(k) = \left(\frac{2\pi}{L}\right)^2 \left(k_1^2 + \ldots + k_d^2\right).$$

Note that this Green's function has the same form as the infinite-space propagator, the only difference being that the continuous-valued momenta are exchanged for an infinite but discrete set.

On the lattice, the eigenfunctions of the finite-differenced Laplacian are given by

$$f_p^N(x) = \exp\left(\imath\frac{2\pi k_1 n_1}{N} + \ldots + \imath\frac{2\pi k_d n_d}{N}\right),$$

where $p(k)$ is the same as before, $x(n) = a\,(n_1, \ldots, n_d)$ and the integers may be chosen to have values in the finite sets $n_\nu = 1, \ldots, N$, and $k_\nu = 1, \ldots, N$. The lattice mode functions satisfy the orthonormality and completeness conditions

$$\sum_{n_1=1}^{N} \cdots \sum_{n_d=1}^{N} f_p^{N*}(x) f_{p'}^N(x) = N^d \delta_{k_1 k_1'} \cdots \delta_{k_d k_d'},$$
$$\sum_{k_1=1}^{N} \cdots \sum_{k_d=1}^{N} f_p^{N*}(x) f_p^N(x') = N^d \delta_{n_1 n_1'} \cdots \delta_{n_d n_d'}.$$

The momentum-space transformation for the field, and its inverse, are written as

$$\tilde{\phi}_i(p) = \frac{1}{N^d} \sum_x^{N^d} f_p^{N*}(x)\ \phi_i(x),$$
$$\phi_i(x) = \sum_p^{N^d} f_p^N(x)\ \tilde{\phi}_i(p),$$

where the sums always consist of N^d terms, as denoted symbolically by the superscripts. On the lattice, p^2 are not the exact eigenvalues of the finite-differenced Laplacian. Instead, the eigenvalues are given by ρ^2/a^2 where

$$\rho_k^2 = 4\left[\sin^2\left(\frac{\pi k_1}{N}\right) + \ldots + \sin^2\left(\frac{\pi k_d}{N}\right)\right].$$

For large N, all the lattice quantities converge to the corresponding continuum quantities. For the transformed field $\tilde{\phi}_i(p)$ the momentum-space lattice Green's function is a function of ρ^2/a^2 rather than p^2,

$$\tilde{G}_N(p) = \left\langle \tilde{\phi}_i^*(p)\tilde{\phi}_i(p) \right\rangle = \frac{1}{\rho_k^2/a^2 + m^2}.$$

Note that this Green's function has the same form as the Green's function in a box given above, but with the infinite set of discrete momenta exchanged for a finite set of corresponding lattice quantities.

---

*E-mail address: delyra@snfma1.if.usp.br

[1]See, for example: B. Freedman, P. Smolensky and D. Weingarten, Phys. Lett. **113B** (1982), 481; I. A. Fox and I. G. Halliday, Phys. Lett. **159B** (1985), 148; M. G. do Amaral and R. C. Shellard, Phys. Lett. **171B** (1986), 285; D. J. E. Callaway, "Triviality Pursuit: Can Elementary Scalar Particles Exist?", Physics Reports **167,** (1988), 241–320.

[2]A detailed discussion of these properties of free fields, and of some of their consequences concerning the definition of the models via the lattice, can be found in an earlier paper[3].

[3]J. L. deLyra, S. K. Foong and T. E. Gallivan, "Differentiability and continuity of quantum fields on a lattice", Phys. Rev. **D43,** (1991), 476–484.

[4]A detailed description of the non-perturbative behavior of the models, and of their relation with the non-linear sigma models, can be found in Sec. III of an earlier paper[5].

[5]J. L. deLyra, S. K. Foong and T. E. Gallivan, "Finite lattice systems with true critical behavior", Phys. Rev. **D46,** (1992), 1643–1657.

[6]The bracket can in principle be zero, since α may be negative, but it differs from the equation of the critical curve $\alpha_R = 0$ by a relative factor of two between the two terms. Therefore, it is not zero near the critical curve, and hence in the continuum limit.