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PARTICLES CREATION BY AN EXTERNAL FIELD IN 2+1 QED

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Abstract

Particles creation effect by a constant electrical field is considered in the frame of QED in $2+1$-dimensions. In this connection exact solutions of the 2+1 Dirac and Klein-Gordon equations in such a field are found. Using them, characteristics of the effect are calculated, namely, the mean numbers of particles created and the vacuum-to-vacuum transition probability. Expressions for these quantities can be presented in a universal form for both dimensions (2+1 and 3+1) and for both kinds of statistics of particles created.

The effect of particles creation from vacuum by an external field ranks among the most intriguing nonlinear phenomena in quantum theory. Its study is theoretically important, since it requires one to go beyond the scope of the perturbation theory, and its experimental observation would verify the validity of the theory in the superstrong field domain. The study of the effect began, in fact, first in 3+1 dimensional QED in connection with the so-called Klein [1] paradox, which revealed the possibility of electron penatration through an arbitrary high barrier formed by an external fielf. Then Schwinger [2] found the vacuum-to-vacuum transition probability in a constant electric field. It became clear that the effect can actually be observed as soon as the external field strength approaches the characteristic value (critical field) $E_c = m^2c^3/|e|\hbar \simeq 1,3 \cdot 10^{16}\ V/cm$. This is the field that produces the work mc^2 when acting along the path equal to the Compton length $\hbar/mc$. Although does not exist now any possibility of creating such fields under laboratory conditions, they can exist in astrophysics, where the characteristic values of electromagnetic fields near pulsars and gravitational fields near black holes are enormous. One can also mention that the Coulomb fields of superheavy nuclei can create the electron-positron pairs. Consideration of diffrent problems in QED connected with vacuum unstability (particles creation effect) and detailed bibliography can be found in the book [3]. Particle creation by an external gravitational field, in analogy with electrodynamics, has also been considered in many papers. A detailed bibliography may be found in [4].

It is interesting to turn again to the particle creation effect and see which kind of modifications appear in the corresponding formulas of 3+1 QED when passing to 2+1 dimensions. In the last years a great attention is devoted to field theoretical models in such dimensions [5,6], in particular, because there probably could exist particles with fractional spin and exotic statistics (anyons), which can have an interest in connection with diffrent applications in physics of planar fenomenos. One can mention here the quantum Hall effect and high temperature conductivity, see, for example [7]. In the present paper we are calculating characteristics of particles creation effect by a constant electrical field in $2+1$-dimensional QED, namely, the mean number of particles created and the vacuum-to-vacuum transition probability. We are using the general approach, which was elaborated in field theory for such kind of calculations [8,9,3]. According to it all the information about the processes of particles scattering and creation by an external field (in zeroth order with respect to the radiative corrections) can be extracted from complete sets of exact solutions of the relativistic wave equations in the external field. (A complet collection of exact solutions of such equations in 3+1 QED is presented in the book [10], one can also find there a detailed bibliography.) That is why we find first a special set of exact solutions in $2+1$-dimensional QED for the constant homogeneous external electric field, which can create particles from the vacuum.

The $2+1$-dimensional Dirac equation in an external electromagnetic field with potentials $A_\mu(x)$ has the form (further $\hbar = c = 1$)

$$(P_\mu\gamma^\mu - m)\,\psi(x) = 0\ , \quad x = (x^\mu) = (t, \mathbf{x}),\ \mathbf{x} = (x^i),\ \ \mu = 0,1,2,\ \ i = 1,2, \tag{1}$$
$$P_\mu = i\partial_\mu - eA_\mu(x)\ , \quad [\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu},\ \ \eta^{\mu\nu} = \mathrm{diag}(1,-1,-1)\ ,$$

where $\psi(x)$ is a two component column. We will use the following representation for the two-dimensional γ-matrices

$$\gamma^0 = \sigma^3,\ \gamma^1 = i\sigma^2,\ \gamma^2 = -i\sigma^1,\quad [\gamma^\mu,\gamma^\nu] = -2i\varepsilon^{\mu\nu\lambda}\gamma_\lambda,\ \ \gamma^{\mu+} = \gamma^0\gamma^\mu\gamma^0\ , \tag{2}$$

where σ are Pauli matrices, $\varepsilon^{\mu\nu\lambda}$ is totally antisymmetric Levi-Civita tensor normalized by $\varepsilon^{012} = 1$, and $[P_\mu, P_\nu] = -ieF_{\mu\nu}$, $\quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Electric field $\mathbf{E} = (E_x, E_y)$ and magnetic field H are defined as $E_x = F_{01},\ E_y = F_{02},\ H = F_{21}$. In the case under consideration, it is possible to construct only one invariant I,

$$I = \frac{1}{2}F_{\mu\nu}F^{\mu\nu} = F^*_\mu F^{*\mu} = H^2 - \mathbf{E}^2,\ \ F^*_\mu = \frac{1}{2}\epsilon_{\mu\nu\lambda}F^{\nu\lambda} = (-H, E_y, -E_x)\ . \tag{3}$$

To solve the Dirac equation (1) it is convenient to make the well known Ansatz

$$\psi(x) = (P_\mu\gamma^\mu + m)\,\phi(x). \tag{4}$$

Then the function ϕ has to obey the squared Dirac equation in $2+1$ dimensions,

$$\left(P^2 - m^2 - eF^*_\mu\gamma^\mu\right)\phi(x) = 0\ , \quad F^*_\mu\gamma^\mu = i(E_x\sigma^1 + E_y\sigma^2 + iH\sigma^3)\ . \tag{5}$$

Consider the constant and homogeneous field with the invariant $I < 0$. This is a particular case of the external field, which can create particles. In this case one can always select the reference frame so that $H = E_x = 0,\ E_y = E \neq 0$. For such a field we will use the potentials $A_0 = A_1 = 0,\ \ A_2 = Ex^0,\ \ F_{02} = E_y = E$. Besides, we select $eE > 0$. Solutions of the squared Dirac equation (5) for such potentials can be expressed in terms of the Weber parabolic cylinder functions [11],

$$ {}^{\pm}_{\pm}\phi_{\mathbf{p},s}(x) = C_{\mathbf{p}}\ {}^{\pm}_{\pm}\phi_{\mathbf{p},s}(x^0)\exp\{i\mathbf{px}\},\ \ s = \pm 1, \tag{6}$$
$$ {}^{-}_{+}\phi_{\mathbf{p},s}(x^0) = D_{\nu-\frac{1+s}{2}}(\mp(1-i)\tau)v_s,\ \ {}^{+}_{-}\phi_{\mathbf{p},s}(x^0) = D_{-\nu-\frac{1-s}{2}}(\mp(1+i)\tau)v_s,$$
$$\sigma^2 v_s = s\ v_s,\quad v_{+1} = \begin{pmatrix} 1 \\ i \end{pmatrix}\ ,\quad v_{-1} = \begin{pmatrix} 1 \\ -i \end{pmatrix}\ ,$$
$$\nu = \frac{i\lambda}{2},\ \ \lambda = \frac{m^2 + p_1^2}{eE},\ \ \tau = \frac{p_2 - eEx^0}{\sqrt{eE}}\ .$$

Solutions of the Klein-Gordon equation, which looks like the eq. (5) at $\gamma \to 0$. follow from (6) at $s = 0,\ v_s \to 1$, and $C_{\mathbf{p}} = \left(2\pi\sqrt[4]{2eE}\right)^{-1}\exp\{-\pi\lambda/8\}$. They can be derived from the corresponding solutions in the $3+1$-dimensional case [8,10] by means of a dimensional reduction.

Solutions of the Dirac equation can be found, using the formula (4),

$$ {}^{\pm}_{\pm}\psi_{\mathbf{p},s}(x) = (P_\mu\gamma^\mu + m)\,{}^{\pm}_{\pm}\phi_{\mathbf{p}s}(x) = C_{\mathbf{p}}\ {}^{\pm}_{\pm}\psi_{\mathbf{p},s}(x^0)\exp\{i\mathbf{px}\}\ , \tag{7}$$
$$ {}^{-}_{+}\psi_{\mathbf{p},+1}(x^0) = (m + ip_1)D_{\nu-1}(\mp(1-i)\tau)v_{+1} \mp \sqrt{eE}(1+i)D_\nu(\mp(1-i)\tau)v_{-1}\ ,$$
$$ {}^{-}_{+}\psi_{\mathbf{p},-1}(x^0) = \mp\frac{m - ip_1}{2\sqrt{eE}}(1-i)\ {}^{-}_{+}\psi_{\mathbf{p},+1}(x^0)\ ,$$
$$ {}^{+}_{-}\psi_{\mathbf{p},-1}(x^0) = (m - ip_1)D_{-\nu-1}(\mp(1+i)\tau)v_{-1} \pm \sqrt{eE}(1-i)D_{-\nu}(\mp(1+i)\tau)v_{+1}\ .$$
$$ {}^{+}_{-}\psi_{\mathbf{p},+1}(x^0) = \pm\frac{m + ip_1}{2\sqrt{eE}}(1+i)\ {}^{+}_{-}\psi_{\mathbf{p},-1}(x^0)\ ,$$

As we can see, only half of them appears to be independent, that means, in fact, that the spin projection can take up only one value, as was also remarked in the case of a free $2+1$ Dirac particle [6]. We select as independent the following sets of solutions

$$ {}_{\pm}\psi_{\mathbf{p}}(x) = {}_{\pm}\psi_{\mathbf{p},\pm 1}(x)\ ,\quad {}^{\pm}\psi_{\mathbf{p}}(x) = {}^{\pm}\psi_{\mathbf{p},\mp 1}(x)\ . \tag{8}$$

Choosing $C_{\mathbf{p}} = \left(4\pi\sqrt{eE}\right)^{-1}\exp\left(-\pi\lambda/8\right)$, one can verify that the relations of orthonormality hold,

$$({}_\zeta\psi_{\mathbf{p}},\ {}_{\zeta'}\psi_{\mathbf{p}'}) = \delta_{\zeta,\zeta'}\delta(\mathbf{p}-\mathbf{p}'),\ \ \left({}^\zeta\psi_{\mathbf{p}},\ {}^{\zeta'}\psi_{\mathbf{p}'}\right) = \delta_{\zeta,\zeta'}\delta(\mathbf{p}-\mathbf{p}'),\ \ \zeta = \pm\,. \tag{9}$$

One can also prove, similarly to the 3 + 1-dimensional case [3,8], that each of the sets (8) forms a complete system of functions.

The solutions ${}_\zeta\psi_{\mathbf{p}}(x)$ are classified as describing particles (+) or antiparticles (−) at $t \to -\infty$, and ${}^\zeta\psi_{\mathbf{p}}$ are classified as describing particles (+) or antiparticles (−) at $t \to +\infty$. Thus, they present so-called IN and OUT sets of solutions. This interpretation can be confirmed in two ways. One of them [8] is to consider the constant electrical field as a limit of a slowly alternating field, which switches out at $t \to \pm\infty$. Another one [9,3] is to check the relations

$$H_D(t)\ {}_\zeta\psi_{\mathbf{p}}(x) = {}_\zeta\mathcal{E}_{\mathbf{p}}\ {}_\zeta\psi_{\mathbf{p}}(x),\ \text{sign}\ {}_\zeta\mathcal{E}_{\mathbf{p}} = \zeta,\quad t \to -\infty\,,$$

$$H_D(t)\ {}^\zeta\psi_{\mathbf{p}}(x) = {}^\zeta\mathcal{E}_{\mathbf{p}}\ {}^\zeta\psi_{\mathbf{p}}(x),\ \text{sign}\ {}^\zeta\mathcal{E}_{\mathbf{p}} = \zeta,\quad t \to +\infty\,, \tag{10}$$

where $H_D = \gamma^0(eA_0\gamma^0 - P_i\gamma^i + m)$ is the corresponding one-particle Hamiltonian and $\mathcal{E}$ is quasi-energy. By using the asymptotic expansion of the Weber parabolic cylinder functions [11], $D_\nu(z) = z^\nu\exp\{-z^2/4\} + O(|z|^{-2})$, one can see that (10) takes place, and ${}_\zeta\mathcal{E}_{\mathbf{p}} = \zeta eE|t|$, ${}^\zeta\mathcal{E}_{\mathbf{p}} = \zeta eEt$.

Now one can find the decomposition coefficients $G\left({}_\zeta|{}^{\zeta'}\right)$ of the OUT solutions in the IN solutions,

$${}^\zeta\psi(x) = {}_+\psi(x)G\left({}_+|{}^\zeta\right) + {}_-\psi(x)G\left({}_-|{}^\zeta\right)\ . \tag{11}$$

To this end, for example, one can consider the asymptotic behaviour of ${}^\zeta\psi(x)$ at $t \to -\infty$, using the corresponding asymptotics of the Weber parabolic cylinder functions [11]. Thus, we obtain

$$G\left({}_\zeta|{}^{\zeta'}\right)_{\mathbf{p}\mathbf{p}'} = g\left({}_\zeta|{}^{\zeta'}\right)_{\mathbf{p}}\delta(\mathbf{p}-\mathbf{p}'),\ \ g\left({}_+|{}^+\right)_{\mathbf{p}} = \exp\left\{\frac{i\pi}{2}(\nu - 1/2)\right\}\frac{\sqrt{\pi}(m - ip_1)}{\Gamma(1+\nu)\sqrt{eE}}\,,$$

$$g\left({}_-|{}^+\right)_{\mathbf{p}} = -\exp\{i\pi\nu\},\ \ g\left({}_-|{}^-\right)_{\mathbf{p}} = -g\left({}_+|{}^+\right)^*_{\mathbf{p}},\ \ g\left({}_+|{}^-\right)_{\mathbf{p}} = g\left({}_-|{}^+\right)_{\mathbf{p}}\ . \tag{12}$$

Having these coefficients, one can extract all the information about the of the particles creation processes [8,9,3]. First of all, let us calculate the mean number $N^+_{\mathbf{p}}$ of electrons with a given momentum $\mathbf{p}$ created by the external field (at the same time this is the corresponding number of the electron-positron pairs created). To this end is convenient to use the volume regularization, so that $\delta(\mathbf{p}-\mathbf{p}') \to \delta_{\mathbf{p},\mathbf{p}'}$. Then

$$N^+_{\mathbf{p}} = \left\{G\left({}_-|{}^+\right)^+ G\left({}_-|{}^+\right)\right\}_{\mathbf{p}\mathbf{p}} = \exp(-\pi\lambda)\,. \tag{13}$$

To get the total number N^+ of the electrons (pairs) created, one has to take the sum of the expression (13) over all the momenta, which can be easily transformed into an integral,

$$N^+ = \sum_{\mathbf{p}} N^+_{\mathbf{p}} = \frac{V_{(2)}}{(2\pi)^2}\int d\mathbf{p}N^+_{\mathbf{p}}\,,$$

where $V_{(2)}$ is two dimensional volume. Here we meet a divergency (integration over p_2), which is related to the infinite time of the electric field action. If we restrict ourseves to a big enough, but finite, time T of the action, we can efficiently replace the integral over p_2 by eET [8,3]. Then

$$N^+ = \frac{V_{(2)}Tm^3}{(2\pi)^2}\left(\frac{E}{E_c}\right)^{\frac{3}{2}}\exp\left\{-\pi\left(\frac{E_c}{E}\right)\right\}\,. \tag{14}$$

where $E_c = m^2/e$ is the critical field strength. The vacuum-to-vacuum transition probability $P_v = \exp\left\{\text{tr}\ln|G\left({}_-|{}^-\right)|^2\right\}$ can be calculated, using both kinds of regularization, with respect to the volume and to the time. Thus, we get the 2 + 1-dimensional analogue of the well known Schwinger formula [2],

$$P_v = \exp\left\{-\frac{V_{(2)}Tm^3}{(2\pi)^2}\left(\frac{E}{E_c}\right)^{\frac{3}{2}}\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}e^{-n\pi\left(\frac{E_c}{E}\right)}\right\}\,. \tag{15}$$

Similar calculations can be made for scalar QED, using the corresponding Klein-Gordon solutions. In particular, it turns out that $N^+_{\mathbf{p}}$ and N^+ have the same form (13) and (14), whereas the vacuum-to-vacuum transition probability P_v differs from (15). It can be calculated in the scalar QED by means of general formula $P_v = \exp\left\{-\text{tr}\ln|G\left({}_-|{}^-\right)|^2\right\}$ and has the form

$$P_v = \exp\left\{-\frac{V_{(2)}Tm^3}{(2\pi)^2}\left(\frac{E}{E_c}\right)^{\frac{3}{2}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\frac{3}{2}}}e^{-n\pi\left(\frac{E_c}{E}\right)}\right\} . \tag{16}$$

Remembering the corresponding expressions for N^+ [8,3] and P_v [2,3] in $3+1$ dimensions, one can write formulas for both dimensions $D+1$, $D=2,3$, and for both statistics in a unique form

$$N^+ = (D-1)^s\frac{V_{(D)}Tm^{D+1}}{(2\pi)^D}\left(\frac{E}{E_c}\right)^{\frac{D+1}{2}}\exp\left\{-\pi\left(\frac{E_c}{E}\right)\right\} , \quad P_v = \exp\left\{-\mu N^+\right\} ,$$

$$\mu = \sum_{n=0}^{\infty}\frac{(-1)^{(1-s)n}}{(n+1)^{\frac{D+1}{2}}}\exp\left\{-n\pi\left(\frac{E_c}{E}\right)\right\} , \quad s = \begin{cases} 0 & \text{scalar particles} \\ 1 & \text{spinor particles}\end{cases} \tag{17}$$

One can see that in $3+1$ dimensions N^+ for scalar particles is one half of the same quantity of spinning particles, whereas they coincide in $2+1$ dimensions. We believe that occurs in virtue of the coincidence of degrees of freedom for spinning and spinless particles in the latter case.

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