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ON THE ZERO MASS LIMIT OF THE NON LINEAR SIGMA MODEL IN FOUR DIMENSIONS

by

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ABSTRACT

The existence of the zero mass limit for the non-linear σ-model in four dimensions is shown to all orders in renormalized perturbation theory. The main ingredient in the proof is the imposition of many current axial vector Ward identities and the tool used is Lowenstein's momentum-space subtraction procedure. Instead of introducing anisotropic symmetry breaking mass terms, which do not vanish in the symmetry limit, it is necessary to allow for "soft" anisotropic derivative coupling in order to obtain the correct Ward identities.

I - INTRODUCTION

In the nonlinear σ-model[1] the chiral symmetry O(N) is realized in a nonlinear manner. This nonlinear realization has been studied in various contexts. Most recently it has been shown[2] how to treat it in two-dimensional space-time, where the model is renormalizable and can be considered as the limit of a classical Heisenberg model when the lattice spacing tends to zero.

In this paper we want to study the nonrenormalizable four dimensional version, which can be obtained from the free linear σ-model described by the Lagrangian

$$\mathcal{L} = \frac{1}{2} \left(\partial_\mu A_i\right)\left(\partial^\mu A_i\right), \quad i = 1, 2, \dots n \tag{1.1}$$

by imposing a constraint which eliminates the field $A_n(x)$ from (1.1), while preserving the O(n) symmetry. As is well known[3] this can be done in many ways. We choose the following condition

$$\sum_{i=1}^{n-1} A_i(x) A_i(x) + \left[\sigma(x)\right]^2 = f^2 \tag{1.2}$$

where $\sigma(x) = A_n(x)$ and f is a constant. Inserting (1.2) into (1.1) we obtain

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu A_i(x)\,\partial^\mu A_i(x)\right) + $$

$$+ \frac{1}{2}\left(f^2 - A^2\right)^{-1}\left(A(x)\cdot\partial_\mu A(x)\right)^2 \tag{1.3}$$

where $A^2 = \sum_{i=1}^{n-1} A_i A_i$, $A\cdot\partial_\mu A = \sum_{i=1}^{n-1} A_i \partial_\mu A_i$ and $(f^2 - A^2)^{-1}$ is understood as a formal power series expansion. Since the Lagrangian (1.3) is not renormalizable and because the symmetry acts nonlinearly on the A_i field the problem of defining a finite symmetry-preserving perturbation expansion in f^{-1} is a nontrivial one, whose solution is presented in this paper.

Taking as usual the A_i field to be pseudoscalar (1.3) has a conserved vector current $\mathcal{V}_\mu^i$:

$$\mathcal{V}_\mu^i = A_k X_{kl}^i \partial_\mu A_l \tag{1.4}$$

where $i = [rs]$ and $X_{kl}^{rs} = \delta_{rk}\delta_{sl} - \delta_{rl}\delta_{ks}$, $r<s$. The conserved axial vector current $\mathcal{A}_\mu^i$ is

$$\mathcal{A}_\mu^i = (f^2 - A^2)^{1/2}\,\overleftrightarrow{\partial}_\mu A_i \tag{1.5}$$

satisfying

$$\partial^\mu \mathcal{A}_\mu^i = 0 \tag{1.6}$$

We may also include a symmetry-breaking term $\mathcal{L}_{SB}$ in (1.3)

$$\mathcal{L}_{SB} = m^2 f\left[(f^2 - A^2)^{1/2} - f\right] \tag{1.7}$$

so as to obtain a partially conserved axial current (PCAC):

$$\partial^\mu \mathcal{A}_\mu^i = - m^2 f A_i \tag{1.8}$$

Nonrenormalizable theories have a notoriously bad high energy behaviour and the nonlinear σ-model shares this property. Before tackling this difficult problem we want to show the existence of the $m^2 \to 0$ limit satisfying (1.6). This problem has been dealt with by several authors. D.Bessis and J.Zinn-Justin[4] are able to show the existence of the zero mass limit, but only on the one loop level. Y.M.P. Lam[5] obtains PCAC to all orders of perturbation theory, but the zero mass limit does not exist. In fact it has up to now been impossible to construct a renormalized field possessing a conserved axial current (CAC) in the zero mass limit to

II - THE CLASSICAL MODEL AND THE TREE APPROXIMATION

In this section we want to discuss some features present on the classical level and in the tree approximation .

The conservation of the axial current is easily obtained using the equation of motion once. From the definition (1.5) we get

$$\partial^\mu \mathcal{A}^i_\mu = \partial^\mu \{ \sigma \overleftrightarrow{\partial}_\mu A_i \} = \sigma \partial^2 A_i - A_i \partial^2 \sigma =$$

$$= \sigma \partial^2 A_i + \frac{1}{4} A_i \sigma^{-3} (\partial_\mu A^2)^2 + \frac{1}{2} A_i \sigma^{-1} \partial^2 (A^2) \tag{2.1}$$

Now using the equation of motion

$$\begin{aligned} \partial^2 A_i = & -\frac{1}{4} \sigma^{-4} (\partial_\mu A^2)^2 A_i - \\ & - \frac{1}{2} \sigma^{-2} \partial^2 (A^2) A_i - m^2 f \sigma^{-1} A_i \end{aligned} \tag{2.2}$$

for $\sigma \partial^2 A_i$ we get

$$\partial^\mu \mathcal{A}^i_\mu = - m^2 f A_i \tag{2.3}$$

If in (1.5) we expand the square-root before taking the divergence, the computation is rather complicated; yet this has to be done on the quantum level. What we learn from the above calculation is that we should use the equation of motion only to calculate $\partial^2 A_i$, but not $\partial^2 \sigma$. This will be done in the following sections.

The tree-approximation Ward identity for $\mathcal{A}^i_\mu$ to be preserved in higher order is

$$\begin{aligned} \partial^\mu \langle T \mathcal{A}^i_\mu(x) \underline{X} \rangle = & - f m^2 \langle T A_i(x) \underline{X} \rangle - \\ & - i \sum_{\alpha=1}^{n} \delta(x - x_\alpha) \delta_{i k_\alpha} \langle T (f^2 - A^2(x_\alpha))^{1/2} \underline{X}_{\hat{\alpha}} \rangle \end{aligned} \tag{2.4}$$

where $\underline{X} = \prod_{\beta=1}^{n} A_{k_\beta}(x_\beta)$ and $\underline{X}_{\hat{\alpha}} = \prod_{\substack{\beta=1 \\ \beta \neq \alpha}}^{n} A_{k_\beta}(x_\beta)$.

Once Eq.(2.4) has been shown to hold, we expand its l.h.s as

$$\partial^\mu \langle T [\sigma \overleftrightarrow{\partial}_\mu A_i](x) \underline{X} \rangle = f \partial^2 \langle T A_i(x) \underline{X} \rangle + \mathcal{O}(f^{-1}) \tag{2.5}$$

The r.h.s. of (2.5) gives (up to a trivial d'Alembertian on an external line) the subtraction prescription for $\langle T A_i(x) \underline{X} \rangle$.

all orders in perturbation theory. In our view the difficulty lies in the procedure usually followed: first one selects rather arbitrarily some subtraction scheme for Green functions and then tries to define currents satisfying the appropriate Ward identities.

What we propose instead is to impose the validity of axial current Ward identities (ACWIs) for an arbitrary number of currents. They will determine how the vertices appearing in the Lagrangian are to be subtracted, once a suitable subtraction scheme for current vertices has been chosen. A minimal requirement a subtraction scheme should satisfy is the removal of ultraviolet divergencies without introducing infrared (IR) divergencies when zero mass propagators are present.

In this respect a scheme proposed by J.H.Lowenstein (6) is very convenient. It evolved from an extension (7) of the BPHZ renormalization [1]: one subtracts not only polynomials of momenta, but also polynomials in mass parameters. Using Lowenstein's prescription [2] one may subtract all, except the highest order term in a polynomial, at $p_\mu = 0$, $m^2 = 0$, which is a great aid if one actually wants to perform explicit calculations. Because the axial current is non polynomial its conservation law can easily be spoiled if anisotropies (8) are not controlled. (On the

other hand the conservation of the bilinear vector current never poses much of a problem).

Up to now it has always been the mass terms, which are anisotropic. This is the reason why they do not vanish in the zero mass limit and thus spoil the ACWI. In contrast our mass terms are always minimally subtracted and consequently vanish when $m^2 \to 0$.

The price we pay is that now the derivative coupling in (1.3) becomes anisotropic and has to be kept under control. To this end we introduce in Sec.III a "soft" way to treat this coupling, which ensures the validity of the equation of motion within normal products and the validity of the correct ACWI. This requires a new definition of what we mean by overlapping graphs and also our anisotropies are generalizations of previous definitions, since we need more lines and more variables for the subtraction operators to act on than there appear in the unsubtracted integrand.
Only external vertices (i.e. vertices on which ends at least one line through which only external momentum flows) can be treated "softly" and internal derivative vertices are always "hard" (i.e. subtracted according to Lowenstein).
It is the purpose of Sec.IV to isolate the anisotropies resulting from "soft" vertices and to show the validity of many current identities for arbitrary graphs.

III - FEYNMANN RULES AND WARD IDENTITIES

We consider the version of the nonlinear σ-model specified by the effective Lagrangian density

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_{int} \\
\mathcal{L}_0 &= \tfrac{1}{2}\partial_\mu A_i \partial^\mu A_i - \tfrac{1}{2} m^2 A^2 \\
\mathcal{L}_{int} &= \mathcal{L}_1 + \mathcal{L}_2 \\
\mathcal{L}_1 &= (m^2 - b) f \left[(f^2 - A^2)^{1/2} - f \right] + \tfrac{1}{2} m^2 A^2 \\
\mathcal{L}_2 &= \tfrac{1}{2} (f^2 - A^2)^{-1} (A \cdot \partial_\mu A)^2
\end{aligned} \tag{3.1}$$

where b is a finite counterterm chosen recursively to enforce a pole of the two point Green function at $p^2 = m^2$.

The N point Green function is defined via the modified Gell-Mann Low formula

$$G^{(N)}(x_1, x_2, \dots x_N; i_1, i_2, \dots, i_N) =$$

$$= \text{finite part of } {}^{(0)}\!\left\langle T \prod_{\alpha=1}^{N} A^{(0)}_{i_\alpha}(x_\alpha) \exp\left[i \int \mathcal{L}^{(0)}_{int}\, d^4x\right] \right\rangle^{(0)} \tag{3.2}$$

where $A^{(0)}$ is the free field given by $\mathcal{L}_0$. Notice the absence of Wick ordering in (3.2). The finite part prescription will be given in this and the following section such that the ACWI

$$\partial^\mu \langle T N[(f^2 - A^2) \overleftrightarrow{\partial}_\mu A_k](x) \, X \rangle = - f (m^2 - b) \langle T A_k(x) \, X \rangle -$$

$$- i \sum_{\alpha=1}^{N} \delta(x - x_\alpha) \delta_{k i_\alpha} \langle T N[(f^2 - A^2)^{1/2}](x_\alpha) \prod_{\substack{\beta=1 \\ \beta \neq \alpha}}^{N} A_{i_\beta}(x_\beta) \rangle \tag{3.3}$$

with $X = \prod_{\beta=1}^{N} A_{i_\beta}(x_\beta)$

holds in every order of the perturbation parameter f^{-1}. The normal-product Green functions appearing in (3.3) are defined as usual[9]: if $\mathcal{O}$ is any monomial in the basic field A_i and their derivatives then

$$\langle T N[\mathcal{O}](x) \, X \rangle = \text{finite part of}$$

$${}^{(0)}\!\left\langle T\, \mathcal{O}^{(0)}(x) \prod_{\alpha=1}^{N} A_{i_\alpha}(x_\alpha) \exp\left[i \int \mathcal{L}^{(0)}_{int}\, d^4x\right] \right\rangle^{(0)} \tag{3.4}$$

and again the finite part respects (3.3).

To describe the subtraction procedure consider the Feynman expansions of (3.2) and (3.4). We obtain (formal) Feynman integrals corresponding to a connected graph G of the type

$$\lim_{\varepsilon \to 0} \int d^4k \; I_G(p, k, m, \varepsilon) \tag{3.5}$$

where $I_G(p,k,m,\varepsilon)$ is of the form

$$I_G(p,k,m,\varepsilon) = \prod_{V\in G} P_V(p,k) \prod_{L\in G} \Delta_{Lij}(l_{ab\sigma}, m_L, \varepsilon) \qquad (3.6)$$

and where

$p = \{p_1, p_2, \ldots, p_s\}$ = basis for external momenta of G

$k = \{k_1, k_2, \ldots, k_m\}$ = basis for internal momenta of G

$m = \{m_1, m_2, \ldots, m_L\}$ with mass m_i assigned to the internal line L=i

$\prod_{V\in G}$ = product over vertices V of G

$\prod_{L\in G}$ = product over lines L of G

$l_{ab\sigma} = -l_{ba\sigma}$ = momentum flowing through line L

$P_V(p,k)$ = polynomial of degree w_i ($w_i \leq 2$) in the momenta p and k flowing into the vertex V.

$$\Delta_{Lij}(l_{ab\sigma}, m_L, \varepsilon) = i\delta_{ij}\left[l^2_{ab\sigma} - m_L^2 + i\varepsilon(\vec{l}_L^{\,2} + m_L^2)\right]^{-1}$$

i, j = isospin indices

For subtraction pruposes we distinguish masses corresponding to different lines. At the end they are all set equal to m^2.

In order to obtain the ACWI (3.3) it is crucial to redefine the concept of overlapping graphs. Due to isospin conservation the lines of G can only end or start on external lines or form closed internal loops. In drawing a vertex it is convenient to make the isospin flow explicit: as shown in Fig.(2.1) each vertex is replaced by a set of new vertices linked by new additional dashed lines through which no isospin flows and whose aim is to identify their endpoints. We now define overlapping graphs as usual[9], but treating all lines, dashed and continuous, on the same footing: two graphs γ_1 and γ_2 are nonoverlapping, if either $\gamma_1 \supset \gamma_2$, $\gamma_2 \supset \gamma_1$ or $\gamma_1 \cap \gamma_2 = \emptyset$. An illustration of this concept is given in Fig.(2.2). Thus two graphs whose continuous lines overlapp in one point only are according to our definition not necessarily overlapping. This property will allow us to derive an equation of motion within normal products, which is free of anomalies.

To avoid misunderstandings later on we include here a definition for dashed lines: a dashed line i_1 is said to belong to the graph γ, if the momentum flowing through i_1 (obtained by algebraically adding to two momenta of the continuous lines on which i_1 ends, i.e. treating the dashed line on equal footing and applying energy-momentum conservation) contains a non zero component of internal momenta of γ. A sum of dashed lines $\mathfrak{I} = \sum_{\alpha=1}^{N} i_\alpha$ is said to belong to γ, if the sum of the momenta $q(i_\alpha)$ flowing through i_α contains a non zero k_γ component:

$$\mathfrak{I} = \sum_{\alpha=1}^{N} i_\alpha \in \gamma, \text{ if } \sum_{\alpha=1}^{N} q(i_\alpha) = A k_\gamma + B p_\gamma,\ A \neq 0 \tag{3.7}$$

Finally we say that two lines i_1, i_2 are equivalent, if there does not exist a graph γ, proper, such that $i_1 \in \gamma,\ i_2 \notin \gamma$.

A - THE SUBTRACTION SCHEME

The subtracted integrand R_G to be associated with I_G depends on the nature of the vertices of G, which are of three types:

1) vertices associated with $\mathcal{L}_1$, the breaking part of $\mathcal{L}$.
2) Internal derivative vertices, i.e. vertices associated with $\mathcal{L}_2$ and without external lines attached to them.
3) External derivative vertices. These are either the special current vertex in the case of the normal-product Green function or vertices associated with $\mathcal{L}_2$ having at least one external line attached.

If G contains only vertices of type 1) or 2), then the subtracted integrand is defined by first factoring out the m^2 and b factors comming from $\mathcal{L}_1$ and then applying Zimmermann's forest formula to the remaining integrand, using generalized "Taylor" operators i.e.[3]

$$R_G = S_G \sum_{U \in \mathcal{F}} \prod_{\gamma \in U} (-\tau^{d(\gamma)}) S_\gamma I_G(U) \tag{3.8}$$

where $\mathcal{F}$ is the set of all forests (non-trivial, non-overlapping $\underbrace{\text{subgraphs}}_{\text{1PI}}$ of G, having non-negative degree of superficial divergence $d(\gamma)$). The generalized "Taylor" operator $\tau^{d(\gamma)}$ is given by

$$\tau^{d(\gamma)} = t^{d(\gamma)-1}_{p^\gamma, m^\gamma} + \sum_{2\sum_{i=1}^{M} r_i + \sum_{i=1}^{N} s_i = d(\gamma)} (m_1^2)^{r_1} \cdots (m_M^2)^{r_M} \frac{\partial^{r_1}}{\partial (m_1^2)^{r_1}} \cdots \frac{\partial^{r_M}}{\partial (m_M^2)^{r_M}} \cdot$$

$$\cdot (p_1^\gamma)^{s_1} \cdots (p_N^\gamma)^{s_N} \frac{\partial^{s_1}}{\partial (p_1^\gamma)^{s_1}} \cdots \frac{\partial^{s_N}}{\partial (p_N^\gamma)^{s_N}} \cdot \Bigg|_{p_i^\gamma = 0,\ m_j^2 = \mu^2} \cdot \frac{1}{\prod_{ij} r_i!\, s_j!} \tag{3.9}$$

where $t^{d(\gamma)-1}_{p^\gamma, m^\gamma}$ is the Taylor operator of order $d(\gamma) - 1$ in the mass parameters $m^2_\gamma = \{m^2_{l_i}, l_i \in \gamma\}$ and in the external momenta $p^\gamma = \{p_1^\gamma, p_2^\gamma, \ldots p_N^\gamma\}$ of γ. S_γ is a substitution operator, shifting from the variables of $\lambda \in \mathcal{U}$ to those of $\gamma \in \mathcal{U}, \lambda \subset \gamma$ and S_G realizes the additional job of replacing all masses m_i^2 by m^2.

If G contains vertices of type 3), the subtraction procedure will depend on the topology of the graph near the vertex ϑ, i.e. it depends on the way in which the lines of ϑ are contracted. It is at this point that softly subtracted derivative vertices from $\mathcal{L}_2$ make their appearence. They are defined in the following way.

Let $k_{1,s}^2, k_{2,s}^2, \ldots\ldots k_{r,s}^2$ be the soft momentum factors. (Here and in the following the subscript s will indicate that the corresponding momentum is soft. Momenta without subscript are hard, i.e. subtracted with the operators (3.9)). A soft momentum if internal[4], depends only on the loop momentum through one of the lines, either dashed[5] or continuous, ending at the given vertex; for convenience we think this momentum factor to be associated with this line rather than the vertex. The subtracted integrand $\left(\prod_{i=1}^{r} k_{i,s}^2 I'_G\right)_R$ where I'_G is such that $I_G \equiv \prod_{i=1}^{r} k_i^2 I'_G$ is defined recursively by[6]

$$\left(\prod_{i=1}^{r} k_{i,s}^2 I'_G\right)_R = S_G \Big\{ \sum_{\mathcal{U} \in \mathcal{F}} \prod_{\gamma \in \mathcal{U}} \left(-\tau^{d(\gamma)}\right) S_\gamma \prod_{i=1}^{r} (k_i^2 - m_i^2) I'_G +$$

$$+ \sum_{i=1}^{r} m_i^2 \Big(\prod_{\substack{j=1 \\ j \neq i}}^{r} k_{j,s}^2 I' \Big)_R - \sum_{i,j=1}^{r} m_i^2 m_j^2 \Big(\prod_{\substack{n=1 \\ n \neq i, n \neq j}}^{r} k_{n,s}^2 I'_G \Big)_R +$$

$$+ \ldots + (-1)^{r+1} \prod_{i=1}^{r} m_i^2 \left(I'_G\right)_R \tag{3.10}$$

where m_i^2 always contributes two powers to $d(\gamma)$, when written to the right of a "Taylor" operator. If k_i^2 is associated with a dashed line, then the mass m_i^2 does not show up in the unsubtracted integrand and the "Taylor" operator $\tau^{d(\gamma)}$ is now of the form (3.9), but depending on $(N+M+1)$ variables, where the extra variable is m_i^2. The momenta k_i^2 are expressed in terms of the variables p^σ, k^σ where σ is the smallest graph $\sigma \in \mathcal{U}$ containing k_i^2. We also allow for the situation when two indices i are equal, representing the momentum factor $(k_{i,s}^2)^2$.

A shorthand version of (3.10) is

$$\prod_{i=1}^{r} (k_{i,s}^2) = \prod_{i=1}^{r} \langle k_i^2 - m_i^2 \rangle + \sum_{i=1}^{r} m_i^2 \prod_{\substack{j=1 \\ j\neq i}}^{r} (k_{j,s}^2) -$$

$$- \sum_{i,j=1}^{r} m_i^2 m_j^2 \prod_{\substack{n=1 \\ n\neq i, n\neq j}}^{r} (k_{n,s}^2) + \dots + (-1)^{r+1} \prod_{i=1}^{r} m_i^2 \qquad (3.10)'$$

For $r = 1$ (3.10)' yields

$$k_{1,s}^2 = \langle k_1^2 - m_1^2 \rangle + m_1^2 \qquad (3.11)$$

Using (3.11) we can rewrite (3.10)' as a product

$$\prod_{i=1}^{r} (k_{i,s}^2) = \prod_{i=1}^{r} \left\{ \langle k_i^2 - m_i^2 \rangle + m_i^2 \right\} \qquad (3.12)$$

with the convention $\langle A \rangle \langle B \rangle \equiv \langle AB \rangle$. We have not written (3.10) as a product, because it would be very cumbersome to put all the operators $\tau^{d(\gamma)}$ in their correct positions. For later purposes it is convenient to rewrite (3.12) as follows

$$\prod_{i=1}^{r} (k_{i,s}^2) = \prod_{i=1}^{r} \left\{ \langle k_i^2 \rangle - \left[\langle m_i^2 \rangle - m_i^2 \right] \right\} \qquad (3.12)'$$

so that all polynomials are hard and all subtractions by which $k_{i,s}^2$ differs from its hard counterpart are contained in $\langle m_i^2 \rangle - m_i^2$.

Let us now come back to our type 3) vertices. What makes external $\mathcal{L}_2$ vertices ϑ difficult is the fact that, in order to obtain the correct ACWI, the polynomial $P_\vartheta(p,k)$ associated with ϑ has to be written as a sum of two terms

$$P_\vartheta(p,k) = P_{\vartheta 1}^{(S)}(p,k) + P_{\vartheta 2}^{(H)}(p,k) \qquad (3.13)$$

where $P_{\vartheta 1}^{(S)}(p,k)$ is subtracted softly, whereas $P_{\vartheta 2}^{(H)}(p,k)$ is subtracted using the hard operators (3.9). Actually it will turn out to be simpler to obtain a decomposition where all polynomials are hard

$$P_\vartheta(p,k) = \left[P_{\vartheta 1}^{(H)}(p,k) + P_{\vartheta 2}^{(H)}(p,k) \right] - \mathcal{M} \qquad (3.13)'$$

where we have used (3.11).

The details of the decomposition (3.13), (3.13)' will be described in Sec. IV. Here we assume it to be known how to effect the decompositions (3.13), (3,13)'. (Since (3.12),(3.12)' have a product structure, the decompositions (3.13), (3.13)' can be obtained for each vertex independently and then inserted into (3.12), (3.12)'.

The finite part prescription in this section is then well defined up to the decomposition (3.13),(3.13)'.

B - EQUATION OF MOTION

In order to derive an equation of motion inside normal products like $N[(A^2)^n\partial^2 A_k](x)$, we add and subtract the mass term $N[(A^2)^n m_k^2 A_k](x)$ thus obtaining a Klein-Gordon operator acting on the field A_k. In momentum space this operator corresponds to a factor $\ell^2 - m_L^2$, where ℓ is the momentum through the line L associated with A_k which is used to cancel the propagator of the line L, i.e. one obtains a graph where the line L is cut. The remaining step is then to identify the result with the Euler derivative of the interaction vertices.

This procedure produces in general anisotropies, i.e., terms not present on the classical level, due to two problems:

a) the mass terms introduced to complete the Klein-Gordon operator tend to be oversubtracted, because the "Taylor" operator (3.9) does not act on masses and momenta in the same way.

b) topologically different graphs, having therefore different

subtraction schemes, can after cutting the line L become topologically identical as illustrated in Fig.(3.1)

We circumvent difficulty a) by introducing masses m_k^2 attached to lines. Of course, since m_k^2 is chained to a line, it cannot be pulled out of the normal product. The difference between $N[(A^2)^n m_k^2 A_k]$ and $m^2 N[(A^2)^n A_k]$ will be absorbed by the derivative coupling $\mathcal{L}_2$ and thus induce the introduction of soft derivative coupling. The problem b) is overcome by our definition of overlapping, i.e. graphs which are not overlapping before cutting line L will always continue nonoverlapping. This is exemplified in Fig. (3.2). It is worth noting that we are able to avoid both difficulties a) and b), because our model contains only fields carrying a conserved quantum number.

Having this in mind it is now straightforward to derive the equation of motion. Consider the object

$$\langle T N[(A^2)^n(\partial^2 + m_L^2) A_k^{V_i}](x)\, \overline{\underline{X}} \rangle \tag{3.14}$$

where the superscript V_i on A_k indicates that A_k is contracted with a field belonging to the vertex V_i. V_i may be either i) the special vertex (3.14)

ii) an external vertex at x_β : $V(\beta)$

iii) a symmetry breaking vertex from $\mathcal{L}_1$

iv) a derivative coupling vertex from $\mathcal{L}_2$

as illustrated in Fig. 3.3. In case i) this contribution will be cancelled by the subtractions. In case ii) one obtains the usual delta terms

$$\langle T N [(A^2)^n (\partial^2 + m_L^2) A_k^{V(\beta)}](x) \overline{X} \rangle = = -i \sum_{\beta=1}^{N} \delta(x - x_\beta) \delta_{k i_\beta} \langle T N [(A^2)^n](x) \prod_{\alpha \neq \beta}^{N} A_{i_\alpha}(x_\alpha) \rangle \tag{3.15}$$

In case iii) we obtain, considering all vertices from $\mathcal{L}_1$

$$m^2 \langle T N [(A^2)^n A_k](x) \overline{X} \rangle - - (m^2 - b) f \langle T N [(A^2)^n (f^2 - A^2)^{-1/2} A_k](x) \overline{X} \rangle \tag{3.16}$$

where we used the fact that the m^2 and b factors in $\mathcal{L}_1$ are always factored out before applying the subtraction scheme.

If V_i is an internal vertex of $\mathcal{L}_2$, we obtain a contribution to the object

$$\langle T N [(A^2)^n [\frac{\delta \mathcal{L}_{2i}}{\delta A_k} - \partial_\mu \frac{\delta \mathcal{L}_{2i}}{\delta A_{k,\mu}}](x) \overline{X} \rangle \tag{3.17}$$

where the momentum polynomial attached to V_i is hard,

since by hypotheses all lines ending at V_i are internal.

Finally, if V_i is an external vertex from $\mathcal{L}_2$, it will have attached to it a certain polynomial with soft and hard components as in (3.13). After cutting the line L we obtain a contribution to (3.17), where $P_{V_i}(p,k)$ has the same decomposition as before cutting the line.

Summing up, we obtain the following equation of motion

$$\begin{aligned}\langle T N [(A^2)^n (\partial^2 + m_L^2) A_k](x) \overline{X} \rangle &= m^2 \langle T N [(A^2)^n A_k](x) \overline{X} \rangle - \\ &- (m^2 - b) f \langle T N [(A^2)^n (f - A^2)^{-1/2} A_k](x) \overline{X} \rangle + \\ &+ \sum_i \langle T N [(A^2)^n (\frac{\delta \mathcal{L}_{2i}}{\delta A_k} - \partial_\mu \frac{\delta \mathcal{L}_{2i}}{\delta A_{k,\mu}})](x) \overline{X} \rangle - \\ &- i \sum_{\beta=1}^{N} \delta(x - x_\beta) \delta_{k i_\beta} \langle T N [(A^2)^n](x) \prod_{\alpha \neq \beta}^{N} A_{i_\alpha}(x_\alpha) \rangle\end{aligned} \tag{3.18}$$

where the hard-soft decomposition (3.13), (3.13)', must be taken into account in the term containing the Euler derivative. We remark that the decomposition (3.13) is dictated by the ACWI, whereas the equation of motion (3.18) is valid for any given decomposition.

C - WARD IDENTITIES

To see how soft momentum factors defined by (3.10) and (3.12) enter in the discussion of the ACWI (3.3) we consider the divergence of the normal product

$$N\left[(A^2)^n \overleftrightarrow{\partial}_\mu A_k\right](x)$$

where the derivative is hard, i.e. subtracted with (3.9). Since the subtractions are made at zero external momenta we have

$$\partial^\mu \langle T N[(A^2)^n \overleftrightarrow{\partial}_\mu A_k](x)\underline{\overline{X}}\rangle = \langle TN[(A^2)^n \overleftrightarrow{\partial}^2 A_k](x)\underline{\overline{X}}\rangle =$$

$$= \langle TN[(A^2)^n(\partial^2+m_k^2)A_k](x)\underline{\overline{X}}\rangle - \langle TN[A_k(\partial^2+m_k^2)(A^2)^n](x)\underline{\overline{X}}\rangle$$

(3.18)

where, in order to obtain an anisotropy-free equation of motion inside the normal product, the mass m_k^2 in the second line of (3.18) is associated with the line corresponding to A_k . We now use (3.11) to rewrite (3.19) as

$$\partial^\mu\langle TN[(A^2)^n \overleftrightarrow{\partial}_\mu A_k](x)\underline{\overline{X}}\rangle = \langle TN[(A^2)^n(\partial^2+m_k^2)A_k](x)\underline{\overline{X}}\rangle -$$

$$-\langle TN[A_k\,\partial_s^2(A^2)^n](x)\underline{\overline{X}}\rangle - m^2\langle TN[(A^2)^n A_k](x)\underline{\overline{X}}\rangle$$

(3.20)

to obtain

$$\partial^\mu\langle TN[(f^2-A^2)^{1/2}\overleftrightarrow{\partial}_\mu A_k](x)\underline{\overline{X}}\rangle =$$

$$=\langle TN[(f^2-A^2)^{1/2}(\partial^2+m_k^2)A_k](x)\underline{\overline{X}}\rangle - \langle TN[A_k\,\partial_s^2(f^2-A^2)^{1/2}](x)\underline{\overline{X}}\rangle -$$

$$- m^2\langle TN[(f^2-A^2)^{1/2}A_k](x)\underline{\overline{X}}\rangle$$

(3.21)

Applying now the equation of motion (3.18) to the first term on the r.h.s. of (3.21) we obtain

$$\partial^\mu\langle TN[(f^2-A^2)^{1/2}\overleftrightarrow{\partial}_\mu A_k](x)\underline{\overline{X}}\rangle = -(m^2-b)f\langle T A_k(x)\underline{\overline{X}}\rangle -$$

$$-i\sum_{\beta=1}^{N}\delta(x-x_\beta)\,\delta_{k i_\beta}\langle TN[(f^2-A^2)^{1/2}](x)\prod_{\alpha\neq\beta}A_{i_\alpha}(x_\alpha)\rangle +$$

$$+\sum_i\langle TN[(f^2-A^2)^{1/2}\left(\frac{\delta\mathcal{L}_{2i}}{\delta A_k}-\partial_\nu\frac{\delta\mathcal{L}_{2i}}{\delta A_{k,\nu}}\right)](x)\underline{\overline{X}}\rangle -$$

$$-\langle TN[A_k\,\partial_s^2(f^2-A^2)^{1/2}](x)\underline{\overline{X}}\rangle$$

(3.22)

Thus in order to obtain PCAC the following identity must be fulfilled

$$\sum_i \langle TN[(f^2-A^2)^{1/2}(\frac{\delta\mathcal{L}_{2i}}{\delta A_k} - \partial_\nu \frac{\delta\mathcal{L}_{2i}}{\delta A_{k,\nu}})](x)\,\overline{X}\,\rangle =$$

$$= \langle TN[A_k \partial_s^2 (f^2-A^2)^{1/2}](x)\,\overline{X}\,\rangle \tag{3.23}$$

The above identity (3.23) furnishes the basic recipe how to subtract the Green functions so that the PCAC Ward identity (3.3) is satisfied. The decomposition (3.13) will be effected in precisely such a way that (3.23) is satisfied as will be shown in the next section.

To illustrate the general procedure we consider here the case where the external derivative vertex V_i is quadrilinear. In this case only the lowest order contribution of (3.23) is relevant

$$f \langle TN[(\frac{\delta\mathcal{L}_2}{\delta A_k} - \partial_\nu \frac{\delta\mathcal{L}_2}{\delta A_{k,\nu}})\Big|_{\text{to order } f^{-2}}](x)\,\overline{X}\,\rangle =$$

$$= -\frac{1}{2f}\langle TN[A_k \partial_s^2 A^2](x)\,\overline{X}\,\rangle \tag{3.24}$$

Evaluating the Euler derivative yields

$$(\frac{\delta}{\partial A_k} - \partial_\nu \frac{\delta}{\partial A_{k,\nu}})\left[\frac{1}{2f^2}(A_i \partial_\alpha A_i)^2\right] = -\frac{1}{2f^2} A_k \partial^2 A^2$$

Thus we obtain the result that the momentum factors

associated with external quadrilinear vertices are completely soft, i.e. $P_2^{(H)}(p,k) = 0$. More precisely the momentum factor must be associated with the line that carries the same isospin index as the external line at the vertex under consideration.

Finally we comment on the vector current

$$A_k\, X^i_{ke}\, \partial_\mu A_e \tag{3.25}$$

Since it is only bilinear, no anisotropies arise and we obtain

$$\partial^\mu \langle TN[A_k X^i_{ke} \partial_\mu A_e](x)\,\overline{X}\,\rangle =$$

$$= X^i_{ke}\left\{\langle TN[A_k(\partial^2+m_k^2)A_e]\,\overline{X}\,\rangle - \langle TN[A_e(\partial^2+m_k^2)A_k]\,\overline{X}\,\rangle =\right.$$

$$= -2i\sum_{\beta=1}^{N} \delta(x-x_\beta)\, X^i_{k r_\beta} \langle T A_k(x) \prod_{\alpha\neq\beta} A_{r_\alpha}(x_\alpha)\rangle \tag{3.26}$$

where the derivative in (3.25) is hard.

IV - SUBTRACTION PRESCRIPTION FOR ARBITRARY GRAPHS

The validity of the equation of motion within normal products and the ACWI has been shown in Sec. III, when all the derivative vertices of $\mathcal{L}_2$ are quadrilinear. Here we will show that it is possible to decompose the subtractions of the derivative vertices into soft and hard parts in such a way that the Green functions satisfy the correct ACWI.

For this purpose we classify external derivative vertices $\mathcal{V}$ with interaction $(A^2)^n(A\cdot\partial_\alpha A)^2$ according to the way in which the lines of $\mathcal{V}$ are contracted, since the subtraction procedure will depend on the topology of the graph around $\mathcal{V}$. Out of the $2n+4$ lines ending on $\mathcal{V}$ let (r_1+2r_2) be external, r_1 flowing into the graph (we call them r_1-lines), $2r_2$ are connected among themselves (called r_2-lines) and $2s=(2n+4)-2(r_1+r_2)$ are contracted to form internal lines (we call them "spikes"). The internal r_1-lines are labeled by overbared integers and we furthermore exhibit a dashed line for each spike labeled by under-bared integers as shown in Fig. 4.1. The purpose of these appearently redundant dashed lines will become clear below.

It will turn out that specifying the contractions

is not enough and that the number of loops formed at $\mathcal{V}$ is the other important topological aspect on which the subtraction procedure depends.

A - CASE $r_1=1$; VERTEX $\mathcal{V}$ WITH ONE EXTERNAL LINE

Before treating the general case consider $r_1=1$, $r_2=0$ in detail. Calling x_1 the end point of the only external line of $\mathcal{V}$ we want to calculate the following Green function

$$\frac{1}{2f^2}\int dy\,\langle T\,\Delta_{k,e}(x_1-y)\,N_?\left\{\frac{\tilde\delta}{\delta A_e}\left[\left(\frac{A^2}{f^2}\right)^n(A\cdot\partial_\alpha A)^2\right]\right\}(y)\,\underline{X}\,\rangle \tag{4.1}$$

where $\underline{X}=\prod_{m=2}^{B}A_{k_m}(x_m)$

and $\tilde\delta/\delta A_e$ indicates the Euler derivative

$$\frac{\tilde\delta}{\delta A_e}\equiv\frac{\delta}{\delta A_e}-\partial_\nu\frac{\partial}{\delta A_{e,\nu}} \tag{4.2}$$

None of the fields in the normal product is contracted directly with an external field $A_{k_m}(x_m)$, $m=2,3,\dots,B$. The question mark on the normal product indicates that its subtraction prescription is to be determined. This process will have to be repeated for all external vertices among $N[\mathcal{L}_2](z_i)$ to fix their subtraction prescription.

Evaluating the Euler derivative in (4.1) yields

$$-\frac{1}{2}\int dy \langle T \Delta_{k_1,e}(x_1-y) N_2 \Big\{ \frac{A_e}{n+1} \partial^2 \Big[\Big(\frac{A^2}{f^2}\Big)^{n+1} \Big] -$$

$$-\frac{1}{2} A_e \partial_\nu \Big[\Big(\frac{A^2}{f^2}\Big)^{n} \Big] \partial^\nu \Big(\frac{A^2}{f^2}\Big) \Big\}(y)\, \underline{X} \rangle \tag{4.3}$$

The subtraction prescription for the above normal product is determined by the lowest order current Green function[7]

$$-\frac{f}{2}\int dy \langle T \partial_\mu \Delta_{k_1,e}(x_1-y) N_2 \Big\{ \frac{A_e}{n+1} \partial^2 \Big[\Big(\frac{A^2}{f^2}\Big)^{n+1} \Big] -$$

$$-\frac{1}{2} A_e \partial_\nu \Big[\Big(\frac{A^2}{f^2}\Big)^{n} \Big] \partial^\nu \Big(\frac{A^2}{f^2}\Big) \Big\}(y)\, \underline{X} \rangle \tag{4.4}$$

whose graph is shown in Fig. 4.2.

We now write an ACWI for (4.4), keeping only terms whose unsubtracted integrand is the same as the one of the divergence of (4.4) (The remaining terms will be dealt with in (4.7) below):

$$\langle T \underline{X} \Big\{ \frac{f}{2}\int dy\, \partial^2 \Delta_{k_1,e}(x_1-y) N_2 \Big\{ \frac{A_e}{n+1} \partial^2 \Big[\Big(\frac{A^2}{f^2}\Big)^{n+1} \Big] -$$

$$-\frac{1}{2} A_e \partial_\nu \Big[\Big(\frac{A^2}{f^2}\Big)^{n} \Big] \partial^\nu \Big(\frac{A^2}{f^2}\Big) \Big\}(y) -$$

$$-\frac{1}{2f}\int dy \sum_{j=1}^{n} (-1)^j \binom{1/2}{j} N\Big\{ \Big[\Big(\frac{A^2}{f^2}\Big)^{j} \Big](x_1) \Big([\partial^2 + m_{k_1}^2] \Delta_{k_1,k_j}(x_1-y) \Big) \cdot$$

$$\cdot \Big[\frac{\tilde{\delta}}{\delta A_{k_j}} \Big(\Big(\frac{A^2}{f^2}\Big)^{n-j} (A \cdot \partial_\alpha A)^2 \Big) \Big](y) \Big\} +$$

$$+ f (-1)^{n+1} \binom{1/2}{n+1} N \Big\{ A_{k_1} \partial_s^2 \Big[\Big(\frac{A^2}{f^2}\Big)^{n+1} \Big](x_1) \Big\} \Big\} \rangle = 0 \tag{4.5}$$

The first two terms in (4.5) correspond to the l.h.s. of Eq. (3.23) and the 3rd one to the r.h.s. of Eq.(3.23). The graphical structure of (4.5) is shown in Fig. 4.3.

At this point we have to know whether the vertex $\mathcal{V}_2'$ at y is internal or external. This second case can only happen if through the line joining $\mathcal{V}_1'$ and $\mathcal{V}_2'$ flows only external momentum, i.e. the graph is not one-particle-irreducible and the number of loops formed at $\mathcal{V}$ in the graph of Fig. 4.2 is less than the maximum $2s$. If this happens one has to apply the considerations of this section A to the vertex $\mathcal{V}_2'$ first. In order to simplify matters we will here and in the following give explicit formulas only for the case when at the vertex under consideration, the maximum number of loops is formed. At the end of this section we will present two recursion formulas (Eq.(4.26) and (4.27)) which do not depend on the number of loops formed at $\mathcal{V}$.

If now the vertex $\mathcal{V}_2'$ at y in the 2nd term of (4.5) is completely internal, the equation of motion yields two hard terms:

$$A_{k_1}\partial^2[(A^2)^{n+1}](x_1) \quad \text{and} \quad A_{k_1}\partial_\nu[(A^2)^n]\,\partial^\nu[A^2]\,(x_1)$$

where we have evaluated the Euler derivative. Thus for the Eq.(4.5) to hold and the necessary cancellations to take place we have to subtract (4.3) in the following way

$$\frac{1}{2}N_2\left\{\frac{A_\ell}{n+1}\,\partial^2\left[\left(\frac{A^2}{f^2}\right)^{n+1}\right]-\frac{1}{2}A_\ell\,\partial_\nu\left[\left(\frac{A^2}{f^2}\right)^n\right]\partial^\nu\left(\frac{A^2}{f^2}\right)\right\}(y)\equiv$$
$$N\left\{(-1)^n\binom{1/2}{n+1}A_\ell\,\partial_s^2\left[\left(\frac{A^2}{f^2}\right)^{n+1}\right]+\left[\frac{1}{2(n+1)}-(-1)^n\binom{1/2}{n+1}\right]A_\ell\left[\left(\frac{A^2}{f^2}\right)^{n+1}\right]\right. -$$
$$\left.-\frac{1}{2}A_\ell\,\partial_\nu\left[\left(\frac{A^2}{f^2}\right)^n\right]\partial^\nu\left(\frac{A^2}{f^2}\right)\right\}(y) \tag{4.6}$$

The Eq.(4.6) above realizes the decomposition (3.13) for the case at hand.

The reasoning leading to (4.6) will be repeated all over again to analyze more complicated situations.

The two terms not written down in the ACWI (4.5) are

$$f(-1)^{n+1}\binom{1/2}{n+1}\left\langle TN\left\{\left[\frac{A^2}{f^2}\right]^{n+1}(\partial^2+m^2_{k_1})A_{k_1}\right\}(x_1)\,\underline{\overline{X}}\right\rangle -$$
$$-\frac{1}{2f}\int dy\sum_{j=1}^{n}(-1)^j\binom{1/2}{j}\left\langle TN\left\{\left(\partial_s^2\left[\left(\frac{A^2}{f^2}\right)^j\right](x_1)\right)\Delta_{k_1,k_j}(x_1-y)\,\times\right.\right.$$
$$\left.\left.\times\left[\frac{\tilde{\delta}}{\delta A_{k_j}}\left(\left(\frac{A^2}{f^2}\right)^{n-j}(A\cdot\partial_\alpha A)^2\right)\right](y)\right\}\underline{\overline{X}}\right\rangle \tag{4.7}$$

whose graphical structure is shown in Fig. 4.4. Their unsubtracted integrands are different from the divergence of (4.5) and consequently they will be cancelled by other graphs in the following fashion [8].

The first term in Fig.4.4 will participate in another Ward identity like (4.5), but with $n \to n'$ and $n' > n$. There it plays the role of one of the terms in the second line of (4.5) with $n \to n'$.

Each of the n terms in the second line of (4.7) will again participate in another Ward identity like (4.5), but now with $n \to n''$ and $n'' < n$ [9]. There it plays the rôle of the term in the last line of (4.5).

B.- CASE $r_2 = 0$, r_1 ARBITRARY

In order to analyze more complex situations it is convenient to go to momentum space. The polynomial associated with $(A^2)^n (A\cdot\partial_\alpha A)^2$ will depend on the number of independent loops formed by the internal lines of $\mathcal{V}$. For this reason we will at the end of this subsection eliminate the polynomial from our considerations in favour of the decomposition (3.12)'. To start with consider the case $r_2 = 0$ and the maximum number of loops, namely $(r_1 + 2s - 1)$. Adopting the routing shown in Fig. 4.1 we get for the polynomial at $\mathcal{V}$

$$-2^n n!\, \tfrac{1}{2} \left\{ \sum_{i=1}^{r_1-\delta_{s,0}\delta_{r_1,2}} k_i^2 + \sum_{j=1}^{s} Q_{2j}^2 \right\} \tag{4.8}$$

where $Q_{2j} = q_{2j-1} + q_{2j}$ and $\sum_{i=1}^{r_1} k_i + \sum_{j=1}^{s} Q_{2j} = 0$.
The subtraction prescription for (4.8) is discovered applying the ACWI to the line with say external momentum p_1. As should be clear from our $r_1 = 1$ example, the only place we get a soft k_1^2 is from a term analogous to the 3rd one in (4.5)[10]:

$$(-1)^{n+1} \binom{1/2}{n+1} A_{k_1} \partial_s^2 \left[\left(\frac{A^2}{\rho^2} \right)^{n+1} \right]$$

where A_{k_1} forms an internal line through which $-p+k_1$ flows. An identical remark is valid when applying the ACWI to the other external lines and we obtain the following prescription (3.13) for the polynomial (4.8)[11]:

$$\begin{aligned} -2^{n-1} n!\, N_2 \left\{ \sum_{i=1}^{r_1-\delta_{s,0}\delta_{r_1,2}} k_i^2 + \sum_{j=1}^{s} Q_{2j}^2 \right\} &\Rightarrow \\ -(2n-1)!! \sum_{i=1}^{r_1-\delta_{s,0}\delta_{r_1,2}} k_{i,s}^2 - \left[\tfrac{1}{2}(2n)!! - (2n-1)!! \right] \sum_{i=1}^{r_1-\delta_{s,0}\delta_{r_1,2}} k_i^2 &- \\ -2^{n-1} n! \sum_{j=1}^{s} Q_{2j}^2 & \end{aligned} \tag{4.9}$$

The prescription (4.9) can be explicitly verified applying the ACWI successivly to all the r_1 external lines of $\mathcal{V}$.

In order to avoid writing down long polynomials, we will use Eq.(3.12)' to transform all momenta into hard ones, isolate the "mass" terms and keep track only of them. We hasten to stress that they belong to the subtractions of the derivative vertices and we put them in quotation marks to avoid confusion with minimally subtracted mass vertices from $\mathcal{L}_1$. Using now (3.12)' in

(4.9) we get

$$(4.9) = -\frac{1}{2}(2m)!!\left\{\sum_{i=1}^{r_1-\delta_{s,0}\delta_{r_1,2}} k_i^2 + \sum_{j=1}^{s} Q_{2j}^2\right\} + $$
$$+ (2m-1)!!\sum_{i=1}^{r_1-\delta_{s,0}\delta_{r_1,2}} m^2(\bar{i}) \qquad (4.10)$$

where

$$m^2(\bar{i}) \equiv S_G \sum_{U\in F}\prod_{\gamma\in U}\left(-\tau^{d(\gamma)}\right)\left[m_{\bar{i}}^2\, I_G/P_v(q,k)\right] -$$
$$- m^2 S_G \sum_{U\in F}\prod_{\gamma\in U}\left(-\tau^{d(\gamma)}\right)\left[I_G/P_v(q,k)\right]$$
$$(4.11)$$

As in (3.12)' $m_{\bar{i}}^2$ contributes two powers to $d(\gamma)$. Eq.(4.10) realizes the decomposition (3.13)'.

To streamline our formulas let us introduce the following notation for the "mass" terms in the decomposition (3.13)' of the vertex of Fig.(4.1)

$$S_m(r_1,r_2;s\,|\,\bar{1},\bar{2},\ldots,\bar{r}_1;\underline{1},\underline{2},\ldots,\underline{s}) \qquad (4.12)$$

where the first index counts the number of r_1 -lines,

the second the number of external pairs and the third the number of spikes. After the vertical bar we have listed the labels of the internal lines: overbared indices for continuous lines and underbared indices for dashed lines. The purpose of these last ones will become evident when $r_2 \neq 0$. The index m indicates that we consider here the case where the maximum of loops is formed at v. In the notation (4.12) to "mass" terms in (4.10) become

$$(2m-1)!!\sum_{i=1}^{r_1-\delta_{s,0}\delta_{r_1,2}} m^2(\bar{i}) = S_m(r_1,0;s\,|\,\bar{1},\bar{2},\ldots,\bar{r}_1;) \qquad (4.13)$$

The hard polynomials will be supressed from now on and we only give the decomposition (3.13)'.

C - ANISOTROPIC DERIVATIVE COUPLING

Eq.(4.13) is a very simple formula telling us how to subtract a derivative vertex with an arbitrary number of r_1 -lines. If r_2 -lines are present a new complication arises. If we apply the ACWI to an r_2 -line, the terms analogous to the second line of (4.5) will now involve soft (k^2) monomials associated with the same line carrying the Klein-Gordon operator, as can be seen from Fig. 4.5. The reason is that the r_2 -line becomes external at v_2 and this vertex will contribute with soft (k^2) terms. But now the continuous line carrying the soft monomial does not appear in the graph of Fig. 4.1; its unsubtracted integrand does not depend on the mass associated with the line carrying the soft monomial in Fig.4.7. Yet in order to obtain the correct ACWI "mass" terms of the type (4.13) have to be present and will be introduced using the seemingly redundant dashed lines labelled by underbared indices.

It is clear from (4.11) that $m^2(\bar{\iota})$ vanishes for a subgraph having $\bar{\iota}$ as an external line. This property will be important also for "mass" terms associated with dashed lines and this was the reason for introducing the definition (3.7).

We now define anisotropic multiindexed "mass" terms $m^2(i_1,i_2,\ldots,i_n)$ associated with the lines $i_1,i_2,\ldots i_n$, which may or not be dashed. We will need "Taylor" operators of the form (3.9) depending on an extra variable $M^2(J)$, where $J=\sum_{\alpha=1}^{n} i_\alpha$. The "Taylor" operators $\tau^{d(\gamma)}$ act of course only on variables belonging to γ and we say that

$$M^2(J)\in\gamma \text{ , iff } J\in\gamma \tag{4.14}$$

Define now

$$\begin{aligned} m^2(i_1,i_2,\ldots,i_n) &\equiv \\ & S_G\sum_{U\in F}\prod_{\gamma\in U}\left(-\tau^{\tilde{d}(\gamma)}_{\ldots,M^2(J)}\right)S_\gamma\left[M^2(J)\,I_G/P_\nu(k,p)\right]\Big|_{M^2(J)=m^2} - \\ & - m^2 S_G\sum_{U\in F}\prod_{\gamma\in U}\left(-\tau^{d(\gamma)}_{\ldots}\right)S_\gamma\left[I_G/P_\nu(k,p)\right] \end{aligned} \tag{4.15}$$

where the dots in $\tau^{d(\gamma)}_{\ldots,M^2(J)}$ indicate the variables listed in (3.9), $d(\gamma)$ is the superficial degree of divergence of $I_G/P_\nu(k,p)$ to which $M^2(J)$ does not contribute and

$$\tilde{d}(\gamma)=\begin{cases} d(\gamma)+2 & \text{, if } J\in\gamma \\ d(\gamma) & \text{, if } J\notin\gamma \end{cases} \tag{4.16}$$

we notice that $M^2(J)$ contributes no m^2 dependence, but only a μ^2 dependence, to the vertex $v^{[12]}$ and (4.15) vanishes if $J\notin\gamma$. Actually (4.15) depends only on J and its l.h.s. should be written $m^2(J)$; but we find the more explicit notation convenient, although it is not unique, since J can in general be written as a sum of lines in several ways. In particular one of the lines may itself be a sum $i_\alpha = \sum_{\beta=1}^{\ell} i_\beta$.

If $J = \bar{i}_1$, where $\bar{i}_1$ is a continuous line, the factorial in the definition (3.9) ensures that, inspite of our introducing a new variable, which already occurs in $\tau^{\alpha(\gamma)}_{\dots}$, (4.15) reduces to (4.11) in this case.

As an example of the use of (4.15) consider a vertex with $r_1 = 2, r_2 = 1, s = 2$ as shown in Fig. 4.6. Applying the ACWI to the r_2-line we obtain graphs with $r_2 = 0$, which we know how to subtract. The resulting subtraction prescription is

$$\mathcal{S}_m(2,1;2\,|\,\bar{1},\bar{2};\underline{1},\underline{2}) = 18\left[m^2(\bar{1}) + m^2(\bar{2})\right] + 6\left[m^2(\underline{1}) + m^2(\underline{2})\right] + 2\left[m^2(\underline{1},\underline{2}) + m^2(\underline{1},\bar{1}) + m^2(\underline{2},\bar{1})\right] \tag{4.17}$$

Had we applied the ACWI to one of the r_1-lines, the same result obtains, but now the knowledge of $\mathcal{V}(r_1=1, r_2=1, s=1)$ and $\mathcal{V}(r_1=0, r_2=1, s=2)$ is needed. Applying again the ACWI to one of the external lines we get

$$\mathcal{S}_m(1,1;1\,|\,\bar{1};\underline{1}) = 2\,m^2(\bar{1}) = 2\,m^2(\underline{1}) \tag{4.18}$$

$$\mathcal{S}_m(0,1;2\,|\,;\underline{1},\underline{2}) = 2\,m^2(\underline{1}) = 2\,m^2(\underline{2}) \tag{4.19}$$

D - TWO RECURSION RELATIONS

Instead of writing long formulas for the general case, we present two recursion relations, which allow one to obtain the subtraction prescription for arbitrary vertices. Consider first the case, when the maximum number of loops is formed at $\mathcal{V}$.

The first one is an r_2-type, i.e. obtained by applying the ACWI to an r_1-line, say $\bar{1}$:

$$\begin{aligned}
&\mathcal{S}_m(r_1,r_2;s\,|\,\bar{1},\bar{2},\dots,\bar{r}_1;\underline{1},\underline{2},\dots,\underline{s}) = \\
&= m^2(\bar{1})\Big\{[2(r_1+r_2+s-1)-3]!! + \sum_{\ell=1}^{r_2}[2\ell-3]!!\,\mathcal{S}_m(r_1,r_2-\ell;s\,|\,\bar{1},\dots,\bar{r}_1;\underline{1},\dots,\underline{s}) + \\
&+ \sum_{k=2}^{r_1+r_2+s-1}\ {\sum_{i_1,i_2,\dots i_{\beta-1}=1}^{s}}'\ {\sum_{j_1,j_2,\dots j_{\alpha_1}=2}^{r_2}}'\ [2(r_1+r_2+s-k)-3]!!\ * \\
&* \Big\{1 + \delta_{\alpha_2,0}\Big[\binom{s}{k-\alpha_1-1} - 1\Big]\Big\}\binom{r_2}{\alpha_2}\ * \\
&* \mathcal{S}_m(\alpha_1,\alpha_2;\beta\,|\,\bar{j}_1,\bar{j}_2,\dots,\bar{j}_{\alpha_1};\underline{i}_1,\underline{i}_2,\dots,\underline{i}_{\beta-1},\underline{i}_\beta)
\end{aligned} \tag{4.20}$$

where α_1, α_2 and β take all values compatible with $\alpha_1 + \alpha_2 + \beta = k$, but subjected to the following restrictions:

$$\left\{ \begin{array}{l} \alpha_1 \leq r_1 - 1 \,,\ \alpha_2 \leq r_2 \,,\ 0 < \beta \leq s + 1 \\ \text{Exclude the term } \alpha_1 = 0, \beta = 1, \ \forall \alpha_2 \end{array} \right\} \tag{4.21}$$

The sum $\sum'^{\,u}_{v_1, v_2, \dots, v_n = a}$ means $\sum^{u}_{\substack{v_1 = a, \dots, v_n = a \\ v_1 < v_2 < \dots < v_n}}$ all variables ranging from a up to u. We always use j_α to indicate r_1 -lines and i_α to label dashed lines ending on spikes. Finally $\underline{i}_\beta$ stands for

$$\underline{i}_\beta = \sum_{\alpha=1}^{\beta-1} \underline{i}_\alpha + \sum_{\alpha=1}^{\alpha_1} \bar{j}_\alpha \tag{4.22}$$

Let us briefly comment about the origin of the various terms in (4.20), where we refer to Fig. 4.3, except that now the vertex v_1 has an arbitrary number of lines. The factors multiplying $m^2(\bar{1})$ arise from the analogon of the third term in Fig.4.3. and the term involving $S_m(r_1, r_2 - \ell; s | \dots)$ from the second term of that figure, where at v_2' we have only r_2 -lines and no internal lines. The index k

in the sum over k stands for the number of lines at vertex v_2', i.e. the interaction at v_2' is $(A^2)^{k-2} (A \cdot \partial_\alpha A)^2$. The restrictions (4.21) express the fact that the sum over k terminates, once all lines of vertex v_1' have been shifted to v_2', leaving a vertex with three lines at v_1'. Furthermore β is always different from zero, since the $\bar{1}$-line becomes internal at v_2'. Finally it can easily be seen that the term $\alpha_1 = 0$, $\beta = 1$, α_2 arbitrary does not contribute and that $\underline{i}_\beta$ is not summed over in (4.20), since it is a dependent variable according to (4.22), i.e. $\underline{i}_\beta$ and the r.h.s. of (4.22) are equivalent. It may be verified that (4.20) is symmetric in the $\bar{j}_\alpha$ -lines.

A second recursion relation may be obtained applying the ACWI to an r_2 -line:

$$\begin{aligned} & S_m(r_1, r_2; s \,|\, \bar{1}, \bar{2}, \dots, \bar{r}_1; \underline{1}, \underline{2}, \dots, \underline{s}) = \\ & \sum_{\ell=1}^{r_2} [2\ell - 3]!! \binom{r_2}{\ell} S_m(r_1 + 1, r_2 - \ell; s \,|\, \bar{1}, \bar{2}, \dots, \bar{r}_1, \overline{r_1 + 1}; \underline{1}, \underline{2}, \dots, \underline{s}) + \\ & + \sum_{j=1}^{r_1} [2(r_1 + r_2 + s - 2) - 3]!! \, m^2(\bar{j}) + \sum_{i=1}^{s} [2(r_1 + r_2 + s - 2) - 3]!! \, m^2(\underline{i}) + \\ & + \sum_{k=3}^{r_1 + r_2 + s - 1} \ \sum'^{\,r_1}_{j_1, j_2, \dots, j_{\alpha_1} = 1} \ \sum'^{\,s}_{i_1, i_2, \dots, i_\beta = 1} [2(r_1 + r_2 + s - k) - 3]!! \binom{r_2}{\alpha_2} \cdot \\ & \cdot S_m(\alpha_1, \alpha_2; \beta \,|\, \bar{j}_1, \bar{j}_2, \dots, \bar{j}_{\alpha_1}; \underline{i}_1, \underline{i}_2, \dots, \underline{i}_\beta) \end{aligned} \tag{4.23}$$

Again we sum over all values of $\alpha_1, \alpha_2, \beta$ compatible with $\alpha_1 + \alpha_2 + \beta = k$ but subjected to the following restrictions:

$$\left\{ \begin{array}{l} 0 < \alpha_1 \le r_1+1 ,\ \alpha_2 \le r_2 - 1 ,\ \beta \le s \\ \text{Exclude the term } \ \alpha_1 = 1, \beta = 0, \ \forall\, \alpha_2 \end{array} \right\} \tag{4.24}$$

and

$$\bar{j}_{\alpha_1} = \sum_{\alpha=1}^{\beta} \underline{i}_\alpha + \sum_{\alpha=1}^{\alpha_1 - 1} \bar{j}_\alpha \ , \quad \overline{r_1+1} = \sum_{i=1}^{s} \underline{i} + \sum_{j=1}^{r_1} \bar{j} \tag{4.25}$$

Notice that the term $k=2$ and $(\alpha_1 = r_1+1, \beta_1 = s)$ has been written explicitly in (4.23).

If the number of loops at v is not maximum, the formulas (4.20) and (4.23) have to be written more explicitly, since in (4.20) the $\bar{1}$ -line does not necessarily become internal at v_2' . We get applying the ACWI to $\bar{1}$:

$$\begin{aligned} & S(r_1, r_2; s \mid \bar{1}, \bar{2}, \dots, \bar{r}_1 ; \underline{1}, \underline{2}, \dots, \underline{s}) = \\ & = m^2(\bar{1}) \, [2(r_1 + r_2 + s - 1) - 3]!! \, + \\ & + \sum_{k=2}^{r_1+r_2+s-1} \ \sum_{i_1, \dots, i_\beta = 1}^{s}{}' \ \sum_{j_1, \dots, j_{\alpha_1} = 1}^{r_1}{}' \ [2(r_1 + r_2 + s - k) - 3]!! \binom{r_2}{\alpha_2} \cdot \\ & \cdot S(\alpha_1, \alpha_2; \beta \mid \bar{j}_1, \bar{j}_2, \dots, \bar{j}_{\alpha_1} ; \underline{i}_1, \underline{i}_2, \dots, \underline{i}_\beta) \end{aligned} \tag{4.26}$$

where, again α_1, α_2 and β take all values compatible with $\alpha_1 + \alpha_2 + \beta = k$ and instead of the restrictions (4.21) we impose:

1) in the terms obeying (4.21), i_β is not an independent variable, but given by (4.22)
2) in the terms not obeying (4.21) i_β is to be summed over as an independent variable and j_1 takes only the value $j_1 = 1$.

Formula (4.26) is simpler in appearance than (4.20), but contains more terms. Some of them may not exist for a particular graph and the corresponding term is to be dropped from the sum in (4.26).

Applying the ACWI to an r_2 -line, we obtain

$$\begin{aligned} & S(r_1, r_2; s \mid \bar{1}, \bar{2}, \dots, \bar{r}_1 ; \underline{1}, \underline{2}, \dots, \underline{s}) = \\ & = \sum_{k=2}^{r_1+r_2+s-1} \ \sum_{i_1, \dots, i_\beta = 1}^{s}{}' \ \sum_{j_1, \dots, j_{\alpha_1} = 1}^{r_1}{}' \ [2(r_1 + r_2 + s - k) - 3]!! \binom{r_1}{\alpha_2} \cdot \\ & \cdot S(\alpha_1, \alpha_2; \beta \mid \bar{j}_1, \bar{j}_2, \dots, \bar{j}_{\alpha_1} ; \underline{i}_1, \underline{i}_2, \dots, \underline{i}_\beta) \end{aligned} \tag{4.27}$$

where, again, we lift restrictions (4.24) and impose that j_{α_1} is an independent variable for the terms not satisfying (4.24) and is given by (4.25) for the terms obeying (4.24).

Using (4.20), (4.23), (4.26) and (4.27) it is now straightforward to obtain the subtraction prescription for any vertex.

To obtain the subtraction prescription for the whole graph one proceeds as follows:

i) obtain $S^{(m)}(r_1^{(m)}, r_2^{(m)}; s^{(m)} | \bar{1}, \ldots, \bar{r}_1^{(m)}; \underline{1}, \ldots, \underline{s}^{(m)})$ for each external derivative vertex v_m

ii) calculate the polynomial $P_{v_m}(p,k)$ according to standard rules

iii) obtain the decomposition (3.13)' by the substitution

$$P_{v_m}(p,k) \longrightarrow$$

$$P_{v_m}^{(H)}(p,k) - S^{(m)}(r_1^{(m)}, r_2^{(m)}; s^{(m)} | \bar{1}, \ldots, \bar{r}_1^{(m)}; \underline{1}, \ldots, \underline{s}^{(m)}) \tag{4.28}$$

iv) to each vertex v_m we have now associated a new numerator given by the r.h.s. of (4.26).

The resulting subtracted integrand R_G is then obtained from a product like (3.12)', where $\langle m_i^2 \rangle - m_i^2$ is replaced by $S^{(m)}(\cdot, \cdot ; \cdot | \cdot ; \cdot)$.

V - CONCLUSION

After having established the existence of the zero mass limit several questions remain. First one can show[10] that our currents satisfy a current algebra. Then one would like to study the properties of the theory in more detail, in particular the structure of counterterms. This is difficult since our Green functions have no simple μ^2-dependence, due to the need of introducing the decomposition (3.13). In particular we do not expect the vertex functions of the theory to satisfy any simple normalization conditions. Since at this point we do not know the structure of counterterms, we also ignore how much the imposition of vector and axial vector Ward identities has cut down the infinite number of renormalization constants of our nonrenormalizable model[14].

In the 2-dimensional renormalizable case one renormalizes the expression $(\not{\ell}^2 - A^2)^{1/2}$ without expanding in powers of A^2 and treating each resulting term separately. Also here one might wonder whether cancellations occur once the square root is reassembled and whether this may improve the high energy behaviour.

FOOTNOTES

[1] A review of BPHZ renormalization can be found in Ref.9.

[2] One may easily verify that the criteria for convergence stated in Ref. 6 are satisfied.

[3] Unless stated otherwise $d(\gamma)$ is the smallest number compatible with ultraviolet divergence, i.e. (3.8) is minimally subtracted.

[4] In subsection B and C we work in x-space and use the notation ∂_s^2 for a soft d'Alembertian, although it may happen that the corresponding momentum is external. In this case there is no distinction between hard and soft.

[5] The necessity of associating soft momenta also to dashed lines will become clear in Sec.IV.

[6] The highest derivative of the operator (3.9) will for hard momenta produce terms like $k^2/(k^2-\mu^2)^n$. If k^2 belongs to only one divergent graph, the difference between hard and soft subtraction schemes is the replacement $k^2/(k^2-\mu^2)^n \to (k^2-\mu^2)^{-(n-1)}$ This is the reason for the name soft.

[7] By "lowest order current Green function" we mean the term indicated in the r.h.s. of (2.5).

[8] If the reader is confused by the following two paragraphs we remind him that we are only describing the cancellation mechanism of the unsubtracted integrand.

[9] It is assumed that the subtraction prescription of vertices with $n'' < n$, which are of lower order in f^{-1} , has already been worked out.

[10] This remark is of course not true, if the number of loops is less than $(r_1 + 2s - 1)$.

[11] As in (3.13) we have only written the polynomial at v_2.

[12] The m^2-dependence from $M^2(J)$ cancels out in (4.5) when $\mathcal{U}$ is the empty forest.

[13] In this case the d'Alembertian in the r.h.s. of (3.23) is associated with the square of an external momentum and the subscript s may be dropped.

[14] One arbitraryness lies in the fact that we have made all, but the last subtraction at $p=0$, $m=0$, which certainly is not necessary.

REFERENCES

1) M.Gell-Mann & M.Levy, Nuovo Cimento 16, 705 (1960).

2) E.Brezin & J.Zinn-Justin, Phys.Rev.Lett. 36, 691 (1976); B.A.Bardeen et al., Phys.Rev. D14, 985 (1976).

3) S.Weinberg, Phys.Rev. 166, 1568 (1968).

4) D.Bessis & J.Zinn-Justin, Phys.Rev. D5, 1313 (1972).

5) Y.M.P.Lam, Phys.Rev. D7, 2950 (1973).

6) J.H.Lowenstein, Commun. Math.Phys.

7) M.Gomes, J.H.Lowenstein, and W.Zimmermann, Comm.Math. Phys. 39, 81 (1974).

8) Y.M.P.Lam, Phys.Rev. D6, 2145 (1972); M.Gomes and J.H. Lowenstein, Nucl.Phys.

9) J.H.Lowenstein, "BPHZ Renormalization", Lectures given at International School of Mathematical Physics, Erice, Sicily; G.Velo & A.S.Wightman (ed.) Academic Press (1976)

10) M.Gomes and R.Köberle, to be published.

FIGURE CAPTIONS

FIGURE 2.1

Graphical representation of a general vertex. The sum in the l.h.s. is over all permutations of the indices.

FIGURE 2.2

Graphs γ_1 and γ_2 are overlapping.

Graphs γ_1 and γ_3 are non-overllaping

FIGURE 3.1

Graphs (b) and (d) have the same unsubtracted integrand, but come from graphs with different subtraction schemes.

FIGURE 3.2

Although graphs (b) and (d) have the same unsubtracted integrand, the dashed lines indicate that they are to be subtracted differently: subgraphs (A) and (B) are overlapping in (d) whereas they don't overlap in (b).

FIGURE 3.3

Graphs contributing to the equation of motion inside a normal product.

FIGURE 4.1

Classification of lines of an external derivative vertex $(A^2)^n(A\cdot\partial_\alpha A)^2$. Lines with an overbar carry isospin and those with an underbar refer to dashed lines, one for each spike.

FIGURE 4.2.

Graph corresponding to expression (4.4) with the cross at x_4 indicating the lowest order current vertex $f\,\partial_\mu A_k(x_4)$.

FIGURE 4.3.

Graphical structure of identity (4.5). A cut line means multiplication by the Klein-Gordon operator corresponding to the momentum flowing through the line. No cut after the current vertex stands for $A_k\,\partial_s^2(A^2)^n$, the momentum of $(A^2)^n$ being written in parentheses. The three black boxes of the figure are identical.

FIGURE 4.4

Graphical structure of the two terms in expression (4.7). The two black boxes are equal to the ones of Fig.4.3.

FIGURE 4.5

Graphs contributing to the ACWI of an r_2-line. The cut line will carry k^2 terms, which may be soft.

FIGURE 4.6

Graph with a vertex $\mathcal{V}(r_1=2, r_2=1, s=2)$ where at $\mathcal{V}$ five loops are formed.

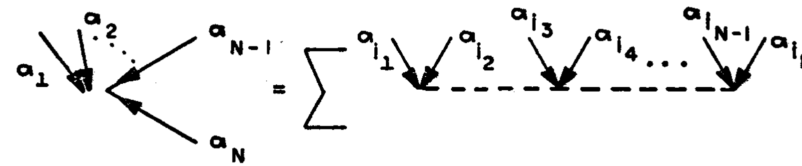

Fig. 2.1

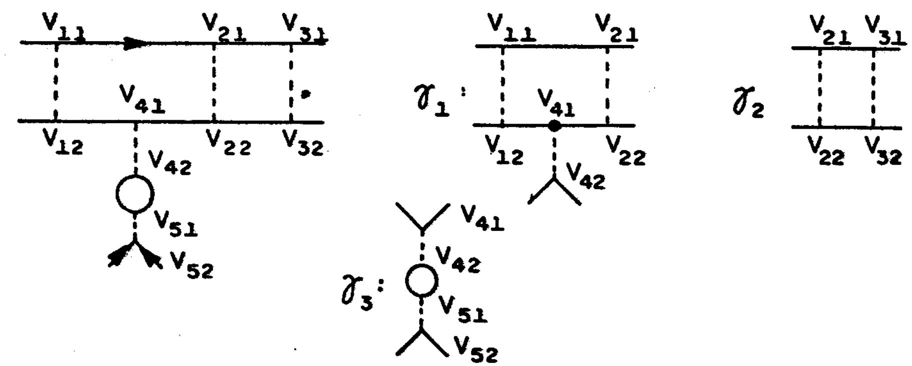

Fig 2.2.

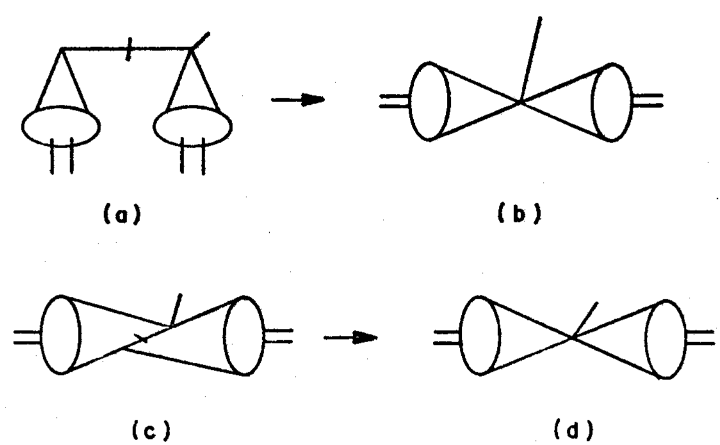

Fig 3.1.

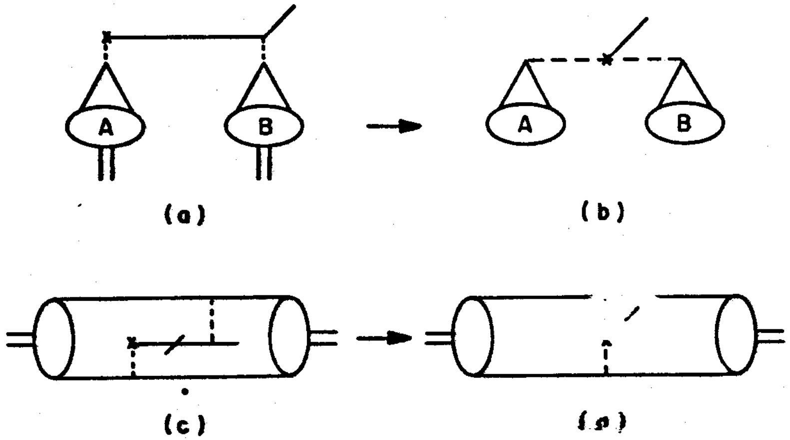

Fig. 3.2

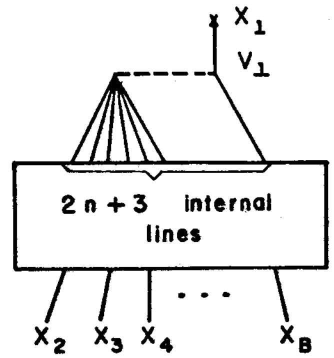

Fig. 4.2

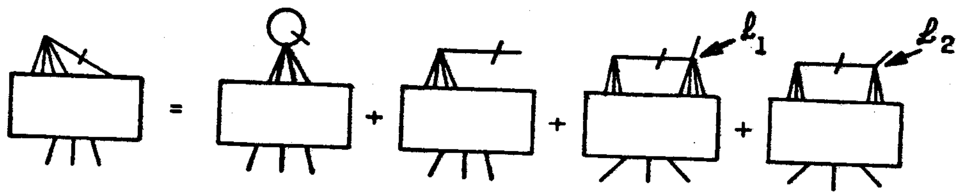

Fig. 3.3

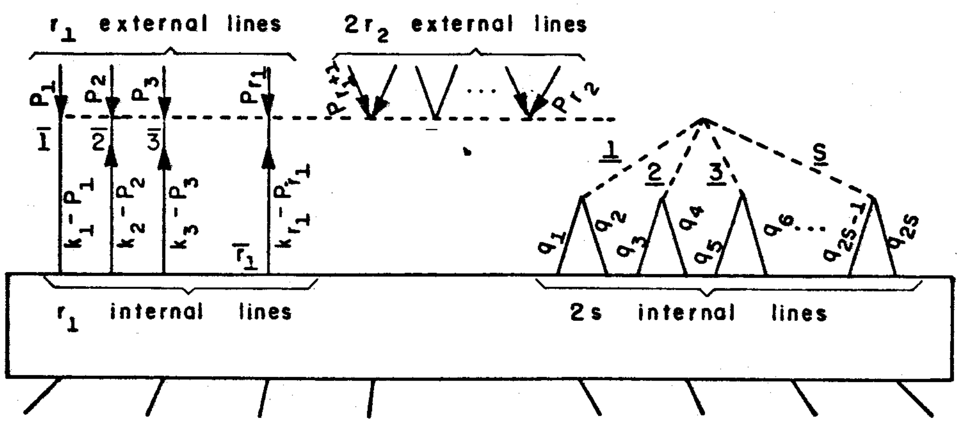

Fig. 4.1

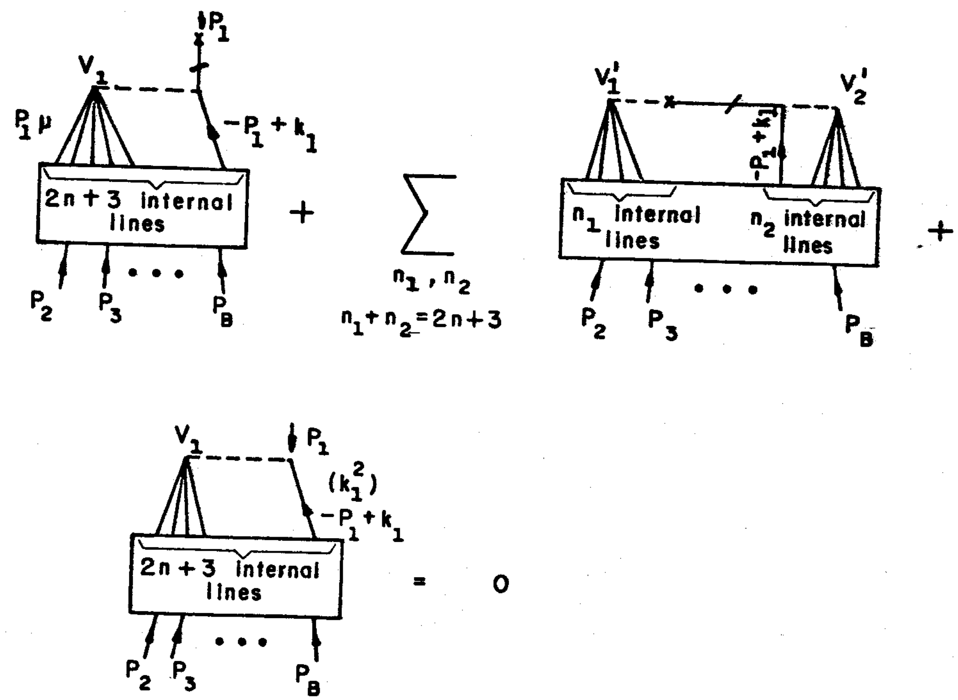

Fig. 4.3

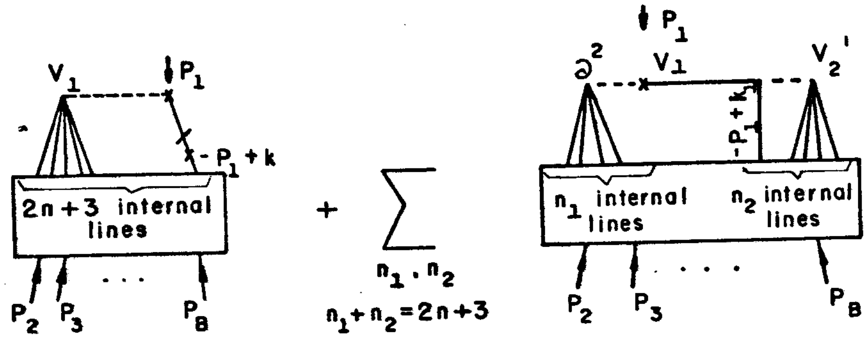

Fig. 4.4

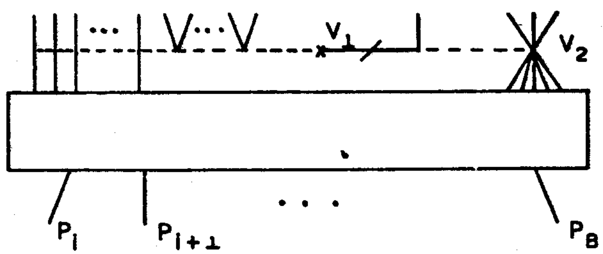

Fig. 4.5

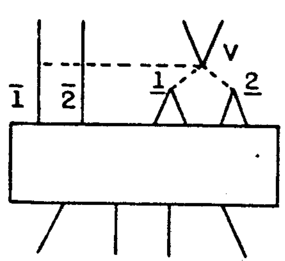

Fig. 4.6