

UNIVERSIDADE DE SÃO PAULO

PUBLICAÇÕES

**INSTITUTO DE FÍSICA
CAIXA POSTAL 66318
05389-970 SÃO PAULO - SP
BRASIL**

IFUSP/P-1177 - A

**EXACT EXPRESSION FOR THE FREE ENERGY OF
ELECTRON - POSITRON PAIRS IN A CONSTANT
MAGNETIC FIELD**

Luiz C. de Albuquerque
Instituto de Física, Universidade de São Paulo

Outubro/1995

Exact Expression for the Free Energy of Electron-Positron Pairs in a Constant Magnetic Field

Luiz C. de Albuquerque*

*Departamento de Física Matemática, Instituto de Física
Universidade de São Paulo
São Paulo SP
Caixa Postal 66.318 - CEP 05389-970
Brasil*

Abstract

We use a modified version of the Schwinger's proper-time effective action to obtain an exact expression for the Gibbs Free Energy of a fermion gas of electron-positron pairs in thermal equilibrium in a constant magnetic field, up to one-loop order.

PACS number(s): 12.20.Ds

*e-mail: claudio@snfma1.if.usp.br

Since the seminal work of Kirzhnits and Linde [1], the restoration of spontaneously broken symmetries due to the action of a thermal bath [2] or a background (electromagnetic) field [3] has been extensively studied. Possible applications are in high-temperature superconductivity and the early universe [4]. However, few studies deal with both effects [5]. Besides, approximations are usually made concerning the high-temperature/electromagnetic field domain.

In this note, we use a simple model to show how to compute an exact expression for the Gibbs Free Energy of a gas of electron-positron pairs (in thermal equilibrium) in a constant background magnetic field, evaluated by Dittrich [6] as a series of terms in a high-temperature expansion.

We use a modified version of the Schwinger's proper-time effective action [7], that has shown to be a powerful tool in the computation of the Casimir energy at zero temperature [8] as well as at finite temperature [9], and which allows the use of an analytic regularization technique [8, 9]. An appendix contains the necessary analytic continuations of the functions required in the regularization process. We hope that the technique developed here can be useful in theories with spontaneously broken symmetries, like the standard model of elementary particles and interactions [10]. Moreover, recently the thermal two-point function for bosonic particles was exactly computed, at one-loop order, using forward scattering amplitudes [11]. We think that our approach can be more suitable for the discussion of fermions.

1. The one-loop effective action (the generating functional of one-loop 1PI, amputated, Green's functions) is given by Schwinger's formula [7] as

$$\Omega^{(1)}[A] = \frac{i}{2} \int_0^\infty \frac{ds}{s} e^{-is(m^2 - it)} \text{Tr} e^{-is\mathcal{H}}, \quad (1)$$

plus counterterms. $\Omega^{(1)}$ describes the effect of an arbitrary number of external electromagnetic field legs on a single fermion loop. In fact, the (fermionic) vacuum persistence amplitude at the one-loop level in the presence of an external field is given by $\langle 0_+ | 0_- \rangle^A = e^{i\Omega^{(1)}(A)}$. s is a ("proper-time") parameter, m is the renormalized mass, and $\mathcal{H}(A)$ is the gauge invariant proper-time Hamiltonian operator, to be specified later. The trace is over the coordinate

and spinorial spaces.

The explicitly gauge invariant representation given by (1) is ill-defined due to ultraviolet divergences at $s = 0$. We employ a modification of formula (1) which adds arbitrary powers of s in the numerator, and so regulate the integral. The modified formula reads¹ [8]

$$\Omega^{(1)}[A, \nu] = \frac{i}{2} \int_0^\infty \frac{ds}{s} s^\nu e^{-is(m^2 - ic)} \text{Tr} e^{-is\mathcal{H}}. \quad (2)$$

The integral is now defined over a restricted region of the complex s -plane. Before taking $\nu \rightarrow 0$, an analytic continuation to the whole complex plane is performed, and subtractions of non-physical terms (if any) are made.

2. We consider a constant magnetic field in the z direction, $F_{12} = -F_{21} = H$. Thus, \mathcal{H} reads

$$\mathcal{H} = -(\gamma \cdot \Pi)^2 = \Pi^2 - eH\sigma_3, \quad (3)$$

where $\Pi_\mu = \frac{1}{i}\partial_\mu - eA_\mu$ and $\sigma_3 := \frac{1}{2}[\gamma_1, \gamma_2]$. The spinorial part of the trace is easily computed by observing that σ_3 has two double-degenerate eigenvalues, 1 and -1. Thus, passing to Euclidean space ($x_0 = it$), we are left with the trace of the operators $\Delta_\pm := m^2 - \partial_0^2 - \partial_3^2 - (\frac{1}{i}\vec{\partial} - e\vec{A})_\perp^2 \pm eH$. Assuming that the system is enclosed in a large normalization box, in momentum space we need the spectrum of the operator $(\vec{k} - e\vec{A})_\perp^2$, which is known to be given by $2eH(n + \frac{1}{2})$ [12], with density of states $\frac{eH}{2\pi}$ (degeneracy per unit area of each Landau level). Also, at finite temperature the antiperiodic condition in the imaginary time for the fermionic degrees of freedom imply that the integral over k_0 is replaced by a discrete sum over odd multiples of $\frac{\pi}{\beta^2}$ ($\beta = 1/T$).

Then, we get from (2):

$$\Omega^{(1)}[H, T, \nu] = L^3 \frac{eH}{4\pi^2} \sum_{n=0}^{\infty} \sum_{l=-\infty}^{\infty} \int_0^\infty ds s^{\nu-1} \int_{-\infty}^{\infty} dk_3 \exp\left\{-\frac{is}{\mu^2} [k_3^2 + m^2 + \frac{\pi^2}{\beta^2} (2l+1)^2 + 2eH(n + \frac{1}{2}) \pm eH]\right\}. \quad (4)$$

We introduced the notation $f(x \pm eH) = f(x + eH) + f(x - eH)$. μ is an arbitrary mass scale

¹Our notation has the following meaning: $\Omega^{(1)}$ is a functional of A and a function of ν .

introduced to keep $\Omega^{(1)}$ with the right dimension, and L^3 is a three-dimensional normalization volume.

Integration over k_3 and use of (A.2) leads to

$$\Omega^{(1)}[H, T, \nu] = L^3 \frac{eH}{4\pi^2} \sqrt{\pi} \Gamma(\nu - \frac{1}{2}) (i)^{-\nu} \mu^{2\nu} \sum_{n=0}^{\infty} \sum_{l=-\infty}^{\infty} \left[m^2 + \frac{\pi^2}{\beta^2} (2l+1)^2 + 2eH(n + \frac{1}{2}) \pm eH \right]^{\frac{1}{2}-\nu}. \quad (5)$$

Defining the function

$$\mathbb{G}_2^{c^2}(s; a_1, a_2) := \sum_{n=0}^{\infty} \sum_{l=-\infty}^{\infty} \left[c^2 + a_1 n + a_2 (2l+1)^2 \right]^{-s}, \quad (6)$$

and using the property

$$\mathbb{G}_2^{c^2+a_1}(s; a_1, a_2) := \mathbb{G}_2^{c^2}(s; a_1, a_2) - \mathbb{Z}_1^{c^2}(s; a_2), \quad (7)$$

where the function $\mathbb{Z}_1^{c^2}(s; a_2)$ is defined in the appendix, eq. (A.10), we can write

$$\Omega^{(1)}[H, T, \nu] = L^3 \frac{eH}{4\pi^2} \sqrt{\pi} \Gamma(\nu - \frac{1}{2}) (i)^{-\nu} \mu^{2\nu} \left[2\mathbb{G}_2^{m^2}(\nu - \frac{1}{2}; 2eH, \frac{\pi^2}{\beta^2}) - \left(\frac{\pi^2}{\beta^2}\right) \mathbb{Z}_1^{\frac{m^2\beta^2}{\pi^2}}(\nu - \frac{1}{2}; 1) \right] \quad (8)$$

Using equations (A.9) and (A.11), we obtain

$$\begin{aligned} \Omega^{(1)}[H, T, \nu] &= (L^3 \beta) \frac{eH}{4\pi^2} (i)^{-\nu} \mu^{2\nu} \\ &\times \left\{ \left[(2eH)^{1-\nu} \zeta_{RG}(\nu-1; \frac{m^2}{2eH}) - \frac{1}{2} m^{2-2\nu} \right] \Gamma(\nu-1) - \frac{2m}{\beta} \left(\frac{\beta}{m}\right)^\nu \sum_{l=1}^{\infty} (-1)^l \left(\frac{l}{2}\right)^{\nu-1} K_{\nu-1}(m\beta l) \right. \\ &\left. + 4 \sum_{n=0}^{\infty} \sum_{l=1}^{\infty} (-1)^l \left[\frac{\beta^2 l^2}{4(m^2 + 2eHn)} \right]^{\frac{\nu-1}{2}} K_{\nu-1}(\beta l \sqrt{m^2 + 2eHn}) \right\} \quad (9) \end{aligned}$$

The polar part of $\Omega^{(1)}[H, T, \nu]$ at $\nu = 0$ comes from the factor $\Gamma(\nu - 1)$ (this contribution comes from the $l = 0$ terms in the summations, the temperature-independent part). We multiply the polar part by ν , use that

$$\left. \frac{\partial}{\partial \nu} \left(\nu F(\nu) \right) \right|_{\nu=0} = F(0), \quad (10)$$

and the analytic continuation of the Gamma function, to obtain

$$\begin{aligned}
V^{(1)}(H, T) = & \frac{(eH)^2}{4\pi^2} \left\{ \left(\frac{m^2}{2eH} \right)^2 + \frac{1}{6} + \left(\left(\frac{m^2}{2eH} \right)^2 - \frac{m^2}{2eH} + \frac{1}{6} \right) \ln \left(\frac{m^2}{2eH} \right) \right. \\
& - 2\zeta'_{GR} \left(-1, \frac{m^2}{2eH} \right) \left. \right\} - \frac{eH}{\beta\pi^2} \left\{ -m \sum_{l=1}^{\infty} \frac{(-1)^l}{l} K_1(m\beta l) \right. \\
& \left. + 2 \sum_{n=0}^{\infty} \sum_{l=1}^{\infty} \frac{(-1)^l}{l} \sqrt{m^2 + 2eHn} K_1(\beta l \sqrt{m^2 + 2eHn}) \right\}, \quad (11)
\end{aligned}$$

where $\zeta'_{GR}(-1, a) = \frac{\partial}{\partial s} \zeta_{RG}(s-1, a)|_{s=0}$. In equation (11), we have fixed the arbitrary scale using the renormalization condition $\mu^2 = m^2$, and used that $\zeta_{GR}(-1, a) = -\frac{1}{2}(a^2 - a + \frac{1}{6})$.

3. The temperature-independent part of $V^{(1)}$ is the usual one-loop effective potential of QED₄ in a constant background magnetic field [7, 13] (the second term in brackets vanish when $T = 0$). The result (11) was derived for *virtual pairs*. Nevertheless, at finite temperature, the effective potential is identical to the Gibbs Free Energy density [14]. In [15] we have interpreted the Schwinger's formula as a sum over zero-point energies. Thus, eq. (11) is the contribution of the vacuum energy to the total Free Energy density of a gas of *real* electron-positron pairs in thermal equilibrium in a constant external magnetic field [6].

4. We emphasize that eq. (11) contains all the finite temperature and magnetic field one-loop corrections to the total Free Energy in an analytic exact form. In [6], the answer was given as a series of terms in a high-temperature expansion. The connection between our result and that obtained in [6] is achieved by summing over n in (11). The relevant part is

$$\begin{aligned}
\bar{V}^{(1)}(H, T) = & \frac{eH}{\beta\pi^2} \left\{ -m \sum_{l=1}^{\infty} \frac{(-1)^l}{l} K_1(m\beta l) \right. \\
& \left. + 2 \sum_{n=0}^{\infty} \sum_{l=1}^{\infty} \frac{(-1)^l}{l} \sqrt{m^2 + 2eHn} K_1(\beta l \sqrt{m^2 + 2eHn}) \right\}. \quad (12)
\end{aligned}$$

Using (A.7), we can write

$$\bar{V}^{(1)}(H, T) = -\frac{eH}{2\pi^2} \sum_{l=1}^{\infty} (-1)^{l+1} \int_0^{\infty} \frac{dx}{x^2} \left[\sum_{n=0}^{\infty} e^{-2eHxn} - \frac{1}{2} \right] e^{-\frac{e^2 l^2}{4x} - m^2 x}. \quad (13)$$

The sum is easily made with the aid of the formula for a sum of a geometric progression of ratio e^{-2eHx} . Thus

$$\bar{V}^{(1)}(H, T) = -\left(\frac{eH}{2\pi} \right)^2 \sum_{l=1}^{\infty} (-1)^{l+1} \int_0^{\infty} \frac{dz}{z^3} e^{-\alpha z - \frac{\gamma}{z}} \coth z, \quad (14)$$

where we have made $z = eHx$, and defined $\alpha := \frac{m^2}{eH}$ and $\gamma := \frac{\beta^2 l^2}{4} eH$. Equation (14) is just the result quoted in the second paper of [6] (after the substitution $\mathcal{L}^{(1)} = -V^{(1)}$). From (14), one could obtain the high-temperature expansion.

5. As is well known [7], an external electric field can lead to an instability of the fermionic vacuum, the pair creation probability being given by the imaginary part of $\Omega^{(1)}$. To find the thermal contribution to the pair creation probability is an interesting problem which we will leave for a further paper.

APPENDIX

Analytic regularization methods deals with the analytic continuation of certain special functions. In this appendix, we explore properties of the Jacobi's Theta function [16] to obtain the analytic continuations of the functions appearing in the text. We believe that these representations are not quoted before in the literature (see, however, [17, 18]).

Let the function

$$G_2^{c^2}(s; a_1, a_2) := \sum_{n=0}^{\infty} \sum_{l=-\infty}^{\infty} [c^2 + a_1 n + a_2(2l+1)^2]^{-s}. \quad (A.1)$$

defined for $\mathcal{R}(s)$ big enough, where $c, a_1 \geq 0$ and $a_2 \geq 0$ are numbers.

Using the following integral representation of the Gamma function

$$\Gamma(z) \alpha^{-z} = \int_0^{\infty} dx x^{z-1} e^{-\alpha x}, \quad \mathcal{R}(z, \alpha) > 0 \quad (A.2)$$

we could write (A.1) as

$$G_2^{c^2}(s; a_1, a_2) := \frac{1}{\Gamma(s)} \int_0^{\infty} dx x^{s-1} e^{-xc^2} \sum_{n=0}^{\infty} e^{-(a_1 n + a_2)x} \Theta \left(-\frac{2a_2 x}{\pi}; \frac{4a_2 x}{\pi} \right), \quad (A.3)$$

where [19]

$$\Theta(z; x) := \sum_{l=-\infty}^{\infty} e^{-\pi l^2 x + 2\pi i l z} \quad (\text{A.4})$$

is the Jacobi's Theta function. Applying the Jacobi transformation

$$\Theta(z; x) := \frac{1}{\sqrt{x}} e^{-\frac{\pi z^2}{x}} \Theta\left(\frac{z}{ix}; \frac{1}{x}\right), \quad (\text{A.5})$$

we obtain from (A.3)

$$\begin{aligned} \mathcal{G}_2^{c^2}(s; a_1, a_2) &:= \frac{1}{\Gamma(s)} \sqrt{\frac{\pi}{4a_2}} \sum_{n=0}^{\infty} \sum_{l=-\infty}^{\infty} (-1)^l \int_0^{\infty} dx x^{s-3/2} e^{-x(c^2+a_1n)} e^{-\frac{\pi^2 l^2}{4a_2 x}} \\ &= R(s) \sum_{n=0}^{\infty} \left\{ \int_0^{\infty} dx x^{s-3/2} e^{-x(c^2+a_1n)} + 2 \sum_{l=1}^{\infty} (-1)^l \int_0^{\infty} dx x^{s-3/2} e^{-x(c^2+a_1n) - \frac{\pi^2 l^2}{4a_2 x}} \right\} \end{aligned} \quad (\text{A.6})$$

where we have separated the $l = 0$ contribution from the sum, and defined $R(s) := \frac{1}{\Gamma(s)} \sqrt{\frac{\pi}{4a_2}}$ for convenience.

Using the integral representation of the modified Bessel function of the second kind [20],

$$\int_0^{\infty} dx x^{\alpha-1} e^{-\beta/x - \gamma x} = 2 \left(\frac{\beta}{\gamma}\right)^{\frac{\alpha}{2}} K_{\alpha}(2\sqrt{\beta\gamma}), \quad \mathcal{R}(\beta, \gamma) > 0 \quad (\text{A.7})$$

and the series representation of the generalized Riemann's zeta-function

$$\zeta_{GR}(s, a) = \sum_{n=0}^{\infty} (n+a)^{-s}, \quad \mathcal{R}(s) > 1, a > 0 \quad (\text{A.8})$$

we obtain the desired result

$$\begin{aligned} \mathcal{G}_2^{c^2}(s; a_1, a_2) &:= \frac{1}{2} \sqrt{\frac{\pi}{a_2}} \left\{ \frac{\Gamma(s-1/2)}{\Gamma(s)} a_1^{1/2-s} \zeta_{GR}\left(s-1/2, \frac{c^2}{a_1}\right) \right. \\ &\quad \left. + \frac{4}{\Gamma(s)} \sum_{n=0}^{\infty} \sum_{l=1}^{\infty} (-1)^l \left[\frac{\pi^2 l^2}{4a_2(c^2+a_1n)} \right]^{s/2-1/4} K_{s-1/2}\left(\pi l \sqrt{\frac{c^2+a_1n}{a_2}}\right) \right\}. \end{aligned} \quad (\text{A.9})$$

From the above representation, we could make the analytic continuation of $\mathcal{G}_2^{c^2}$ to the whole complex s -plane, since the functions $\Gamma(z)$, $K_{\alpha}(z)$ and $\zeta_{GR}(s, a)$ are well known. Thus, ζ_{GR} possesses only a simple pole at $s = 1$, with residue $+1$, and its analytic continuation is

given by the Hermite's formula. On the other hand, $\Gamma(z)$ possesses simple poles located at $z = 0, -1, -2, \dots$, and its analytic continuation can be easily done with the aid of the simple relation $z\Gamma(z) = \Gamma(z+1)$. Then, $\mathcal{G}_2^{c^2}$ is a meromorphic function; in particular, it is analytic at the origin.

The analytic continuation of the function defined by

$$\mathcal{Z}_1^{c^2}(s, a) := \sum_{l=-\infty}^{\infty} [c^2 + a(2l+1)^2]^{-s}, \quad (\text{A.10})$$

goes along the same lines. The result is

$$\mathcal{Z}_1^{c^2}(s, a) := \frac{1}{2} \sqrt{\frac{\pi}{a}} \left\{ \frac{\Gamma(s-1/2)}{\Gamma(s)} c^{1-2s} + \frac{4}{\Gamma(s)} \sum_{l=1}^{\infty} (-1)^l \left(\frac{\pi^2 l^2}{4ac^2}\right)^{s/2-1/4} K_{s-1/2}\left(\frac{\pi l c}{a}\right) \right\}. \quad (\text{A.11})$$

The pole structure of $\mathcal{Z}_1^{c^2}$ is dictated by $\Gamma(s-1/2)$. $\mathcal{Z}_1^{c^2}$ is a meromorphic function analytic at the origin; in particular, $\mathcal{Z}_1^{c^2}(0, a) = 0$.

The author is grateful to Carlos Farina, Jossif Frenkel, Victor Rivelles and Nami Svaiter for many stimulating discussions, to Nelson Alves and M. V. Cougo-Pinto for reading the manuscript, and also to the Departamento de Física Matemática for their kind hospitality. This work was supported by CNPq (Brazilian Council of Research).

References

- [1] D. A. Kirzhnits and A. Linde, Phys. Lett. **B42** (1972) 471.
- [2] For an early review and original references, see D. J. Gross, R. D. Pisarski, and L. G. Yaffe, Rev. Mod. Phys. **53** (1980) 43.
- [3] A. Salam and J. Strathdee, Nucl. Phys. **B90** (1975) 203; G. Ghika and M. Visinescu, II Nuovo Cimento **A46** (1978) 25.
- [4] M. Dine et al, Phys. Rev. **D46** (1992) 550.
- [5] J. Chakrabarti, Phys. Rev. **D24** (1981) 2232; ib. **D28** 2657; M. Reuter and W. Dittrich, Phys. Lett. **B144** (1984) 99.

- [6] W. Dittrich, Phys. Rev. **D19** (1979) 2385; ib. Phys. Lett. **B83** (1979) 67.
- [7] J. Schwinger, Phys. Rev. **82** (1951) 664.
- [8] M. V. Cougo-Pinto, C. Farina, and A. J. Seguí-Santonja; Lett. Math. Phys. **30** (1994) 169.
- [9] M. V. Cougo-Pinto, C. Farina, and A. J. Seguí-Santonja; Lett. Math. Phys. **30** (1994) 169.
- [10] Currently under investigation.
- [11] A. P. de Almeida, J. Frenkel, and J. C. Taylor, Phys. Rev. **D45** (1992) 2081.
- [12] L.D. Landau and E.M. Lifshitz, *Quantum Mechanics*, Pergamon Press, 1980.
- [13] W. Dittrich and M. Reuter, *Effective Lagrangians in Quantum Electrodynamics*, chapter 6; Springer-Verlag, 1985.
- [14] J.J. Binney, N.J. Dowrick, A.J. Fisher, and M.E.J. Newman, *The Theory of Critical Phenomena*, Oxford University Press, 1993.
- [15] Luiz C. de Albuquerque, C. Farina, S. J. Rabello and A. Vaidya, Lett. Math.Phys. **34** (1995) 373.
- [16] J. Ambjorn and S. Wolfram, Ann. Phys. **147** (1983) 1.
- [17] E. Elizalde, J. Phys. **A22** (1989) 931; K. Kirsten, J. Math. Phys. **35** (1994) 459.
- [18] H. Boschi-Filho and C. Farina, to appear in Phys. Lett. **A**.
- [19] E. T. Whittaker and E. N. Watson, *A Course of Modern Analysis*, Cambridge University Press, 1978.
- [20] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products*, Academic Press, 1965

