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**INSTITUTO DE FÍSICA
CAIXA POSTAL 66318
05389-970 SÃO PAULO - SP
BRASIL**

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**ALGEBRA OF NON-LOCAL CHARGES IN
SUPERSYMMETRIC NON-LINEAR SIGMA
MODELS**

L.E. Saltini and A. Zadra
Instituto de Física, Universidade de São Paulo

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Algebra of Non-Local Charges in Supersymmetric Non-Linear Sigma Models

L.E. Saltini* and A. Zadra†

Instituto de Física da Universidade de São Paulo
C.P. 66318, São Paulo, SP, 05389-970, Brazil

Abstract

We propose a graphic method to derive the classical algebra (Dirac brackets) of non-local conserved charges in the two-dimensional supersymmetric non-linear $O(N)$ sigma model. As in the purely bosonic theory we find a cubic Yangian algebra. We also consider the extension of graphic methods to other integrable theories.

1 Introduction

Non-linear sigma models [1-3] are prototypes of a remarkable class of integrable two dimensional models which contain an infinite number of conserved local and non-local charges [4-7]. The algebraic relations obeyed by such charges are supposed to be an important ingredient in the complete solution of those models [8-11]. The local charges form an Abelian algebra. Opposing to that simplicity, the algebra of non-local charges is non-Abelian and actually non-linear [12-23].

In ref.[29] the $O(N)$ sigma model was investigated and a particular set of non-local charges – called *improved* charges – was found to satisfy a cubic algebra related to a Yangian structure. In this work we intend to extend that result to the corresponding supersymmetric case [30-32]. The introduction of supersymmetry might have rendered a much more involved algebra [33]. However, it has been conjectured [29,32] that, in the sigma model, the algebra of supersymmetric non-local charges would remain the same as in the bosonic theory and we shall present results that confirm such conjecture.

This paper is organized as follows. In Sect.2 we briefly review the results from the purely bosonic theory. A graphic technique to compute charges and their algebra is introduced in Sect.3. In Sect.4 we discuss the supersymmetric model and the main results of the paper. Another application of graphic rules is shown in Sect. 5 concerning the $O(N)$ Gross-Neveu model. Sect.6 is left for conclusions while an appendix contains examples of the graphic technique.

*Work supported by FAPESP. E-mail: lsaltini@uspif.if.usp.br

†Work partially supported by CNPq. E-mail: azadra@uspif.if.usp.br

2 Bosonic model - a review

The two-dimensional non-linear $O(N)$ sigma model can be described by the constrained Lagrangean

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_i \partial^\mu \phi_i \quad , \quad \sum_{i=1}^N \phi_i^2 = 1 \quad . \quad (1)$$

Associated to the $O(N)$ symmetry we have a matrix-valued conserved curvature-free current

$$\begin{aligned} (j_\mu)_{ij} &= \phi_i \partial_\mu \phi_j - \phi_j \partial_\mu \phi_i \quad , \quad \partial_\mu j^\mu = 0 \quad , \\ f_{\mu\nu} &= \partial_\mu j_\nu - \partial_\nu j_\mu + 2[j_\mu, j_\nu] = 0 \quad , \end{aligned} \quad (2)$$

whose components satisfy the algebra [29]

$$\begin{aligned} \{(j_0)_{ij}(x), (j_0)_{kl}(y)\} &= (I \circ j_0)_{ij,kl}(x) \delta(x-y) \\ \{(j_1)_{ij}(x), (j_0)_{kl}(y)\} &= (I \circ j_1)_{ij,kl}(x) \delta(x-y) + (I \circ j)_{ij,kl}(x) \delta'(x-y) \\ \{(j_1)_{ij}(x), (j_1)_{kl}(y)\} &= 0 \quad . \end{aligned} \quad (3)$$

where I is the $N \times N$ identity matrix. Above we have introduced the intertwiner field

$$(j)_{ij} = \phi_i \phi_j \quad (4)$$

and the $O(N)$ o-product defined in ref. [29] as

$$(A \circ B)_{ij,kl} \equiv A_{ik} B_{jl} - A_{il} B_{jk} + A_{jl} B_{ik} - A_{jk} B_{il} \quad . \quad (5)$$

This model is known to have infinite non-local conserved charges. The standard set of charges can be iteratively built up by means of the potential method of Brézin *et. al.* [5]. However, in ref. [29] an alternative set of *improved* charges $\{Q^{(n)}, n = 0, 1, 2, \dots\}$ has been defined and it was shown that they obey the non-linear algebra

$$\{Q_{ij}^{(m)}, Q_{kl}^{(n)}\} = (I \circ Q^{(n+m)})_{ij,kl} - \sum_{p=0}^{m-1} \sum_{q=0}^{n-1} (Q^{(p)} Q^{(q)} \circ Q^{(m+n-p-q-2)})_{ij,kl} \quad . \quad (6)$$

These charges were named *improved* because they brought up an algebraic improvement: the non-linear part of the algebra is simply cubic, as opposed to the algebra of the standard charges previously used in the literature [14]. The Jacobi identity and other properties of the improved cubic algebra were thoroughly discussed in ref. [29]. But there is a way to abbreviate that algebra, which is the first among the new results of this paper and which will be presented now.

We shall define a Hermitean generator of improved charges

$$Q(\lambda) \equiv I + i \sum_{n=0}^{\infty} \lambda^{n+1} Q^{(n)} \quad , \quad (7)$$

where λ will be called the spectral parameter. Therefore one can summarize the algebra (6) as follows:

$$i\{Q_{ij}(\lambda), Q_{kl}(\mu)\} = (f(\lambda, \mu) \circ Q(\lambda) - Q(\mu))_{ij,kl} \quad , \quad (8)$$

where

$$f(\lambda, \mu) \equiv \Re e \left(\frac{Q(\lambda)Q(\mu)}{\lambda^{-1} - \mu^{-1}} \right) = \frac{I - \sum_{m,n=0}^{\infty} \lambda^{m+1} \mu^{n+1} Q^{(m)} Q^{(n)}}{\lambda^{-1} - \mu^{-1}} \quad (9)$$

The quadratic non-linearity encoded in $f(\lambda, \mu)$ can be related to the known Yangian structure that underlies this model [17-26,29]. The advantage in writing the algebra as in (8) is not only aesthetic. Recalling the monodromy matrix of standard charges, and its algebra expressed in terms of the classical r -matrix,

$$\begin{aligned} T(\lambda) &= \exp \sum_{n \geq 0} \lambda^{n+1} \hat{Q}^{(n)} \quad , \\ \{T(\lambda) \otimes T(\mu)\} &= [r(\lambda, \mu), T(\lambda) \otimes T(\mu)] \quad , \\ r(\lambda, \mu) &= \frac{I_a \otimes I_a}{\lambda^{-1} - \mu^{-1}} \quad , \quad [I_a, I_b] = f_{abc} I_c \quad , \end{aligned} \quad (10)$$

we remark that the generator $Q(\lambda)$ and the f -matrix play similar rôles to those of the monodromy matrix and classical r -matrix in the standard approach [17-26]. We do not fully understand the relationship between (8) and (10) but we expect to be able to use this analogy to establish a precise translation between the different sets of charges [35]. We also hope that a complete knowledge about the conserved charges and their algebra will become an decisive ingredient in off-shell scattering calculations.

Now let us consider the graphic methods announced in the Introduction. We recall that in ref. [29] the improved charges were constructed by means of an iterative algebraic algorithm that uses $Q^{(1)}$ as a step-generator, as indicated by the relation

$$(I \circ Q^{(n+1)}) = \text{linear part of } \{Q^{(1)}, Q^{(n)}\} \quad (11)$$

“After a tedious calculation” the authors in ref. [29] managed to construct the charges $Q^{(n)}$ and their algebra up to $n = 5$. In the next section we will present a *graphic* method that makes the calculation simpler, less tedious and convenient for a further supersymmetric extension.

3 Graphic rules for the bosonic model

Let us associate white and black semicircles to the $O(N)$ current components,

$$j_0 \iff \text{white semicircle} \quad j_1 \iff \text{black semicircle} \quad (12)$$

a continuous line and an oriented line to the identity and the anti-derivative operator respectively,

$$I \iff \text{continuous line} \quad 2\partial^{-1} \iff \text{oriented line} \quad (13)$$

The operator ∂^{-1} above follows the same convention adopted in ref. [29],

$$\partial^{-1} A(x) = \frac{1}{2} \int dy \epsilon(x-y) A(y) \quad , \quad \epsilon(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ +1, & x > 0 \end{cases} \quad (14)$$

Below one finds some diagrams and the corresponding expressions:

$$\begin{aligned} 2j_0 \partial j_0^{\dagger} &\iff \text{white semicircle} \leftarrow \text{white semicircle} \\ 2\partial j_1 j_0 &\iff \text{black semicircle} \leftarrow \text{white semicircle} \\ 4\partial j_0^{\dagger} \partial j_0^{\dagger} &\iff \text{white semicircle} \leftarrow \text{white semicircle} \\ 4j_1 \partial(j_0 \partial j_0^{\dagger}) &\iff \text{black semicircle} \leftarrow \text{white semicircle} \leftarrow \text{white semicircle} \end{aligned} \quad (15)$$

We have noticed [29] that every improved charge can be written as an integral over symmetrized chains of j_0 's and j_1 's connected by the operator $2\partial^{-1}$. Therefore we can associate a diagram to each improved charge, as exemplified by the second non-local charge $Q^{(2)}$:

$$\begin{aligned} Q^{(2)} &= \int dx \left[2j_0 + j_0(2\partial j_1^{\dagger}) + j_1(2\partial j_0^{\dagger}) + j_0(2\partial(j_0 \partial j_0^{\dagger})) \right] \\ &\iff 2\text{white semicircle} + \text{white semicircle} \leftarrow \text{black semicircle} + \text{black semicircle} \leftarrow \text{white semicircle} + \text{white semicircle} \leftarrow \text{white semicircle} \leftarrow \text{white semicircle} \end{aligned} \quad (16)$$

If one is interested in constructing charges, there is an iterative graphic procedure, inspired by the lessons taken from ref. [29] and which will be described now. Consider the following properties:

(a) The improved non-local charges have the general form $Q^{(n)} = \int dx J^{(n)}$ where $J^{(n)}$ is a combination of terms which one can always write as

$$j_{\mu} \quad \text{or} \quad 2(j_{\mu} \partial S + \partial S^{\dagger} j_{\mu}) \quad , \quad (17)$$

where S is some chain and S^{\dagger} its transposed.

(b) The algebraic definition of improved charges is

$$(I \circ Q^{(n+1)})_{ij,kl} = \text{linear part of } \{Q_{ij}^{(1)}, Q_{kl}^{(n)}\} \quad (18)$$

and we note that the linear part of $\{Q_{ij}^{(1)}, Q_{kl}^{(n)}\}$ comes exclusively from terms like

$$\int dx \left(\{Q_{ij}^{(1)}, (j_{\mu})_{ka}\} 2\partial S_{al} - (k \leftrightarrow l) \right) \quad (19)$$

(c) Using the definition of $Q^{(1)}$ and the elementary current algebra and dropping non-linear terms, we verify that

$$\int dx \{Q_{ij}^{(1)}, (j_0)_{ka}\} 2\delta_{al}^{(k+l)} = \left(I \circ \int dx [j_1 2\delta^1 + j_0 2\delta^0 (j_0 2\delta^1) + 2j_1 S] \right)_{ij,kl} \quad (20)$$

$$\leftarrow \boxed{S} \quad \Longrightarrow \quad \leftarrow \boxed{S} + \leftarrow \leftarrow \boxed{S} + 2 \otimes \rightarrow \boxed{S}$$

$$\int dx \{Q_{ij}^{(1)}, (j_1)_{ka}\} 2\delta_{al}^{(k+l)} = \left(I \circ \int dx [2j_0 j_1 2\delta^1 + j_0 2\delta^0 (j_1 2\delta^1)] \right)_{ij,kl} \quad (21)$$

$$\leftarrow \boxed{S} \quad \Longrightarrow \quad 2 \otimes \rightarrow \boxed{S} + \leftarrow \leftarrow \boxed{S}$$

where some new symbols were introduced,

$$j \iff \otimes \quad \frac{1}{2} \partial \iff \rightarrow \quad \rightarrow = - \iff \frac{1}{2} \partial 2\delta^1 = I \quad (22)$$

The previous expressions justify the following prescription:

- i) We start from the diagram of $Q^{(n)}$.
- ii) Then we replace the left "tip" of each chain according to the rules:

$$\begin{aligned} \leftarrow & \text{ is replaced by } \leftarrow + \leftarrow \leftarrow + 2 \otimes \rightarrow \\ \leftarrow & \text{ is replaced by } 2 \otimes \rightarrow + \leftarrow \leftarrow \end{aligned} \quad (23)$$

- iii) The resulting diagram corresponds to $Q^{(n+1)}$.

We remark that the substitution rules above can be directly read from the following basic brackets:

$$\{Q_{ij}^{(1)}, (j_0)_{kl}\} = (I \circ j_1 - 2\delta_{j_0}^1 j_0 - \partial j)_{ij,kl} + \dots \quad (24)$$

$$\{Q_{ij}^{(1)}, (j_1)_{kl}\} = (I \circ j_0 j_1 - 2\delta_{j_0}^1 j_1)_{ij,kl} + \dots \quad (25)$$

In addition, one should not forget the constraints satisfied by the $O(N)$ current j_μ , given below,

$$[j_\mu, j]_+ = j_\mu \quad \otimes + \otimes = \leftarrow \quad (26)$$

$$\int 2j_1 j S = \int (j_1 S + j \partial S) \quad 2 \otimes = \leftarrow + 2 \otimes \rightarrow \quad (27)$$

The half-white/half-black semicircle means j_0 or j_1 generically. We have tested the efficiency of this method: comparing to the explicit algebraic algorithm in ref. [29] we have taken

much less time and space to construct the improved charges. For the sake of clarity we have gathered a few examples in appendix A.

We have also developed a diagrammatic technique to calculate the algebra itself. It can be seen as a set of contraction rules between the chains that constitute the charges. Indeed, in computing the algebra of non-local charges we have to consider all possible "contractions" (i.e. Dirac brackets) between symmetrized chains. After some partial integrations we end up with elementary contractions of the following general kind:

$$\int \delta_{ia}^1(x) \delta T_{bj}(x) \{ (j_\mu)_{ab}(x), (j_\nu)_{cd}(y) \} \delta U_{kc}(y) \delta V_{dl}(y) - (i+j) - (k+l) \quad (28)$$

The current algebra (3) tells us that a contraction $\{j_\mu(x), j_\nu(y)\}$ may produce a current-like term $(I \circ j_\alpha) \delta(x-y)$ or a Schwinger term. Let us discuss the first kind, in which case (28) produces 4 terms:

$$\begin{aligned} & \int dx [(\delta_{ia}^1 \delta U' \circ \delta T' j_\alpha \delta V) + (\delta T' \delta U' \circ \delta_{ia}^1 j_\alpha \delta V) + \\ & + (\delta_{ia}^1 j_\alpha \delta U' \circ \delta T' \delta V) + (\delta T' j_\alpha \delta U' \circ \delta_{ia}^1 \delta V)]_{ij,kl} \end{aligned} \quad (29)$$

We can associate each of the 4 terms above to one of the 4 possible contractions between the 2 pairs of symmetrized chains. In the presence of a Schwinger term we must take into account extra contributions involving the intertwiner and partial integrations. In any event, the contractions between chains can be resumed by the following rules:

Step 1: Choice

In calculating $\{Q^{(m)}, Q^{(n)}\}$ we take one chain from $Q^{(m)}$ and other from $Q^{(n)}$. Then we pick up the "internal" current components we intend to contract. This is symbolized by the generic diagram below:

$$\boxed{S} \leftarrow \leftarrow \boxed{T} \quad \boxed{U} \leftarrow \leftarrow \boxed{V} \quad (30)$$

Step 2: Isolation

In each chain we must "localize" the current components chosen in step 1. This was explicitly made in (28) by means of partial integrations. Within the diagrams this is achieved by inverting some arrows until all of them are pointing towards the chosen semicircle (i.e. the current component we are isolating). Eventually a minus sign will be picked up, depending on the number of inversions. Finally we have this sort of diagrams:

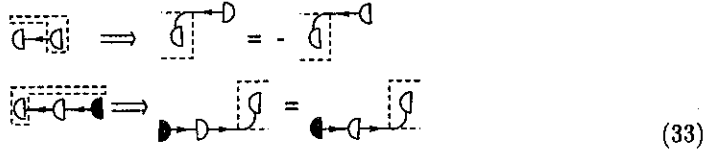
$$\boxed{S} \leftarrow \leftarrow \boxed{T} \quad \boxed{U} \leftarrow \leftarrow \boxed{V} \quad (31)$$

Step 3: Bending

The next step is just a graphic bending of chains, as a preparation to the final contraction. The chains from (31) should be bended in the following way:



Notice that the sub-chains T and U were transposed as eq. (31) demands. Actually the graphic bending implies the transposition, as exemplified below

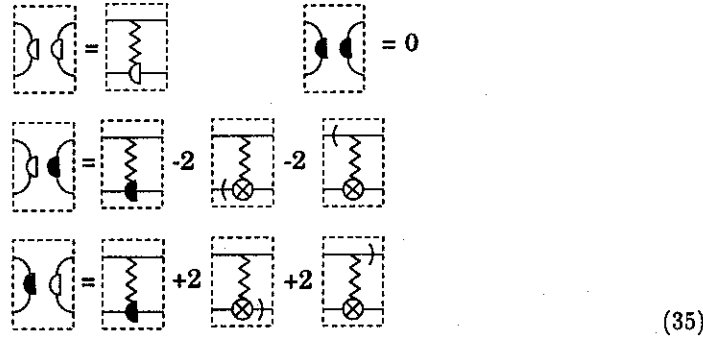


where the transposed current components are naturally represented as

$$\begin{aligned} j_0^i &= D = -\bar{D} = -j_0 \\ j_1^i &= \bar{D} = -D = -j_1 \end{aligned} \quad (34)$$

Step 4: Contraction

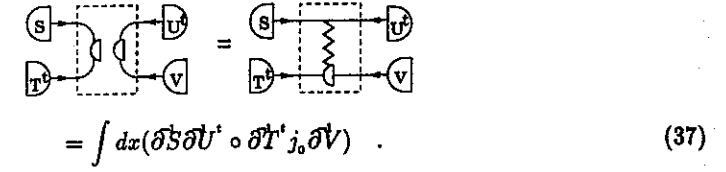
Finally we perform the contraction in (32) according to the rules below:



where we introduced a symbol corresponding to the o-product,

$$\text{Spring} = \int (A \circ B) = \int (B \circ A) \quad (36)$$

For instance, a typical contraction between j_0 components would be



Of course one must repeat all steps for every possible contraction.

The current j_μ obeys another constraint [27] involving the o-product, namely

$$(j_\mu \circ j) = 0 = \text{Spring} \quad (38)$$

which must also be taken into account.

We mention that the elementary contractions in (35) are nothing but the graphic representation of the current algebra (3), where the diagrams containing the intertwiner field come from Schwinger terms followed by partial integrations. These rules were applied to compute various brackets and in all cases the algebra (6) was confirmed. One can also find an example in appendix A.

The most remarkable outcome of this graphic procedure is that it poses an easy and straightforward way to the supersymmetric extension – and possibly other generalizations.

4 Supersymmetric model

The supersymmetric non-linear $O(N)$ sigma model is defined [30-32] by the Lagrangean

$$\mathcal{L}_s = \frac{1}{2} \partial_\mu \phi_i \partial^\mu \phi_i + \frac{i}{2} \bar{\psi}_i \not{\partial} \psi_i + \frac{1}{8} (\bar{\psi}_i \psi_i)^2 \quad (39)$$

where ϕ_i are scalars and ψ_i are Majorana fermions satisfying the constraints

$$\sum_{i=1}^N \phi_i^2 - 1 = 0 \quad , \quad \sum_{i=1}^N \phi_i \psi_i = 0 \quad (40)$$

We also have a conserved $O(N)$ current J_μ which can be split into bosonic and fermionic parts

$$\begin{aligned} J_\mu &= j_\mu + b_\mu \quad , \quad \partial_\mu J^\mu = 0 \quad , \\ (j_\mu)_{ij} &= \phi_i \partial_\mu \phi_j - \phi_j \partial_\mu \phi_i \quad , \\ (b_\mu)_{ij} &= -i \bar{\psi}_i \gamma_\mu \psi_j \quad . \end{aligned} \quad (41)$$

whose curvature obeys the equation

$$F_{\mu\nu} = \partial_\mu J_\nu - \partial_\nu J_\mu + 2[J_\mu, J_\nu] = -(\partial_\mu b_\nu - \partial_\nu b_\mu) \quad (42)$$

Even though its curvature is not null, one can construct non-local conserved charges out of J_μ [30,31]. Here we shall deal with an algebraic procedure to derive these charges. Therefore it is necessary to start from the elementary $O(N)$ current algebra, listed below,

$$\begin{aligned} \{(j_0)_{ij}(x), (j_0)_{kl}(y)\} &= [(I \circ j_0) - (j \circ b_0)]_{ij,kl}(x) \delta(x-y) , \\ \{(j_0)_{ij}(x), (j_1)_{kl}(y)\} &= (I \circ j_1)_{ij,kl}(x) \delta(x-y) + (I \circ j)_{ij,kl}(y) \delta'(x-y) , \\ \{(j_1)_{ij}(x), (j_1)_{kl}(y)\} &= 0 , \\ \\ \{(b_0)_{ij}(x), (b_0)_{kl}(y)\} &= [(I \circ b_0) - (j \circ b_0)]_{ij,kl}(x) \delta(x-y) , \\ \{(b_0)_{ij}(x), (b_1)_{kl}(y)\} &= [(I \circ b_1) - (j \circ b_1)]_{ij,kl}(x) \delta(x-y) , \\ \{(b_1)_{ij}(x), (b_1)_{kl}(y)\} &= [(I \circ b_0) - (j \circ b_0)]_{ij,kl}(x) \delta(x-y) , \end{aligned} \quad (43)$$

$$\begin{aligned} \{(j_0)_{ij}(x), (b_0)_{kl}(y)\} &= (j \circ b_0)_{ij,kl}(x) \delta(x-y) , \\ \{(j_0)_{ij}(x), (b_1)_{kl}(y)\} &= (j \circ b_1)_{ij,kl}(x) \delta(x-y) , \\ \{(j_1)_{ij}(x), (b_0)_{kl}(y)\} &= 0 , \\ \{(j_1)_{ij}(x), (b_1)_{kl}(y)\} &= 0 , \end{aligned}$$

where the intertwiner and the \circ -product were already defined in (4) and (5). The $O(N)$ local charge and the first non-local charge are given [30,31] by the integrals

$$\begin{aligned} Q^{(0)} &= \int dx (j_0 + b_0) \\ Q^{(1)} &= \int dx (j_1 + 2b_1 + 2(j_0 + b_0) \partial^1 (j_0 + b_0)) . \end{aligned} \quad (44)$$

Some other supersymmetric *standard* non-local charges can be found in the literature [30-32]. However, as in the bosonic case, we are searching for *improved* charges satisfying the simplest algebra. Using the algebraic method proposed in ref. [29] we have computed the improved charges and their brackets up to $n = 3$, finding the *same cubic algebra* given by (6). Calculation is hopelessly longer than in the bosonic theory, but we have been able to develop some graphic rules which rather simplified our work. This diagrammatic method is a direct extension of the one proposed for the bosonic theory. For instance, one can show that the supersymmetric step-generator $Q^{(1)}$ satisfies the following algebraic relations:

$$\{Q_{ij}^{(1)}, (j_0)_{kl}\} = (I \circ j_1 - 2\delta_{j_0}^1 j_0 - 2\delta_{b_0}^1 j_0 - \partial j) + \dots \quad (45)$$

$$\{Q_{ij}^{(1)}, (j_1)_{kl}\} = (I \circ 2j_0 j - 2\delta_{j_0}^1 j_1 - 2\delta_{b_0}^1 j_1) + \dots \quad (46)$$

$$\{Q_{ij}^{(1)}, (b_0)_{kl}\} = (I \circ 2b_1 - 2\delta_{j_0}^1 b_0) - 2\delta_{b_0}^1 b_0 + \dots \quad (47)$$

$$\{Q_{ij}^{(1)}, (b_1)_{kl}\} = (I \circ 2b_0 - 2\delta_{j_0}^1 b_1) - 2\delta_{b_0}^1 b_1 + \dots \quad (48)$$

As in the bosonic model (recall eqs.(23-25)) these relations lead us to the proper transformation rules for the construction of charges. One can use the following iterative procedure:

i) We propose the symbolic notation

$$\begin{aligned} j_0 &\iff \curvearrowright , & j_1 &\iff \curvearrowleft \\ b_0 &\iff \triangleleft , & b_1 &\iff \blacktriangleleft \\ 2\delta^1 &\iff \leftarrow , & \frac{1}{2}\partial &\iff \rightarrow \\ j &\iff \otimes \end{aligned} \quad (49)$$

ii) We take the diagram associated to $Q^{(n)}$ and replace the left "tip" of each chain as follows:

$$\begin{aligned} \curvearrowright &\text{ is replaced by } \curvearrowleft + \curvearrowright\curvearrowright + \curvearrowleft\curvearrowright + 2\otimes\rightarrow \\ \blacktriangleleft &\text{ is replaced by } 2\otimes\leftarrow + \curvearrowright\blacktriangleleft + \curvearrowleft\blacktriangleleft \\ \triangleleft &\text{ is replaced by } 2\blacktriangleleft + \curvearrowright\triangleleft + \curvearrowleft\triangleleft \\ \leftarrow &\text{ is replaced by } 2\leftarrow + \curvearrowright\leftarrow + \curvearrowleft\leftarrow \end{aligned} \quad (50)$$

These transformations are a direct translation of eqs. (45-48).

iii) After using the constraints on j_μ and b_μ you will have the diagram of $Q^{(n+1)}$.

In order to calculate the algebra between the non-local charges, one should follow the same algorithm (choice, isolation, bending and contraction), using the contraction rules below

5 Improved charges in the $O(N)$ Gross-Neveu model

This model consists of an N -plet of Majorana fermions transforming as a fundamental representation of the $O(N)$ group, with a quartic interaction. Its Lagrangean reads [7]

$$\mathcal{L}_{GN} = \frac{i}{2} \bar{\psi}_i \not{\partial} \psi_i + \frac{1}{8} (\bar{\psi}_i \psi_i)^2, \quad (55)$$

and it can be regarded as the limit of null bosonic field ($\phi_i \rightarrow 0$) in the supersymmetric model (39) – notice that the constraints (40) disappear. The Noether current associated to the $O(N)$ rotations is

$$(b_\mu)_{ij} = -i \bar{\psi}_i \gamma_\mu \psi_j, \quad \partial_\mu b^\mu = 0 \quad (56)$$

and it satisfies the curvature-free condition

$$\partial_\mu b_\nu - \partial_\nu b_\mu + \{b_\mu, b_\nu\} = 0 \quad (57)$$

and the algebraic relations

$$\begin{aligned} \{(b_0)_{ij}(x), (b_0)_{kl}(y)\} &= (I \circ b_0)_{ij,kl}(x) \delta(x-y) \\ \{(b_0)_{ij}(x), (b_1)_{kl}(y)\} &= (I \circ b_1)_{ij,kl}(x) \delta(x-y) \\ \{(b_1)_{ij}(x), (b_1)_{kl}(y)\} &= (I \circ b_0)_{ij,kl}(x) \delta(x-y) \end{aligned} \quad (58)$$

As before, we may construct an infinite number of conserved non-local currents using the potential algorithm: we consider a conserved current $B_\mu^{(n)}$ and the corresponding potential $\xi^{(n)}$,

$$B_\mu^{(n)} = \epsilon_{\mu\nu} \partial^\nu \xi^{(n)} \quad (59)$$

Then we define the current $B_\mu^{(n+1)}$ as

$$B_\mu^{(n+1)} = 2(\partial_\mu + b_\mu) \xi^{(n)} \quad (60)$$

The properties (56) and (57) imply that $B_\mu^{(n+1)}$ is also conserved. Starting out with $B_\mu^{(0)} = b_\mu$ we find an infinite number of conserved charges $Q^{(n)} = \int dx B_0^{(n)}$. After applying this algorithm to build up some of them, it is straightforward to check that this method is equivalent to the following graphic procedure: one chooses some symbols to represent b_0 and b_1 , for instance,

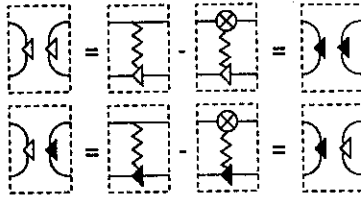
$$b_0 \iff \leftarrow \quad b_1 \iff \blacktriangleleft \quad (61)$$

then one takes the sequence of chains associated to $Q^{(n)}$ and uses the replacement rules for left-tips

$$\begin{aligned} \leftarrow & \text{ is replaced by } 2 \leftarrow + \leftarrow \leftarrow \\ \blacktriangleleft & \text{ is replaced by } 2 \blacktriangleleft + \blacktriangleleft \blacktriangleleft \end{aligned} \quad (62)$$

On the other hand, this is precisely the limit $\phi_i \rightarrow 0$ of the transformation rules (50) in the supersymmetric theory.

This provides an alternative derivation of the graphic rules to construct charges in the Gross-Neveu model. Moreover it implies that those charges defined by the algorithm (60) are actually the improved charges and thus they must obey the cubic algebra (6).



$$\begin{aligned} \leftarrow \blacktriangleleft &= \leftarrow \blacktriangleleft \ominus \leftarrow \blacktriangleleft \oplus \leftarrow \blacktriangleleft \\ \leftarrow \blacktriangleleft &= \leftarrow \blacktriangleleft \oplus \leftarrow \blacktriangleleft \ominus \leftarrow \blacktriangleleft \\ \leftarrow \blacktriangleleft &= \leftarrow \blacktriangleleft \oplus \leftarrow \blacktriangleleft \ominus \leftarrow \blacktriangleleft \\ \leftarrow \blacktriangleleft &= \leftarrow \blacktriangleleft \oplus \leftarrow \blacktriangleleft \ominus \leftarrow \blacktriangleleft \\ \leftarrow \blacktriangleleft &= \leftarrow \blacktriangleleft \oplus \leftarrow \blacktriangleleft \ominus \leftarrow \blacktriangleleft \end{aligned} \quad (51)$$

which is the graphic version of the algebra (43). We also have a new constraint,

$$b_\mu j = j b_\mu = 0 \iff \leftarrow \otimes = \otimes \blacktriangleleft \quad (52)$$

to be added to the list in (26,27,38). As before, the half-white/half-black triangle means b_0 or b_1 in general.

We have used this procedure to construct several charges and confirmed the algebra (6). This is actually the main result of this paper, confirming previous conjectures.

To complete the algebraic analysis, we have also considered the conserved supersymmetry current and charge, given by

$$\begin{aligned} \mathcal{J}_\mu &= \not{\partial} \phi_i \gamma_\mu \psi_i, \\ \mathcal{Q} &= \int dx \mathcal{J}_0. \end{aligned} \quad (53)$$

Using the equations of motion, we have checked that

$$\{\mathcal{Q}, Q^{(n)}\} = 0, \quad n \geq 0 \quad (54)$$

which means that every non-local charge is invariant under supersymmetry on-shell – as already pointed in ref. [32]. Therefore the non-local charges in the supersymmetric sigma model are all bosonic. However we must stress that this is not a general property: for instance, in ref. [34] one finds an integrable supersymmetric theory – the supersymmetric two boson hierarchy – containing fermionic non-local charges whose graded algebra exhibits cubic terms similar to those of eq. (6). It would be very interesting to develop graphic rules for this kind of model [35].

6 Conclusions and final remarks

Diagrammatic methods are frequently used in physics to simplify long calculations and so we have proposed a graphic procedure to construct and compute the algebra of non-local charges in non-linear sigma models. Applying such procedure we have been able to verify that the (improved) non-local charges in the supersymmetric $O(N)$ sigma model obey a cubic Yangian-like algebra which can be expressed as in eq. (8).

One could easily recover the bosonic model from the supersymmetric theory by taking the no-fermion limit $\psi_i \rightarrow 0$. Moreover, the $O(N)$ invariant Gross-Neveu model can be obtained after erasing the bosonic fields ($\phi_i \rightarrow 0$). The improved charges in these models may be different but their algebra is exactly the same. In the Gross-Neveu model, the improved charges could be computed by means of two different methods and therefore the corresponding graphic rules could be confirmed and further understood. We expect to find a similar confirmation in the sigma models but the presence of the intertwiner field within the diagrams has impeded us so far.

It is also interesting to consider the inclusion of Wess-Zumino terms, which modify the current algebra (see for instance ref. [28,29]) and derive the corresponding graphic rules. This problem and the general application of diagrammatic methods to integrable theories is presently under investigation [35].

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A Diagrammatic rules: examples

A.1 Improved non-local charges $Q^{(n)}$ in the non-linear sigma model

Let us apply the graphic method to construct some of the charges in the bosonic model:

i) $n = 0$

According to our conventions the $O(N)$ local charge is represented as

$$Q^{(0)} = \int dx j_0 \iff \text{Ⓛ} \quad (63)$$

ii) $n = 1$

Using the transformation rules (23) we obtain

$$\text{Ⓛ} \Rightarrow \text{Ⓜ} + \text{Ⓛ} \text{---} \text{Ⓛ} + 2 \text{Ⓢ} \text{---} \text{Ⓛ} \quad (64)$$

The last term is zero, since it should be read as $\int dx j \delta I = 0$. Therefore we have the following diagram for $Q^{(1)}$:

$$\text{Ⓜ} + \text{Ⓛ} \text{---} \text{Ⓛ} \quad (65)$$

which means that the first non-local charge is written as

$$Q^{(1)} = \int dx (j_1 + 2j_0 \delta^1_{j_0}) \quad (66)$$

iii) $n = 2$

Now we take the first chain from $Q^{(1)}$ and apply the replacement rule (23) again,

$$\text{Ⓜ} \Rightarrow 2 \text{Ⓢ} \text{---} \text{Ⓢ} + \text{Ⓛ} \text{---} \text{Ⓜ} \quad (67)$$

Next we transform the second chain, replacing its left tip according to (23)

$$\text{Ⓛ} \text{---} \text{Ⓛ} \Rightarrow \text{Ⓜ} \text{---} \text{Ⓛ} + \text{Ⓛ} \text{---} \text{Ⓛ} \text{---} \text{Ⓛ} + 2 \text{Ⓢ} \text{---} \text{Ⓢ} \text{---} \text{Ⓛ} \quad (68)$$

Recalling the property (22),

$$2 \text{Ⓢ} \text{---} \text{Ⓢ} \text{---} \text{Ⓛ} = 2 \text{Ⓢ} \text{---} \text{Ⓢ} \quad (69)$$

adding all contributions and remembering the constraint

$$2(\text{Ⓢ} \text{---} \text{Ⓢ} + \text{Ⓢ} \text{---} \text{Ⓢ}) = 2 \text{Ⓛ} \quad (70)$$

the resulting diagram is

$$2 \text{Ⓛ} + \text{Ⓛ} \text{---} \text{Ⓜ} + \text{Ⓜ} \text{---} \text{Ⓛ} + \text{Ⓛ} \text{---} \text{Ⓛ} \text{---} \text{Ⓛ} \quad (71)$$

and therefore the second non-local charges reads

$$Q^{(2)} = \int dx [2j_0 + 2j_0 \delta_{j_1}^1 + 2j_1 \delta_{j_0}^1 + 4j_0 \delta(j_0 \delta_{j_0}^1)] \quad (72)$$

Notice that the proper use of the constraints has rendered a charge free of intertwiners. This property has been checked up to $n = 7$ and we expect it to hold for every n .

A.2 Graphic derivation of $\{Q^{(1)}, Q^{(1)}\}$ in the non-linear sigma model

In this case we must consider all possible contractions between the sequence of chains from $Q^{(1)}$,

$$\bullet + \text{---} \bullet \quad (73)$$

which altogether sums 9 contractions. Here they are:

i)

$$\boxed{\bullet \bullet} = 0 \quad (74)$$

ii)

$$\boxed{\text{---} \bullet} = \boxed{\bullet \text{---}} = \text{---} \bullet + 2 \text{---} \bullet + 2 \text{---} \bullet \quad (75)$$

The first term corresponds to

$$\text{---} \bullet = \int (I \circ 2j_1 \delta_{j_0}^1) \quad (76)$$

and the second one reads

$$2 \text{---} \bullet = 2 \text{---} \bullet = \int (I \circ 2j_0) \quad (77)$$

The last term vanishes because it contains a derivative of the identity matrix,

$$2 \text{---} \bullet = 2 \int (\partial I \circ j \delta_{j_0}^1) = 0 \quad (78)$$

iii)

$$\boxed{\text{---} \text{---}} = - \boxed{\text{---} \text{---}} = - \text{---} \text{---} = \text{---} \text{---} = \text{---} \text{---} + 2 \text{---} \text{---} + 2 \text{---} \text{---} \quad (79)$$

Notice that we have computed two minus signs: one from the inversion of an arrow (along the isolation step) and other from the transposition of j_0 (during the bending step). The first contribution is

$$\text{---} \bullet = \int (2\delta_{j_0}^1 \circ j_1) \quad (80)$$

The second one vanishes due to the constraint (38)

$$2 \text{---} \bullet = 2 \text{---} \bullet = 0 \quad (81)$$

and the third term is null too,

$$2 \text{---} \bullet = \int dx (2\delta_{j_0}^1 \circ j \partial I) = 0 \quad (82)$$

Let us proceed with the remaining contractions:

iv)

$$\boxed{\text{---} \bullet} = - \boxed{\text{---} \bullet} = - \text{---} \bullet = - \text{---} \bullet + 2 \text{---} \bullet = - \text{---} \bullet \quad (83)$$

v)

$$\boxed{\text{---} \text{---}} = \text{---} \text{---} = - \text{---} \text{---} = - \text{---} \text{---} + 2 \text{---} \text{---} = \text{---} \text{---} + 2 \text{---} \text{---} \quad (84)$$

vi)

$$\boxed{\text{---} \text{---}} = - \text{---} \text{---} = - \text{---} \text{---} \quad (85)$$

vii)

$$\boxed{\text{---} \text{---}} = - \text{---} \text{---} = - \text{---} \text{---} \quad (86)$$

viii)

$$\boxed{\text{---} \text{---}} = - \text{---} \text{---} = - \text{---} \text{---} = \text{---} \text{---} \quad (87)$$

ix)

$$\boxed{\text{---} \text{---}} = - \text{---} \text{---} = - \text{---} \text{---} \quad (88)$$

Adding all the non-vanishing contributions we end up with the following terms:

$$2 \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad (89)$$

The linear part of the algebra is clearly recognized

$$\begin{aligned} & 2 \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \\ &= \int dx (I \circ 2j_0 + 2j_0 \delta j_1 + 2j_1 \delta j_0 + 4j_0 \delta(j_0 \delta j_0)) \\ &= (I \circ Q^{(2)}) \end{aligned} \quad (90)$$

The remaining three diagrams provide the surface term that corresponds to the cubic part of the algebra:

$$\begin{aligned} & - \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = - \rightarrow \left[\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \right] \\ &= - \int dx \frac{1}{2} \partial^8 (\delta j_0 \delta j_0 \circ \delta j_0) = -(Q^{(0)} Q^{(0)} \circ Q^{(0)}) \end{aligned} \quad (91)$$

Finally we obtain the expected answer,

$$\{Q^{(1)}, Q^{(1)}\} = (I \circ Q^{(2)}) - (Q^{(0)} Q^{(0)} \circ Q^{(0)}) \quad (92)$$

This commented example may seem rather long, not revealing the actual power of the graphic method. But we can assure the reader that, after practicing the contraction rules for a while – and skipping the intermediate comments – we have been able to compute many Dirac brackets very efficiently.

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