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Particles in Degenerate Metric Spaces

Luís A. Cabral* and Victor O. Rivelles**

Instituto de Física, Universidade de São Paulo,

Caixa Postal 66318, 05389-970 São Paulo, SP, Brazil

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Abstract

We study the evolution of particles in a space which has a degenerate metric. When the particle propagates into a region where the metric is degenerate that region is transparent for low energy particles and opaque to particles with high energy. We also present the constraint structure of such theories.
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Singularities have a fundamental role in general relativity [1]. However it is not clear how to incorporate them into a quantum gravity theory. Surely singularities are an important feature in quantum gravity as demonstrated by processes like Hawking radiation. However our inability to follow the black hole evaporation process till its very end shows our present limitations. The singularities which give origin to black holes manifest themselves in the curvature tensor. The effect of this sort of singularity on the propagation of quantum particles has recently been studied [2]. There is however another sort of singularity for which the curvature tensor itself is not singular. They appear when the metric tensor is degenerate and therefore has no inverse. If the degeneracy occurs on a set of measure zero then the curvature remains bounded and the topology of the space-time manifold can change [3]. Such singularities are milder than curvature singularities and perhaps it should be easier to handle them in a quantum gravity theory.

In general relativity degenerate metric appears in the Palatini formulation. There we start with an action in terms of a tetrad e^a_μ and a Lorentz connection ω^{ab}_μ. The action is $S = \frac{1}{2}\int e^a e^b R^{cd} \epsilon_{abcd}$ where e^a is the tetrad one-form and $R = d\omega + \omega^2$ is the curvature two-form. The space-time metric is then $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$ where η_{ab} is the tangent space Minkowski metric. This action and the equations of motion which follow from it do not depend on the inverse tetrad. So they are well defined even when the tetrad e^a_μ is degenerate. Degenerate metrics or tetrads have also appeared in other contexts. Ashtekar formulation of general relativity [4] and gauge theoretic formulations of several gravity theories in two and three dimensions [5] also allow degenerate tetrads. Also strings can propagate in degenerate metric backgrounds [6]. It has also been suggested that there would be a topological phase of quantum gravity in which the tetrad vanishes and diffeomorphism invariance is unbroken [7].

All this indicates that degenerate tetrads may have an important role in quantum gravity. At the classical level the main consequence of the degenerate tetrads is to allow topology change of the space-time manifold [3,8]. Since this is not observed in nature there must exist a suppression mechanism for it, probably at the quantum level. On the other side quantum effects involving degenerate tetrads were not studied. In this paper we will consider the simplest situation we can think of in order to explore the effects of degenerate tetrads. We will study the propagation of quantum non-relativistic particles in a classical gravitational background in which the metric is degenerate in a set of points or in a finite region. We will show that we can make sense of such situations giving strong support that such degenerate configurations should be taken into account in a proper formulation of quantum gravity. In particular we will show that when a particle moves in a degenerate region in one dimension the transmission coefficient decreases with increasing particle energy. When we go to higher dimensions we find a very rich constraint structure which will be briefly discussed.

Let us consider first the one-dimensional case. Assume a non-relativistic particle of mass m moving in a one-dimensional space with $ds^2 = N(x)dx^2,\ \ N(x) \geq 0$ [9]. If $N(x)$ does not vanish in any point then it is possible to find a non-singular coordinate transformation which takes the metric $N(x)$ into 1. In the degenerate case there is no such a transformation. The equation of motion which follows from the action $S = \frac{1}{2}m \int dt N \dot{x}^2$ is

$$N\ddot{x} + \frac{1}{2}N'\dot{x}^2 = 0 \tag{1}$$

where dot stands for time derivative and prime for space derivative. Notice that neither the action nor the equation of motion depends on the inverse metric. By multiplying the equation of motion by $\dot{x}$ it is easy to see that $N\dot{x}^2$ is a constant of motion and we can write an energy-like equation

$$\dot{x}^2 + V(x) = 0,\ \ V(x) = -\frac{A}{N(x)} \tag{2}$$

where A is a positive constant. Near the points where the metric is degenerate the particle feels an attractive force towards those points. As the particle approaches the degenerate points $V(x)$ goes to infinity and the particle velocity also goes to infinity. These points are then singular in the sense that the space is not geodesically complete since after a finite time the evolution of the particle is not defined.

We could also consider a finite region where the metric is degenerate. For simplicity assume that

$$N(x) = \begin{cases} 1, & |x| > a \\ \epsilon, & |x| < a \end{cases} \tag{3}$$

in the limit $\epsilon \to 0$. By solving the particle equation of motion Eq.(1) in each region we find that its velocity is constant outside the region where the metric is degenerate and goes to infinity inside that region (when the limit $\epsilon \to 0$ is taken). The degenerate region is also singular.

Consider now a quantum particle propagating in a classical space with a degenerate metric. We will consider the system as quantum mechanical nonsingular if the evolution of any state is defined for all times. Let us first consider the case of a degenerate region with the metric given by Eq.(3). The relevant Laplace-Beltrami operator is

$$\nabla^2\psi = \frac{1}{N}\psi'' - \frac{1}{2}\frac{N'}{N^2}\psi' \tag{4}$$

and it depends on the inverse metric. We will then look for wave functions for finite ϵ and then take the limit $\epsilon \to 0$ if it is well behaved. It turns out that this limit is well defined in all cases we considered. For an incident particle of energy E the scattering states are described by the following wave function

$$\psi(x) = \begin{cases} e^{ikx} - ikae^{-2ika-ikx}/(1-ika), & x < a \\ e^{-ika}\left[1 + ikx/(1-ika)\right], & -a < x < a \\ e^{-2ika+ikx}/(1-ika), & x > a \end{cases} \tag{5}$$

where $k^2 = 2mE/\hbar^2$. There are no bound states. We have used continuity of the wave function and its first derivative at the borders of the degenerate region. Then the S-matrix is unitary. Its matrix elements can be easily computed and are given by $S_{12} = S_{21} = e^{-2ika}/(1-ika)$ and $S_{11} = S_{22} = -ikae^{-2ika}/(1-ika)$. Unlike curvature singularities degenerate metric singularities do not lead to a non-unitary S-matrix. Notice also that although the probability density $|\psi|^2$ does not vanish inside the degenerate region the probability of finding the particle there $\int dx\sqrt{N}|\psi|^2$ vanishes. So even at the quantum level we can not detect the particle inside the degenerate region. We find then that the quantum theory is nonsingular.

The transmission coefficient has a peculiar behaviour. From Eq.(5) we find that $|T|^2 = 1/(1+k^2a^2)$ so that it goes to one at low energies and goes to zero at high energies. Then a degenerate region is transparent for low energy particles and opaque for high energy particles (in a non-relativistic regime). This provides a distinctive signature for the detection of such regions. Usually we would expect $|T|^2$ to have the opposite behaviour. If we think of something like a potential well we know that at low energies the wavelength of the particle is too large to fit into the well so its transmission coefficient should be very low while for high energies a large number of particle wavelengths can fit into the well so its transmission coefficient should be higher. We can have some understanding of what is happening in the degenerate region if we consider a potential barrier $V(x) = V_0$ inside the region and zero outside. If we let V_0 depend on the energy of the incident particle in such a way that $V_0 \approx E$, with $V_0 - E > 0$, we get the same wave function Eq.(5) [10]. We have also analysed the higher dimensional case and found a similar behaviour [11].

We can now consider the former results in the limit in which the degenerate region reduces to a point, i.e. $a \to 0$. In this limit the wave function Eq.(5) reduces to $\psi = e^{ikx}$ for any x and the particle does not feel any effect of this particular degenerate metric [12]. In fact we can consider a more general class of metrics degenerate at a point. Consider a degenerate metric $N(x)$ that goes to zero as $|x|^p$ for some positive p. Using Eq.(4) we find that near the origin the wave function behaves as $\psi(x) = x^s$ with $s = 0$ or $s = p/2 + 1$ for both $x > 0$ and $x < 0$. The wave function is normalizable, near the origin, for both values of s and any p. We can also check that the first derivative of the wave function is continuous at the origin. Then in general degenerate metrics at a point are also nonsingular at the quantum level showing how mild they are. This analysis can be extended to higher dimensions on the same lines as in [2].

Let us now consider the propagation of classical particles in a degenerate space in higher dimensions. For simplicity let us consider the situation where we have a D dimensional space whose metric is degenerate with rank $D-1$ so that it can be written in the form $g_{ij} = g_i g_j$ with $i, j = 1, \ldots, D$. Usually this means that the space is in fact one dimensional but as we will see there is more to it. The Lagrangian $L = \frac{1}{2}mg_{ij}\dot{x}^i\dot{x}^j$ gives rise to a set of $D-1$ constraints

$$\phi_\alpha = g_1 P_\alpha - g_\alpha P_1, \;\; \alpha = 2, \ldots, D \tag{6}$$

where P_i are the canonical momenta. The Hamiltonian is then found to be $H = P_i\dot{x}^i - L + \lambda^\alpha\phi_\alpha$ where λ^α are Lagrange multipliers. Requiring that the constraints Eq.(6) do not evolve in time gives a new set of constraints

$$\frac{P_1^2}{mg_1^2}\left(\partial_\alpha g_1 - \partial_1 g_\alpha\right) + \lambda^\beta\{\phi_\alpha, \phi_\beta\} = 0 \tag{7}$$

where the brackets stand for Poison brackets. We can find also that

$$\Gamma_{\alpha\beta} \equiv \{\phi_\alpha, \phi_\beta\} = P_1[g_\alpha(\partial_\beta g_1 - \partial_1 g_\beta) + g_\beta(\partial_1 g_\alpha - \partial_\alpha g_1) + g_1\partial_{[\alpha}g_{\beta]}] \tag{8}$$

We can now consider two possibilities for the g_i. If they satisfy $\partial_{[i}g_{j]} = 0$ then $\Gamma_{\alpha\beta} = 0$ and we have $D-1$ first class constraints. The Lagrangian can be reduced to $L = \frac{1}{2}m\dot{u}^2$ by performing the coordinate transformation $du = g_i dx^i$ and it describes a particle with just one degree of freedom. The local Lagrangian symmetry generated by the first class constraints Eq.(6) is

$$\begin{aligned} \delta x^1 &= -\theta^\alpha g_\alpha \\ \delta x^\alpha &= \theta^\alpha g_1 \end{aligned} \tag{9}$$

where $\theta(t)$ is the parameter of the transformation.

If $\partial_{[i}g_{j]} \neq 0$ then $\Gamma_{\alpha\beta} \neq 0$ and our analysis will be dependent on D. For odd dimensional spaces ($D \geq 3$) $\Gamma_{\alpha\beta}$ has an inverse. Then we can find out the Lagrange multipliers λ^α in Eq.(7) so that all $D-1$ constraints ϕ_α are second class. Then the number of degrees of freedom is $(D+1)/2$. If D is even we first analyse the case $D = 2$. We find that $P_1 = P_2 = 0$ and that they are first class constraints. They generate a local Lagrangian symmetry of the action of the form

$$\delta x^i = \theta\dot{x}^i - \frac{1}{2}\frac{\dot{\theta}\epsilon^{ij}g_j}{\epsilon^{lm}\partial_l g_m} \tag{10}$$

There are no degrees of freedom. If D is even and $D \geq 4$ then $\Gamma_{\alpha\beta}$ has no inverse. This means that some of the Lagrange multipliers are dependent and that there exists at least one linear combination of the constraints which is a first class constraint. We find that there exists only one first class constraint given by

$$\phi = \epsilon^{\mu_2\mu_3\ldots\mu_D}\phi_{\mu_2}\Gamma_{\mu_3\mu_4}\ldots\Gamma_{\mu_{D-1}\mu_D} \tag{11}$$

and the remaining second class constraints are

$$\psi_{\alpha'} = g_1 P_{\alpha'} - g_{\alpha'}P_1, \;\; \alpha' = 2, \ldots, D-1 \tag{12}$$

Then there is one first class constraint ϕ and $D-2$ second class constraints $\psi_{\alpha'}$ so that the theory has $D/2$ degrees of freedom.

When the metric is degenerate of arbitrary rank a geometrical construction can be done when $\partial_{[i}g_{j]} = 0$ [13]. Assume that we start with the Lagrangian $L = \frac{1}{2}mg_{ij}\dot{x}^i\dot{x}^j$. The action is invariant under $\delta x^i = -\epsilon^\alpha K^i_\alpha$ if the K^i are the Killing vectors (α runs over the number of Killing vectors) which generate an isometry $(\mathcal{L}_K g)_{ij} = 0$, with $\mathcal{L}$ being the Lie derivative. Notice that we do not need the inverse metric. Now assume that the metric g_{ij} is degenerate of rank $D-n$ and $U^i_\alpha, \alpha = 1, \ldots, D-n$, are the null eigenvectors of this metric. The canonical momenta are $P_i = mg_{ij}\dot{x}^j$ so that $\phi_\alpha \equiv P_i U^i_\alpha = 0$. The ϕ_α are a set of $D-n$ constraints and generalizes Eq.(6) for which $n=1$. The algebra generated by the null eigenvectors when $\partial_{[i}g_{j]} = 0$ is $[U_\alpha, U_\beta] = f^\gamma_{\alpha\beta}U_\gamma$ which entails the constraint algebra $\{\phi_\alpha, \phi_\beta\} = -f^\gamma_{\alpha\beta}\phi_\gamma$. This means that the ϕ_α give rise to a first class constraint algebra and we find n degrees of freedom. In the case $\partial_{[i}g_{j]} \neq 0$ the isometry condition is no longer satisfied and the preceeding analysis can not be applied.

We have also studied relativistic particles and strings in degenerate metric backgrounds. We have been able to perform the classification of the constraints and we will report on that elsewhere [11]

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* E-mail: lacabral@if.usp.br

** E-mail: vrivelles@if.usp.br

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[9] If we do not assume $N(x) \geq 0$ then signature changing spaces can be taken into account. They are relevant in quantum cosmology and are being subject of intensive investigation. See F. Embacher, preprint UWThPh-1995-11, gr-qc/9504040 and references therein.

[10] It should be noticed that a potential well with $V(x) = V_0$ inside the region and zero outside gives the same wave function Eq.(5) for $E > 0$ and $V_0 \approx E \approx 0$.

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[12] We should notice that this result does not depend on the order in which we take the limits $\epsilon \to 0$ and $a \to 0$.

[13] A similar treatment in another context can be found in C. R. Ordóñez and J. M. Pons, Phys. Rev. D **45** 3706 (1992); preprint hep-th/9308078.