

UNIVERSIDADE DE SÃO PAULO

**INSTITUTO DE FÍSICA
CAIXA POSTAL 66318
05389-970 SÃO PAULO - SP
BRASIL**

PUBLICAÇÕES

IFUSP/P-1200

Agno. 3057-08

**ASYMPTOTIC BEHAVIOR OF IRREDUCIBLE
EXCITATORY NETWORKS OF GRADED-RESPONSE
NEURONS**

**K. Pakdaman¹, C. Grotta-Ragazzo², C.P. Malta³ and
J.-F. Vibert¹**

- 1) B3E, INSERM U 263, ISARS
Falcuté de Médecine Saint-Antoine
27, rue Chaligny, 75571 Paris Cedex 12, France
- 2) Instituto de Matemática e Estatística
Universidade de São Paulo
CP 66281, 05389-970, São Paulo, BRASIL
- 3) Instituto de Física, Universidade de São Paulo

Fevereiro/1996

Asymptotic behavior of irreducible excitatory networks of graded-response neurons

K. Pakdaman†, C.P. Malta‡, C. Grotta-Ragazzo* and J.-F. Vibert†

† B3E, INSERM U 444, ISARS, UPMC
Faculté de Médecine Saint-Antoine
27, rue Chaligny
75571 Paris Cedex 12 FRANCE

‡ Instituto de Física
Universidade de São Paulo
CP 66318, 05389-970 São Paulo, BRASIL

* Instituto de Matemática e Estatística
Universidade de São Paulo
CP 66281, 05389-970 São Paulo, BRASIL
and
Mathematics Department, Princeton University
Fine Hall, Washington Road
Princeton, NJ 18540 USA

Correspondence should be addressed to:

K. Pakdaman
B3E, INSERM U444
Faculté de Médecine Saint-Antoine
27, rue Chaligny
75571 Paris Cedex 12, FRANCE
tel: 33-1-44738430
fax: 33-1-44738462
email: pakdaman@b3e.jussieu.fr

Abstract: We study the asymptotic behavior of irreducible excitatory networks of graded-response neurons. In these networks, the trajectories of most solutions tend to the equilibria. We derive sufficient conditions for the network to be globally asymptotically stable. When the network is multistable, we provide a description of the basin boundaries. The results hold for systems with and without delay.

1 Introduction

The asymptotic behavior of networks of analog graded-response neurons (GRNs) has been analyzed in a number of studies for example (Hopfield, 1984; Li *et al.*, 1988; Hirsch, 1989; Marcus *et al.*, 1991; Forti, 1994).

Hirsch (1989) showed that almost all trajectories of irreducible excitatory networks of GRNs tend to the set of equilibria. This result is mainly due to the fact that the flow generated by the system is strongly order preserving. We continue along this line of investigation in order to obtain more information on the asymptotic behavior of the trajectories. The results are presented in sections 2 through 4. In section 5 we discuss the case of excitatory connections with finite transmission time (delay).

2 The graded-response model

In the network composed of n GRNs, each unit is described by its activation at time t , denoted by $a_i(t)$, and a sigmoidal output function $\sigma_i(a_i)$. A decay rate γ_i is also implemented in the model (Cowan and Ermentrout, 1978; Hopfield, 1984). Neuron i receives a constant input K_i . W_{ij} represents the connection weight between neurons j and i . The behavior of an n -neuron network is governed by the following system of ordinary differential equations (ODEs):

$$\frac{da_i}{dt}(t) = -\gamma_i a_i(t) + K_i + \sum_{j=1}^n W_{ij} \sigma_j(a_j(t)) \quad 1 \leq i \leq n. \quad (1)$$

For all $x \in \mathbb{R}^n$, there exists a unique real function $a(t, x)$ from \mathbb{R} to \mathbb{R}^n , such that $a(0, x) = x$ and $a(t, x)$ satisfies system (1) for all $t \in \mathbb{R}$.

Throughout this paper we assume that the following hypotheses are satisfied.

Hypothesis H1

The neuron transfer function σ_i is a smooth strictly increasing function, bounded between two real numbers m_i and M_i , such that there is a unique point p_i such that $\sigma_i''(p_i) = 0$.

Hypothesis H2

The connection matrix $W = [W_{ij}]$ is positive ($W_{ij} \geq 0$ for all i, j) and irreducible *i.e.* it does not leave invariant any proper nontrivial subspace generated by a subset of the standard basis vectors for \mathbb{R}^n (Smith, 1988; Hirsch, 1989).

3 Asymptotic behavior

For $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ and $y = (y_1, \dots, y_n)^T \in \mathbb{R}^n$ (the suffix T indicates the transpose), we say that x is larger (resp. strictly larger) than y , denoted $x \geq y$ (resp. $x \gg y$) when

$x_i \geq y_i$ (resp. $x_i > y_i$) for all $1 \leq i \leq n$. Finally $x > y$ indicates that $x \geq y$ and $x \neq y$.

An important property of system (1) is that it generates a strongly order preserving flow:

$$\text{For } (x, y) \in \mathbb{R}^{2n}, \text{ if } x > y \text{ then } a(t, x) \gg a(t, y) \text{ for all } t > 0. \quad (2)$$

Due to this property, trajectories have a strong tendency to converge to equilibria.

Almost quasi convergence (Hirsch, 1989). *Almost all trajectories of system (1) approach the set of equilibria as $t \rightarrow +\infty$.*

We describe in more details the asymptotic behavior of system (1). We introduce the following notations: $p = (p_1, \dots, p_n)^T \in \mathbb{R}^n$, $\Gamma = \text{diag}(\gamma_1, \dots, \gamma_n)$, $\gamma = \min_{1 \leq i \leq n}(\gamma_i)$, the input vector: $K = (K_1, \dots, K_n)^T \in \mathbb{R}^n$, the output vector: $\sigma(x) = (\sigma_1(x_1), \dots, \sigma_n(x_n))^T$, the bounds of the output vector $m = (m_1, \dots, m_n)^T$ and $M = (M_1, \dots, M_n)^T$ with $m \leq \sigma(x) \leq M$ for all $x \in \mathbb{R}^n$.

Let $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$, we define the following $n \times n$ matrix: $V(x) = [W_{ij}\sigma_j'(x_j)]$. $V(x)$ is an irreducible positive matrix, thus it admits a largest real eigenvalue denoted by $\lambda(x)$. The map $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^+$ which associates $\lambda(x)$ to a vector x , is decreasing for $x \geq p$, that is, $\lambda(p) \geq \lambda(x) \geq \lambda(x')$ when $p \leq x \leq x'$. Let $u \in \mathbb{R}^n$ with $u \gg 0$ and $c \in \mathbb{R}$, we have $\lim_{c \rightarrow +\infty} \lambda(x + c.u) = 0$, for all $x \in \mathbb{R}^n$. Similar results hold for $x \leq p$.

Global stability (1). *If $\gamma > \lambda(p)$, system (1) is globally asymptotically stable (for all $K \in \mathbb{R}^n$).*

Proof. Let $x \in \mathbb{R}^n$, we have $V(x) \leq V(p)$ (component-wise order), so that $0 < \lambda(x) \leq \lambda(p) < \gamma$. Thus all eigenvalues of the matrix $-\Gamma + V(x)$ have negative real parts.

Global stability (2). *For $\gamma < \lambda(p)$, $K \in \mathbb{R}^n$, and $u \in \mathbb{R}^n$ with $u \gg 0$ (component-wise order), there are two real numbers c_- and c_+ , such that for either $c < c_-$ or $c > c_+$, system (1) with input vector $K + c.u$ is globally asymptotically stable.*

Proof. Let $K \in \mathbb{R}^n$ be an input vector. An equilibrium point x of system (1) with input K satisfies:

$$-\Gamma x + W\sigma(x) + K = 0. \quad (3)$$

We have thus:

$$y_m(K) \leq x \leq y_M(K) \quad (4)$$

where $y_m(K) = \Gamma^{-1}(W \times m + K)$ and $y_M(K) = \Gamma^{-1}(W \times M + K)$. Let $u \in \mathbb{R}^n$, with $u \gg 0$, we denote by $K(c) = K + c.u$ for $c \in \mathbb{R}$. The map $c \rightarrow y_m(K(c))$ from \mathbb{R} to \mathbb{R}^n is increasing. We have $y_m(K(c)) = y_m(K) + c.\Gamma^{-1}u$, therefore $\lim_{c \rightarrow +\infty} \lambda(y_m(K(c))) = 0$. We have $y_m(K(c)) \leq x$, where x is an equilibrium of system (1) with input $K(c)$. For large enough c , we have $y_m(K(c)) \geq p$, and therefore $\lambda(x) \leq \lambda(y_m(K(c)))$. So that for large enough c , we obtain $\lambda(x) < \gamma$.

Corollary. *There are input vectors K_- and K_+ , such that for either $K \ll K_-$ or $K \gg K_+$ (component-wise order), system (1) is globally asymptotically stable.*

We assume $\gamma < \lambda(p)$, and we define $\kappa = \{K \in \mathbb{R}^n : K \ll K_- \text{ or } K \gg K_+\}$, and $\mathcal{R}_0 = \mathbb{R}^n - \kappa$.

Finite equilibria. *There is a negligible subset \mathcal{Q} , of \mathcal{R}_0 such that for all $K \in \mathcal{R} = \mathcal{R}_0 - \mathcal{Q}$ system (1) has a finite number $q \geq 1$ of equilibria. All the equilibria are hyperbolic. If $q = 1$ the system is globally asymptotically stable. If $q \neq 1$, then necessarily we have $q \geq 3$.*

Proof. *The first statement is a direct application of Sard's theorem (Li et al., 1988). The second one is proved in the next section.*

4 Basin boundaries

From this point on we assume $K \in \mathcal{R}$. We denote by \mathcal{E} the set of the equilibria, by $B(y)$ the basin of attraction of a stable equilibrium point y , and by $\partial B(y)$ the boundary of $B(y)$

Lemma. *There are two equilibrium points x_m and x_M such that: i) $x_m \leq x_M$ ii) $a(t, x) \rightarrow x_m$ as $t \rightarrow +\infty$ for all $x \leq x_m$ and iii) $a(t, x) \rightarrow x_M$ as $t \rightarrow +\infty$ for all $x \geq x_M$.*

Proof of the lemma. We denote by x_1, x_2, \dots, x_q the equilibria of system (1). We suppose that there is no equilibrium larger than x_q . As the union of the basins of attraction of the equilibria is an open and dense subset of \mathbb{R}^n , we have necessarily $[[x_q, +\infty]] = \{x \in \mathbb{R}^n : x \gg x_q\} \subset B(x_q)$. For any equilibrium x_i , we can pick up $x \in \mathbb{R}^n$ such that $x \gg x_i$ and $x \gg x_q$, so that $a(t, x) \gg x_i$ for all $t > 0$ and $a(t, x) \rightarrow x_q$ as $t \rightarrow +\infty$. From this we deduce that $x_q \geq x_i$. In fact, if $x_q \neq x_i$, we have necessarily $x_q \gg x_i$. Therefore we have $x_q = x_M$. The existence of x_m is proved in the same way.

We assume $q \geq 3$. The following results are deduced from the description of the phase portrait of strongly order preserving flows (Smith, 1988).

Largest and smallest equilibria. *There are two locally asymptotically stable equilibrium points x_m and x_M , such that:*

1. $x_m \ll y \ll x_M$ for all $y \in \mathcal{E} - \{x_m, x_M\}$
2. there is a codimension one, Lipschitz, unordered, positively invariant manifold H_m such that $\phi \in B(x_m)$ if and only if $\phi < H_m$ (i.e. there is $u \in H_m$ such that $\phi \leq u$ and $\phi \neq u$)
3. there is a codimension one, Lipschitz, unordered, positively invariant manifold H_M such that $\phi \in B(x_M)$ if and only if $\phi > H_M$

Basin boundaries. Let $y \in \mathcal{E} - \{x_m, x_M\}$, if y is locally asymptotically stable, then there are two codimension one, Lipschitz, unordered, positively invariant manifolds H_1 and H_2 , such that:

1. H_1 is below H_2 (i.e. for all $u \in H_1 - (H_1 \cap H_2)$ (resp. $u \in H_2 - (H_1 \cap H_2)$), $u < H_2$ (resp. $u > H_1$))
2. the lower boundary of $B(y)$ defined as $\partial_- B(y) = \{\phi \in \partial B(y) : \phi < B(y)\}$, satisfies $\partial_- B(y) \subset H_1$
3. the upper boundary of $B(y)$ defined as $\partial_+ B(y) = \{\phi \in \partial B(y) : \phi > B(y)\}$ satisfies $\partial_+ B(y) \subset H_2$
4. $\partial B(y) - (\partial_- B(y) \cup \partial_+ B(y)) \subset H_1 \cap H_2$

Characterization of the basin boundaries. Let $y \in \mathcal{E} - \{x_m, x_M\}$, if y is locally asymptotically stable, then there are two stable equilibria x and x' such that $x \ll y \ll x'$, and there are no stable equilibria in the ordered open intervals $[[x, y]] = \{\phi : x \ll \phi \ll y\}$ and $[[y, x']] = \{\phi : y \ll \phi \ll x'\}$.

1. $H_1 \cap [[x, y]]$ contains an odd number of unstable equilibria
2. $H_2 \cap [[y, x']]$ contains an odd number of unstable equilibria

Oscillations on the basin boundaries. Under the same conditions, solutions in $H_1 \cap [[x, y]]$ oscillate around the corresponding unstable points.

5 Networks with delay

Finite transmission times arising in hardware implementation of GRN networks, as well as possible applications of networks with delay have motivated a number of studies on the dynamics of GRN networks with delay. Taking delays into account, modifies system (1) into the following system of delay differential equations (DDEs):

$$\frac{da_i}{dt}(t) = -\gamma_i a_i(t) + K_i + \sum_{j=1}^n W_{ij} \sigma_j(a_j(t - \tau_{ij})) \quad 1 \leq i \leq n \quad (5)$$

where τ_{ij} represents the delay between units j and i . Let $\tau_j = \max_{1 \leq i \leq n} \{\tau_{ij} : W_{ij} \neq 0\}$, then $S = C[-\tau_1, 0] \times \dots \times C[-\tau_n, 0]$ is the phase space for DDE (5). For any initial condition ϕ in S , there exists a unique solution of DDE (5) defined for all $t \geq 0$. A component-wise order is associated with the product space S . DDE (5) generates a strictly order preserving semiflow (Smith, 1987; Roska *et al.*, 1992). Therefore the stability of the locally stable equilibria is not affected by the presence of delay and the description of the basin boundaries remains valid (Takáč, 1991).

Acknowledgment: This work was partially supported by USP/COFECUB under project U/C 9/94. One of us (CPM) is also partially supported by CNPq (the Brazilian Research Council).

6 References

- Cowan, J. & Ermentrout G.B. (1978). Some aspects of the 'eigenbehavior' of neural nets. In S.A. Levin (Ed.) *Studies in Mathematical Biology, part I*, 67-117, The Mathematical Association of America.
- Forti, M. (1994). On global asymptotic stability of a class of nonlinear systems arising in neural network theory. *Journal of Differential Equations*. **113**, 246-264.
- Hirsch, M.W. (1989). Convergent activation dynamics in continuous time networks. *Neural Networks*. **2**, 331-350.
- Hopfield, J.J. (1984). Neurons with graded response have collective computational properties like those of two-state neurons. *Proceedings National Academy of Sciences USA*. **81**, 3088-3092.
- Li, J.-H., Michel, A.N. & Porod, W. (1988). Qualitative analysis and synthesis of a class of neural networks. *IEEE Transactions on Circuits and Systems*. **35**, 976-986.
- Marcus, C.M., Waugh, F.R. & Westervelt, R.M. (1991). Nonlinear dynamics and stability of analog neural networks. *Physica D*. **51**, 234-247.
- Roska, T., Wu, C.F., Balsi, M. & Chua, L.O. (1992). Stability and dynamics of delay-type general and cellular neural networks. *IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications*. **39**, 487-490.
- Smith, H. (1987). Monotone semiflows generated by functional differential equations. *Journal of Differential Equations*. **66**, 420-442.
- Smith, H.L. (1988). Systems of ordinary differential equations which generate an order preserving flow: a survey of results. *SIAM Review*. **30**, 87-113.
- Takáč, P. (1991). Domains of attraction of generic omega-limit sets for strongly monotone semi-flows. *Zeitschrift für Analysis und ihre Anwendungen*. **10**, 275-317.