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PUBLICAÇÕES

IFUSP/P-1206

RPA APPROXIMATION FOR A CHIRAL GROSS-NEVEU SYSTEM

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Março/1996

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Submitted to Phys. Rev. D.

March 17, 1996

Abstract

We linearize and study the small oscillations regime (RPA approximation) of the mean-field equations which describe the time evolution of the one-body dynamical variables of an uniform system described by Chiral Gross-Neveu model, obtained in a previous work [1]. In this approximation we obtain an analytical solution for the time evolution of the one-body dynamical variables. The two-fermion physics can be explored through this solution. The condition for the existence of bound states is examined.

PACS Numbers : 03.65.Ge, 03.65.Nk, 03.70.+k, 05.30.Fk, 21.60.Jz

*Supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP), Brazil

1 Introduction

In a previous work [1] we obtained in mean-field (Gaussian) approximation the effecti dynamics of one-fermion and pairing densities of a off-equilibrium spatially uniform (1+ dimensional self-interacting fermion system described by chiral Gross-Neveu model (CGN [2]. These dynamical equations acquire the structure of (colisionless) kinetic equations. Th determine the time evolution of the one-fermion densities of this system for a given initi condition. Spatial uniformity (translation and parity invariance) is assumed in our derivatio

Studing the static solutions of these equations in order to renormalize the theory [we found the well-known effective potential obtained from the $1/N$ expansion by Gross a Neveu [2]. We also showed that other static results which have been discussed in the literatu [2, 3, 4] such as dynamical mass generation due to chiral symmetry breaking and dimension transmutation phenomenon can be retrieved from this formulation. Finally we calcul numerically in [1] the mean-field time evolution of the one-body dynamical variables initial displaced from equilibrium.

In this work we explore a particular application of the renormalized nonlinear mean-fie equations obtained in [1]. We consider the near equilibrium dynamics around the stationa solution. From linearized version of these equations we show that the two-fermion (qua fermion) physics can be studied. In particular, one can solve this equations analytically a find two-fermion (quasi-fermion) bound state solution.

This paper is organized as follows. In Sec.II we linearize the mean-field dynamical equ tions which describe the time evolution of a off-equilibrium spatially uniform (1+1) dime sional self-interacting fermion system described by chiral Gross-Neveu model (CGNM). self-consistent renormalization scheme is necessary [1, 5]. In Sec.III, making use of an an ogy with scattering theory [6], we obtain a closed analytical solution for the time evoluti of the one-fermion densities in this regime. Studyng the two-fermion physics in Sec.IV, find the condition for the existence of bound states. Finally, the Sec.V is devoted to a fi discussion and conclusions.

2 Linearization of the mean-field kinetic equations

We begin this section by reviewing our approach which describe a formal treatment of t kinetics of a self-interating quantum field. This approach was developed earlier for the no relativistic nuclear many-body dynamics by Nemes and Toledo Piza [7] and was more recen applied in the quantum-field theoretical context to the self-interacting $\lambda\phi^4$ theory in (1+ dimensions [8]. The generical ideia is to focus on the time evolution of the one-fermi and pairing densities. These observables are kept under direct control when one works va tionally using a Gaussian functional *ansatz* and will therefore be refered to as Gaussian bservables.

We consider an off-equilibrium, spatially uniform, (1+1) dimensional system of relativistic, self-interacting fermions described by chiral Gross-Neveu model (CGNM) [2]. The Hamiltonian density is given by

$$\mathcal{H}_{\text{CGNM}} = \sum_{i=1}^{N} \left\{ \bar{\psi}^i \left[-i\gamma_1 \partial_1 \right] \psi^i \right\} - \frac{g^2}{2} \left\{ \left[\sum_{i=1}^{N} \bar{\psi}^i \psi^i \right]^2 - \xi \left[\sum_{i=1}^{N} \bar{\psi}^i \gamma_5 \psi^i \right]^2 \right\} , \tag{1}$$

where ξ is a constant which indicates whether the model is invariant under discrete γ_5 transformation ($\xi = 0$) or under the Abelian chiral $U(1)$ group ($\xi = 1$). In the form considered here, this is a massless fermion theory in (1+1) dimensions with quartic interaction. The model contains N species of fermions coupled symmetrically. In the Heisenberg picture, the ψ^i are complex Dirac spinors

$$\begin{aligned} \psi(x) &= \sum_{\mathbf{k}} \left(\frac{m}{k_0} \right)^{1/2} \left[a_{\mathbf{k}}(t) u_1(\mathbf{k}) \frac{e^{i\mathbf{k}x}}{\sqrt{L}} + c^{\dagger}_{\mathbf{k}}(t) u_2(\mathbf{k}) \frac{e^{-i\mathbf{k}x}}{\sqrt{L}} \right] \\ \bar{\psi}(x) &= \sum_{\mathbf{k}} \left(\frac{m}{k_0} \right)^{1/2} \left[a^{\dagger}_{\mathbf{k}}(t) \bar{u}_1(\mathbf{k}) \frac{e^{-i\mathbf{k}x}}{\sqrt{L}} + c_{\mathbf{k}}(t) \bar{u}_2(\mathbf{k}) \frac{e^{i\mathbf{k}x}}{\sqrt{L}} \right] , \end{aligned} \tag{2}$$

where $a^{\dagger}_{\mathbf{k}}$ and $a_{\mathbf{k}}$ ($c^{\dagger}_{\mathbf{k}}$ and $c_{\mathbf{k}}$) are fermion creation and annihilation operators associated to positive(negative)-energy solution $u_1(\mathbf{k})$ ($u_2(\mathbf{k})$) of the Dirac equation.

This model is essentially equivalent to the Nambu-Jona-Lasinio model [9], except for the fact that in (1+1) dimensions it is renormalizable. Moreover, it is one of the very few known field theories which are assymptotically free. To leading order in $1/N$ expansion [2], the CGNM exhibits a number of interesting phenomena, like spontaneous symmetry breaking, dynamical fermion mass generation and dimensional transmutation.

The state of this system (assumed spatially uniform) is given in terms of a many-body density operator $\mathcal{F}$ of unit trace. Our implementation of the mean-field (Gaussian) approximation consists in approximating this object by a truncated many-body density operator $\mathcal{F}_0(t)$, also of unit trace, written as the most general hermitian Gaussian functional of the field operators consistent with the assumed uniformity of the system [10]. It will thus be written as the exponential of a general quadratic form in the field operators which can be reduced to diagonal form by suitable canonical transformation. For this purpose we define the transformed quasi-fermion operators from the Bogolyubov transformation below

$$\begin{bmatrix} a_{\mathbf{k}} \\ c_{\mathbf{k}} \\ a^{\dagger}_{-\mathbf{k}} \\ c^{\dagger}_{-\mathbf{k}} \end{bmatrix} = \begin{bmatrix} X_{11} & X_{21} & Y^*_{11} & Y^*_{21} \\ X_{12} & X_{22} & Y^*_{12} & Y^*_{22} \\ Y_{11} & Y_{21} & X^*_{11} & X^*_{21} \\ Y_{12} & Y_{22} & X^*_{12} & X^*_{22} \end{bmatrix} \begin{bmatrix} \beta_{\mathbf{k},1} \\ \beta_{\mathbf{k},2} \\ \beta^{\dagger}_{-\mathbf{k},1} \\ \beta^{\dagger}_{-\mathbf{k},2} \end{bmatrix} , \tag{3}$$

where we have used the parity symmetry of the uniform system to make the Bogolyubov parameters $X_{\lambda',\lambda}(\mathbf{k})$ and $Y_{\lambda',\lambda}(\mathbf{k})$ for $\lambda = 1, 2$ dependent only on the magnitude of $\mathbf{k}$. This transformation is canonical if we impose the unitary condition to the Bogolyubov transformation $\mathcal{X}_{\mathbf{k}}(t)$ defined in (3)

$$\mathcal{X}^{\dagger}_{\mathbf{k}} \mathcal{X}_{\mathbf{k}} = I_4 \quad \text{and} \quad \mathcal{X}_{\mathbf{k}} \mathcal{X}^{\dagger}_{\mathbf{k}} = I_4 . \tag{4}$$

The Bogolyubov transformation defined in Eq.(3) breaks both chiral and charge symmetries of the CGNM. We restrict the following development to a special Bogolyubov transformation (to be called Nambu transformation [9]) which breaks the chiral symmetry of our system only. The elements of this Nambu transformation, parametrized consistently with unitary conditions (4), are given by

$$\begin{aligned} &X_{12} = X_{21} = 0 \quad \text{and} \quad Y_{11} = Y_{22} = 0 \ ; \\ &X_{11} = X_{22} = \cos\varphi_{\mathbf{k}} \quad \text{and} \quad Y_{12} = -Y_{21} = \sin\varphi_{\mathbf{k}} e^{i\gamma_{\mathbf{k}}} . \end{aligned} \tag{5}$$

Now, the Gaussian truncated density operator $\mathcal{F}_0(t)$ acquires a particularly simple form when expressed in terms of the Nambu quasi-fermion operators, namely

$$\mathcal{F}_0(t) = \prod_{\mathbf{k},\lambda} \left[\nu_{\mathbf{k},\lambda} \beta^{\dagger}_{\mathbf{k},\lambda}(t) \beta_{\mathbf{k},\lambda}(t) + (1 - \nu_{\mathbf{k},\lambda}) \beta_{\mathbf{k},\lambda}(t) \beta^{\dagger}_{\mathbf{k},\lambda}(t) \right] , \tag{6}$$

where $\nu_{\mathbf{k},\lambda}$ for $\lambda = 1, 2$ are the Nambu (quasi-fermion) occupation numbers.

With the help of Eq.(3) it is an easy task to express $\bar{\psi}(x)$ and $\psi(x)$ [Eq.(2)] in terms of $\beta^{\dagger}_{\mathbf{k},\lambda}(t)$ and $\beta_{\mathbf{k},\lambda}(t)$ for $\lambda = 1, 2$. In doing so, one finds that the plane waves of $\bar{\psi}(x)$ and $\psi(x)$ are modified by a complex, moment-dependent redefinition of m involving the Nambu parameters $\varphi_{\mathbf{k}}(t)$ and $\gamma_{\mathbf{k}}(t)$. The complex character of these parameters is actually crucial in dynamical situations, where the imaginary parts will allow for the description of time-odd (velocity-like) properties. Finally, the mean values of the Gaussian observables are parametrized in terms of the $\varphi_{\mathbf{k}}(t)$ and $\gamma_{\mathbf{k}}(t)$ and of the occupation numbers $\nu_{\mathbf{k},\lambda}(t) = Tr\left[\beta^{\dagger}_{\mathbf{k},\lambda}(t)\beta_{\mathbf{k},\lambda}(t)\mathcal{F}(t)\right]$ for $\lambda = 1, 2$.

The next step is to obtain the mean-field time evolution for the mean values of the Gaussian observables in the context of the initial-value problem. In other words, we want the mean-field equations of motion for the Nambu parameters $\varphi_{\mathbf{k}}(t)$, $\gamma_{\mathbf{k}}(t)$ and for the quasi-particle occupation numbers $\nu_{\mathbf{k},\lambda}(t)$. In Ref. [1] we obtained

$$\dot{\nu}_{\mathbf{k},1} = 0 \quad \text{and} \quad \dot{\nu}_{\mathbf{k},2} = 0 \quad ; \tag{7}$$

$$\left[i\dot{\varphi}_{\mathbf{k}} + \dot{\gamma}_{\mathbf{k}} \sin\varphi_{\mathbf{k}} \cos\varphi_{\mathbf{k}}\right] e^{-i\gamma_{\mathbf{k}}} = \frac{Tr\left(\left[\beta_{-\mathbf{k},1}\beta_{\mathbf{k},2}, H_{\text{CGNM}}\right]\mathcal{F}_0\right)}{(1 - \nu_{\mathbf{k},1} - \nu_{\mathbf{k},2})} \quad . \tag{8}$$

Equation (7) shows that the occupation numbers of the Nambu quasi-particles are constant. This is a general feature of the mean-field approximation. The complex equation of motion (8) describes the time evolution of the Nambu parameters. From the right-hand side of the Eq. (8), we see that to obtain the time evolution of the Nambu parameters, we have to express the CGNM Hamiltonian in the Nambu basis.

From Hamiltonian density (1) we can explicitly evaluate the Hamiltonian of the system by integration over all one-dimensional space. This involves, in particular, choosing a representation for the γ-matrices. Here we have to be careful, since a bad choice of representation can spoil manifest translational invariance (see Appendix A of Ref.[1]). We choose the Pauli-Dirac representation for the γ-matrices, namely

$$\gamma_0 = \sigma_3 \quad ; \quad \gamma_1 = i\sigma_2 \quad \text{and} \quad \gamma_5 = \gamma_0\gamma_1 = \sigma_1 \quad . \tag{9}$$

Substituting the CGNM Hamiltonian written in Nambu basis in the dynamical equation (8), we obtain an explicit dynamical equation which describes the time evolution of the Nambu parameters. The calculation of traces is lengthy but straightforward. Taking the case $N = 1$ for simplicity and splitting the complex equation (8) into real and imaginary parts we have

$$\begin{aligned}
\dot{\nu}_{\mathbf{k},1} &= 0 \quad \text{and} \quad \dot{\nu}_{\mathbf{k},2} = 0 \quad ; \\
\dot{\varphi}_{\mathbf{k}} &= \sin\gamma_{\mathbf{k}} \frac{|\mathbf{k}|}{k_0}\left[m - \left(\frac{g^2 m}{4\pi}\right)(\xi+1)(I_1+I_2)\right] \\
\dot{\gamma}_{\mathbf{k}} \sin 2\varphi_{\mathbf{k}} &= \frac{2\sin 2\varphi_{\mathbf{k}}}{k_0}\left[\mathbf{k}^2 + \left(\frac{g^2 m^2}{4\pi}\right)(\xi+1)(I_1+I_2)\right] + \\
&\quad + 2\cos 2\varphi_{\mathbf{k}} \cos\gamma_{\mathbf{k}} \frac{|\mathbf{k}|}{k_0}\left[m - \left(\frac{g^2 m}{4\pi}\right)(\xi+1)(I_1+I_2)\right] \quad ,
\end{aligned} \tag{10}$$

where I_1 and I_2 are the divergent integrals below

$$\begin{aligned}
I_1 &= \int \frac{d\mathbf{k}'}{k'_0} \cos 2\varphi_{\mathbf{k}'}(1 - \nu_{\mathbf{k}',1} - \nu_{\mathbf{k}',2}) \\
I_2 &= \int \frac{d\mathbf{k}'}{k'_0}\frac{|\mathbf{k}'|}{m} \sin 2\varphi_{\mathbf{k}'} \cos\gamma_{\mathbf{k}'}(1 - \nu_{\mathbf{k}',1} - \nu_{\mathbf{k}',2}) \quad .
\end{aligned} \qquad (1\text{:}$$

We see that the above results contain divergent integrals. A renormalization procedu is therefore required. In general, renormalization procedures consist in combining diverge terms with the bare mass and coupling constants of the theory to define finite (or renorma ized) values of these quantities. In other words, the bare mass and coupling constants a chosen to be cut-off dependent in a way that will cancel the divergent terms. In the prese case, however the divergent integrals (11) involve the dynamical variables themselves in tl integrand, so that their degree of divergence is not directly available. In order to handle th situation we will use a self-consistent renormalization procedure inspired in Ref.[5].

This technique consist in solve a self-consistency problem for the static solutions of tl dynamical equations (10). The static solutions are determined by the solution to the equ tions

$$\sin\gamma_{\mathbf{k}}|_{eq}\left[1 - \left(\frac{g^2}{4\pi}\right)(\xi+1)(I_1+I_2)\right] = 0 \qquad (1\text{:}$$

$$\tan 2\varphi_{\mathbf{k}}|_{eq} = \frac{-|\mathbf{k}|m\left[1 - (g^2/4\pi)(\xi+1)(I_1+I_2)\right]}{\left[(\mathbf{k})^2 + (g^2 m^2/4\pi)(\xi+1)(I_1+I_2)\right]} \cos\gamma_{\mathbf{k}}|_{eq} \quad . \qquad (1\text{:}$$

In order to proceed we introduce a regularizing momentum cut-off Λ and neglect contr butions that vanish in the limit $\Lambda \to \infty$. The renormalized coupling constant for the CGN can be obtained from the minimization of the CGNM vacuum energy density with respect t m, namely $\frac{\delta}{\delta m}\left[Tr\left(H_{\text{CGNM}}\mathcal{F}_0^{\text{vacuum}}(t)\right)\right] = 0$. From this calculation we obtain (see Refs.[1, 4]

$$g^2 = \frac{4\pi}{(\xi+1)}\left[\ln\left(\frac{\Lambda^2}{m^2}\right)\right]^{-1} \quad . \qquad (1$$

We next assume that the integrals I_1 and I_2 have a logarithmic divergence of the type

$$\begin{aligned} I_1 &= a + b\ln\left(\frac{\Lambda^2}{m^2}\right) \\ I_2 &= c + d\ln\left(\frac{\Lambda^2}{m^2}\right) \ , \end{aligned} \tag{15}$$

where a, b, c and d are finite, cut-off independent constants. Substituting (14) and the *ansatz* (15) in static equation (13) we obtain

$$\tan 2\varphi_{\mathbf{k}}|_{\mathrm{eq}} = \frac{-(-1)^n m|\mathbf{k}|[1-(b+d)]}{[(\mathbf{k})^2 + m^2(b+d)]} \ , \tag{16}$$

where the divergence problem is controlled, since b and d are cut-off independent.

We now must verify if the integrals I_1 and I_2 are self-consistent with the ansatz (15). Substituting (16) into (11) we verify that I_1 and I_2 really have a logarithmic divergence (see Appendix C of Ref.[1]). Moreover, from this calculation, we obtain the values of the constants a, b, c and d. We find $b = 1$, while d remains arbitrary. Substituting these results into (16) we obtain the renormalized static solution which describes the broken chiral symmetry phase of our system in the mean-field approximation as

$$\begin{aligned} &\tan 2\varphi_{\mathbf{k}}|_{\mathrm{eq}} = \frac{(-1)^n m|\mathbf{k}|d}{[\mathbf{k}^2 + (1+d)m^2]} \quad \text{with} \quad d \neq 0 \ ; \\ &\gamma_{\mathbf{k}}|_{\mathrm{eq}} = n\pi \quad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots \ . \end{aligned} \tag{17}$$

We also obtain in Ref.[1] that the connection between particle mass m and quasi-particle mass m_{ef} is given by

$$m_{ef} = (1+d)m \ . \tag{18}$$

We observe that our results contain one free parameter, say d. This is altogether reasonable since our starting point was a massless fermions theory which was determined by one dimensionless coupling constant g. We end up with a theory determined by one free parameter d after the self-consistent renormalization procedure. Therefore, the renormalization procedure effectively replaces the dimensionless coupling constant g by a free parameter d

associated to a mass scale [see Eq.(18)]. This is analogous to the phenomenon of dimensic transmutation found by D. J. Gross and A. Neveu [2] in the case of an $1/N$ expansion.

Finally, we write the renormalized dynamical equations which describe the mean-f time evolution of our system in broken chiral phase ($d \neq 0$ or $m_{ef} \neq m$)

$$\begin{aligned} \dot{\nu}_{\mathbf{k},1} &= 0 \quad \text{e} \quad \dot{\nu}_{\mathbf{k},2} = 0 \\ \dot{\varphi}_{\mathbf{k}} &= (-1)md\frac{|\mathbf{k}|}{k_0}\sin\gamma_{\mathbf{k}} \\ \dot{\gamma}_{\mathbf{k}}\sin 2\varphi_{\mathbf{k}} &= \frac{2\sin 2\varphi_{\mathbf{k}}}{k_0}\left[(\mathbf{k})^2 + m^2(1+d)\right] + \\ &- 2md\frac{|\mathbf{k}|}{k_0}\cos 2\varphi_{\mathbf{k}}\cos\gamma_{\mathbf{k}} \ . \end{aligned} \tag{}$$

In order to study the small oscillation regime of the kinetic equations about the vacu we next linearize the above kinetic equations around the static solution (17) with take $\nu_{\mathbf{k},}$ $\nu_{\mathbf{k},2} = 0$. We begin by introducing the displacement away from equilibrium of the dynam variables $\varphi_{\mathbf{k}}$ and $\gamma_{\mathbf{k}}$

$$\begin{aligned} \varphi_{\mathbf{k}} &= \varphi_{\mathbf{k}}|_{\mathrm{eq}} + \delta\varphi_{\mathbf{k}} \\ \gamma_{\mathbf{k}} &= \gamma_{\mathbf{k}}|_{\mathrm{eq}} + \delta\gamma_{\mathbf{k}} \ , \end{aligned} \tag{}$$

where the static solution $\varphi_{\mathbf{k}}|_{\mathrm{eq}}$ and $\gamma_{\mathbf{k}}|_{\mathrm{eq}}$ are obtained from the Eq.(17)

$$\begin{aligned} \sin 2\varphi_{\mathbf{k}}|_{\mathrm{eq}} &= \frac{m|\mathbf{k}|d}{k_0[\mathbf{k}^2 + (1+d)^2m^2]^{1/2}} \\ \cos 2\varphi_{\mathbf{k}}|_{\mathrm{eq}} &= \frac{[\mathbf{k}^2 + (1+d)m^2]}{k_0[\mathbf{k}^2 + (1+d)^2m^2]^{1/2}} \\ \gamma_{\mathbf{k}}|_{\mathrm{eq}} &= 0 \quad \text{with} \quad d \neq 0 \ . \end{aligned} \tag{}$$

The quantities $\delta\varphi_{\mathbf{k}}$ and $\delta\gamma_{\mathbf{k}}$ will be treated as (first-order) small displacements. Functi of the dynamical variables are expanded also to first-order around the equilibrium solut

(21). Therefore, we must linearize the divergent integrals (11) around equilibrium. Taking $\gamma_{\mathbf{k},\lambda} = 0$ we have

$$
\begin{aligned}
I_1 &= I_1^{(0)} + I_1^{(1)} + \mathcal{O}(\delta\varphi_{\mathbf{k}})^2 \\
&= \int \frac{d\mathbf{k}'}{k_0'} \cos 2\varphi_{\mathbf{k}'}|_{eq} - 2 \int \frac{d\mathbf{k}'}{k_0'} \sin 2\varphi_{\mathbf{k}'}|_{eq}\, \delta\varphi_{\mathbf{k}'} \\
I_2 &= I_2^{(0)} + I_2^{(1)} + \mathcal{O}[(\delta\varphi_{\mathbf{k}})^2, (\delta\gamma_{\mathbf{k}})^2, (\delta\varphi_{\mathbf{k}}\delta\gamma_{\mathbf{k}})] \\
&= \int \frac{d\mathbf{k}'}{k_0'} \frac{|\mathbf{k}'|}{m} \sin 2\varphi_{\mathbf{k}}|_{eq} + 2 \int \frac{d\mathbf{k}'}{k_0'} \frac{|\mathbf{k}'|}{m} \cos 2\varphi_{\mathbf{k}}|_{eq}\, \delta\varphi_{\mathbf{k}'} \; .
\end{aligned}
\tag{22}
$$

The linearized form of kinetic equations for $\delta\varphi_{\mathbf{k}}$ and $\delta\gamma_{\mathbf{k}}$ are then obtained as

$$
\delta\dot{\varphi}_{\mathbf{k}} = -\frac{m|\mathbf{k}|d}{k_0}\delta\gamma_{\mathbf{k}} \tag{23}
$$

$$
\begin{aligned}
\delta\dot{\gamma}_{\mathbf{k}}\frac{m|\mathbf{k}|d}{k_0} &= 4[\mathbf{k}^2 + (1+d)^2m^2]\delta\varphi_{\mathbf{k}} + \\
&-4\left(\frac{g^2}{4\pi}\right)(\xi+1)|\mathbf{k}| \int d\mathbf{k}' \frac{|\mathbf{k}'|}{[(\mathbf{k}')^2 + (1+d)^2m^2]^{1/2}} \delta\varphi_{\mathbf{k}'} \;,
\end{aligned}
\tag{24}
$$

where the renormalization procedure (14) controls the logarithmic divergence of the integral appearing in (24) [see below Eqs.(35), and (39)]. Substituting (24) into (23) we obtain finally

$$
\begin{aligned}
&\delta\ddot{\varphi}_{\mathbf{k}} + 4[\mathbf{k}^2 + (1+d)^2m^2]\delta\varphi_{\mathbf{k}} + \\
&-4\left(\frac{g^2}{4\pi}\right)(\xi+1)|\mathbf{k}| \int d\mathbf{k}' \frac{|\mathbf{k}'|}{[(\mathbf{k}')^2 + (1+d)^2m^2]^{1/2}} \delta\varphi_{\mathbf{k}'} = 0.
\end{aligned}
\tag{25}
$$

As usual in small oscillation treatments, these are a linear oscillator equation. Note that the last term couples different momenta. The solution to this problem involves determining the normal modes of small oscillation and their frequencies. This is done by looking for solutions of Eqs. (23), and (24) which are of the form

$$
\begin{aligned}
\delta\varphi_{\mathbf{k}} &= \Psi_{\mathbf{k}}\, e^{i\omega t} \\
\delta\gamma_{\mathbf{k}} &= \Gamma_{\mathbf{k}}\, e^{i\omega t} \;,
\end{aligned}
\tag{26}
$$

where $\Psi_{\mathbf{k}}$ and $\Gamma_{\mathbf{k}}$ are time-independent amplitudes. Substituting Eq. (26) into Eqs. (23) and (24) results in equations for these amplitudes:

$$
i\omega\Psi_{\mathbf{k}} + \frac{m|\mathbf{k}|d}{k_0}\Gamma_{\mathbf{k}} = 0 \tag{27}
$$

$$
\begin{aligned}
&i\omega\frac{m|\mathbf{k}|d}{k_0}\Gamma_{\mathbf{k}} - 4[\mathbf{k}^2 + (1+d)^2m^2]\Psi_{\mathbf{k}} + \\
&+4\left(\frac{g^2}{4\pi}\right)(\xi+1)|\mathbf{k}| \int d\mathbf{k}' \frac{|\mathbf{k}'|}{[(\mathbf{k}')^2 + (1+d)^2m^2]^{1/2}} \Psi_{\mathbf{k}'} = 0 \; .
\end{aligned}
\tag{28}
$$

3 Analytical solution for the linear mean-field motion equations

In the last section, we have obtained the linearized version of the mean-field equations of motion which describe the time evolution of one-fermion densities of our system in the broken chiral symmetry phase ($d \neq 0$). They describe small amplitude motion of this system around the vacuum. We will next show how this equations can be solved analytically.

First, we rewrite the oscillation amplitude equations (27) and (28) as

$$
\Gamma_{\mathbf{k}} = -\frac{i\omega}{md}\frac{k_0}{|\mathbf{k}|}\Psi_{\mathbf{k}} \tag{29}
$$

$$
\frac{(k_0^{ef})^2}{k_0^{ef}}\Psi_{\mathbf{k}} - \left(\frac{g^2}{4\pi}\right)(\xi+1)\frac{|\mathbf{k}|}{(k_0^{ef})} \int d\mathbf{k}' \frac{|\mathbf{k}'|}{(k_0^{ef})'}\Psi_{\mathbf{k}'} = \frac{\omega^2}{4k_0^{ef}}\Psi_{\mathbf{k}} \;, \tag{30}
$$

where $(k_0^{ef})^2 = [\mathbf{k}^2 + (1+d)^2m^2] = [\mathbf{k}^2 + m_{ef}^2]$.

We look for a solution $\Psi_{\mathbf{k}}$ to the Eq. (30). The crucial point is to realize that we have a Lippmann-Schwinger-like equation with separable potential term [11]

$$\langle \mathbf{k}|V|\mathbf{k}'\rangle = v(\mathbf{k})v^*(\mathbf{k}') = \left(\frac{g^2}{4\pi}\right)(\xi+1)h(\mathbf{k})h(\mathbf{k}')$$

$$\text{where} \quad h(\mathbf{k}) = \frac{|\mathbf{k}|}{k_0^{ef}} \ . \tag{31}$$

The solution of such an equation is well know from scattering theory [6], and the Lippmann-Schwinger equation can be solved in closed form. A general solution to $\Psi_{\mathbf{k}}$ will have two terms. The first one is the free solution ($g=0$ vanishing potential) and represents an incident wave. The second term is the non trivial part (when $g \neq 0$) which couples different momenta, and is associated with the scattered wave.

Thus

$$\frac{|\mathbf{k}|}{k_0^{ef}}\Psi(\mathbf{k},\mathbf{q};\omega) = \alpha\delta(\mathbf{q}-\mathbf{k}) + $$

$$+\frac{1}{[(\mathbf{k}_0^{ef})^2-\omega^2/4+i\epsilon]}\left(\frac{g^2}{4\pi}\right)(\xi+1)\frac{\mathbf{k}^2}{k_0^{ef}}\int d\mathbf{k}'\frac{|\mathbf{k}'|}{(k_0^{ef})'}\Psi(\mathbf{k}',\mathbf{q};\omega) \quad , \tag{32}$$

where $\mathbf{q}$ is interpreted as the relative momentum for two incident quasi-fermions and α is an overall phase factor. We choose the outgoing wave condition ($+i\epsilon$) as solution of Eq.(30), but we could have chosen e.g. the incoming wave condition ($-i\epsilon$) or Van Kampen wave condition [12] or another condition.

Integrating the Eq.(32) with respect to $\mathbf{k}$

$$\int d\mathbf{k}\frac{|\mathbf{k}|}{(k_0^{ef})}\Psi(\mathbf{k},\mathbf{q};\omega) = \frac{\alpha}{\left\{1-(g^2/4\pi)(\xi+1)\int\frac{d\mathbf{k}'}{(k_0^{ef})'}\frac{(\mathbf{k}')^2}{[((k_0^{ef})')^2-\omega^2/4+i\epsilon]}\right\}} \tag{33}$$

and substituting this result back into (32) yields a general solution for $\Psi(\mathbf{k},\mathbf{q};\omega)$

$$\frac{|\mathbf{k}|}{k_0^{ef}}\Psi(\mathbf{k},\mathbf{q};\omega) = \alpha\delta(\mathbf{q}-\mathbf{k}) + \frac{\alpha k_0^{ef}}{[(\mathbf{k}_0^{ef})^2-\omega^2/4+i\epsilon]}\left(\frac{|\mathbf{k}|}{k_0^{ef}}\right)\frac{1}{\Delta^+(\omega)}\left(\frac{|\mathbf{k}|}{k_0^{ef}}\right) \ , \tag{34}$$

where $\Delta^+(\omega)$ is given by

$$\Delta^+(\omega) = \left(\frac{4\pi}{g^2}\right)\frac{1}{(\xi+1)} - \int\frac{d\mathbf{k}}{(k_0^{ef})}\frac{\mathbf{k}^2}{[(k_0^{ef})^2-\omega^2/4+i\epsilon]} \ . \quad ($$

The oscillation amplitude $\Gamma(\mathbf{k},\mathbf{q};\omega)$ is obtained from the Eqs.(29) and (34) and read:

$$\Gamma(\mathbf{k},\mathbf{q};\omega) = -\frac{i\omega\alpha k_0}{mdk_0^{ef}}\{\delta(\mathbf{q}-\mathbf{k})+$$

$$+\frac{k_0^{ef}}{[(k_0^{ef})^2-\omega^2/4+i\epsilon]}\left(\frac{|\mathbf{k}|}{k_0^{ef}}\right)\frac{1}{\Delta^+(\omega)}\left(\frac{|\mathbf{k}|}{k_0^{ef}}\right)\Bigg\} \ . \quad ($$

Finally, substituting (34) into (30) we obtain the oscillation frequencies ω

$$\omega = 2q_0^{ef} = 2[q^2+m_{ef}^2]^{1/2} \ , \quad ($$

where $\mathbf{q}$ is relative momentum for two incident quasi-fermion with mass $m_{ef} = (1+d)m$.

We observe that we can understand the factor 2 in the frequences of oscillation ω [Eq.(37)] as related to the treatment of harmonic oscillators in terms of the symplectic grou given by Goshen and Lipkin [13]. It can be interpreted classically by noticing that, since harmonic oscillators the frequency does not depend on the amplitude of the motion, if a of independent particles in a harmonic field is symmetrically stretched out of equilibrium will subsequently pulsate with frequency 2ω, where ω is the frequency of oscillation of independent particles.

4 Bound states from the small oscillations regime

In this section we will examine the condition for existence of bound states in the sm oscillation regime around the stationary solution (vacuum) of our fermionic system.

From Eqs.(30) and (31) we verify that the potential term which describes the time evo tion of our system in this regime is separable. Again, in analogy with scattering theory, can evaluate the corresponding T matrix [6]. We find

$$T(\mathbf{k},\mathbf{k}';\omega) \propto h(\mathbf{k}')\frac{1}{\Delta^+(\omega)}h(\mathbf{k}) \quad (:$$

with $h(\mathbf{k})$ given by (31) and $\Delta^+(\omega)$ given by (35). The bound states are given by the poles of the T matrix. Therefore, we search for the zeros of $\Delta^+(\omega)$. It is clear that the integral in $\Delta^+(\omega)$ contains a logarithmic divergence. To keep it under control, we use the renormalization procedure of the coupling constant given by (14). Substituting (14) into (35) we get

$$\Delta^+(\omega) = \ln\left(\frac{\Lambda^2}{m^2}\right) - \int_{-\Lambda}^{+\Lambda} \frac{d\mathbf{k}}{[\mathbf{k}^2 + m_{ef}^2]^{1/2}} \frac{\mathbf{k}^2}{[\mathbf{k}^2 + m_{ef}^2 - \omega^2/4 + i\epsilon]} \quad , \qquad (39)$$

where we introduce the regularizing momentum cut-off Λ.

In the interval $0 \leq \omega \leq 2m_{ef}$ the integral of $\Delta^+(\omega)$ is well defined and we can set $\epsilon = 0$. A straightforward calculation yields

$$\Delta^+(\omega) = 2\left[f(\omega) - j(d)\right] \qquad (40)$$

$$\text{where } f(\omega) = \left[\frac{4m_{ef}^2}{\omega^2} - 1\right]^{1/2} \arctan\left\{\left[\frac{4m_{ef}^2}{\omega^2} - 1\right]^{-1/2}\right\}$$

$$j(d) = \ln\left[\frac{2}{|1+d|}\right] \quad . \qquad (41)$$

The Fig.1 shows the zero of the $\Delta^+(\omega)$ as function of d. In this calculation $\mathbf{q} = 0$, therefore ω is the mass of the bound state. Obviously, when $(1+d) = 0$ (free system, see Ref.[1]) there is no bound state. We see from Fig.1 that a bound state of quasi-fermions occurs when $0.74 \leq (1+d) \leq 2$, and that the mass of this bound state will vary in the interval $0 \leq \omega \leq 2m_{ef}$.

D. Gross and A. Neveu obtain $M_\sigma = 2M_F$ [2] for the mass of the σ particle in leading-$1/N$ aproximation, where M_F is equivalent to m_{ef}. They argue that in higher order they might find that

$$M_\sigma = 2M_F\left[1 + \mathcal{O}(1/N)\right] \quad .$$

From Fig.1 we verify that $\omega = 2m_{ef}$ corresponds to $(1+d) = 2$. Observing that $j(1+d=2) = 0$, we may conclude that $j(d)$ can be see as a contribuition of higher order to the Gross-Neveu result.

We believe that in the limit $N \to \infty$ the function $j(d) \to 0$. On the other hand, when N is finite ($N = 1$ for instance), the mass ω of the bound state depends of the renormalized coupling constant d as shown in Fig.1. This dependence can not be obtained from $1/N$ approximation.

Therefore, we can conclure that to $N \to \infty$ the $1/N$ approximation and our mean-f approximation are equivalents. On the other hand, when N is finite our approxima permits to obtain the higher order contribution to the Gross-Neveu result [2].

It is important to observe that the higher order term obtained in the Eqs.(40), (41) from our approach contains no necessarily all terms of $1/N$ order, since the mean-f approximation is not an $1/N$ expansion.

Surprisingly, in Ref.[11] A. Kerman and C-Y. Lin, while studying the bosonic $\lambda\phi^4$ th through a time-dependent variational approach, have obtained for the $\Delta^+(\omega)$ function analogous structure as the one we have found.

Finally, when $\omega > 2m_{ef}$ the integrand of $\Delta^+(\omega)$ has a singularity at $\mathbf{k} = \pm\sqrt{\omega^2/4 -$ From theory of residues we obtain

$$\Delta^+(\omega) = \left(1 - \frac{4m_{ef}^2}{\omega^2}\right)^{1/2} \ln\left[\frac{1 + (1 - 4m_{ef}^2/\omega^2)^{1/2}}{1 - (1 - 4m_{ef}^2/\omega^2)^{1/2}}\right] + $$

$$- \ 2\ln\left(\frac{2}{|1+d|}\right) - i\pi\left[1 - \frac{4m_{ef}^2}{\omega^2}\right]^{1/2} \quad .$$

Now $\Delta^+(\omega)$ does not have any zeros. The interesting point here is to observe that

$$\lim_{\omega\to\infty} \Delta^+(\omega) \longrightarrow \ln\left(\frac{\omega^2}{m_{ef}^2}\right) \longrightarrow \infty \quad ,$$

so that in the large frequency limit ($\omega \to \infty$) the T matrix goes asymptotically to zero. thus recover, in the present approximation, the assymptotically free character of the CGN

5 Discussion and conclusions

In Ref.[1] we showed a way to treat the initial-values problem in quantum field theory, bot mean-field approximation and in a richer approximation allowing for dynamical correlati effects. Although the formalism is quite general, we have specialized it to the treatment relativistic many-fermion system described by Chiral Gross-Neveu model (CGNM).

We obtained the renormalized kinetic equations which describe the effective dynamic the set of one-body variables in the mean-field approximation and in broken chiral symm phase to a relativistic uniform (1+1) dimensional fermion system described by CGNM.

ihowed that previous static results such as dynamical mass generation due the chiral symme- ;ry breaking and an analogous phenomenon of dimensional transmutation, can be retrieved 'rom this formalism in mean-field approximation.

In this work, we have linearized the mean-field kinetic equations obtained in Ref.[1] around :he stationary solution (vacuum). The two-quasi-fermion physics can be analytically inves- :igated from this approach. In particular, we have solved this equations completely. From :hese solutions, we have reinterpreted the near equilibrium physics of our system as a problem of quasi-fermion scattering and have found the condition for the existence of a quasi-fermion bound state.

We verify that for N finite (in this work $N = 1$) the bound state mass obtained from our approach contains a term which depends of the renormalized coupling constant as can be seen in the Fig.1. In the case of an $1/N$ expansion [2] this dependence can not be found, so in the limit $N \to \infty$ this term goes to zero. Therefore, to small N, our approach permits to obtain the higher order contribuition to the $1/N$ expansion.

Finally, it is important to observe that the higher order term obtained to the bound state mass from our approach contains no necessarily all terms of $1/N$ order, since the mean-field approximation is not an $1/N$ expansion.

Acknowledgments

One of the authors (P.L.N.) was supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Brazil; and by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP), Brazil.

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Figure Captions

Figure 1 - The curve represents the mass ω of the quasi-fermion bound state in sma oscillation regime for our system as a function of the renormalized coupling constant d.