UNIVERSIDADE DE SÃO PAULO

INSTITUTO DE FÍSICA
CAIXA POSTAL 66318
05389-970 SÃO PAULO - SP
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HIGHER SPIN CONSTRAINTS AND THE SUPER $(W_{\infty/2} \oplus W_{(1+\infty)/2})$ ALGEBRA IN THE SUPER EIGENVALUE MODEL

L.O. Buffon[1], **D. Dalmazi**[2] and **A. Zadra**[1]
[1]Instituto de Física, Universidade de São Paulo

[2]UNESP, Campus de Guaratinguetá
CP. 205, Guaratinguetá, São Paulo, Brazil
dalmazi@grt000.uesp.ansp.br

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Higher spin constraints and the super $\left(W_{\frac{\infty}{2}} \oplus W_{\frac{1+\infty}{2}}\right)$ algebra in the super eigenvalue model

L. O. Buffon[1], D. Dalmazi[2] and A. Zadra[1]

[1] *Instituto de Física, Universidade de São Paulo*
CP 66318, 05389-970, São Paulo, Brazil
lobuffon@uspif.if.usp.br, azadra@uspif.if.usp.br

[2] *UNESP, Campus de Guaratinguetá*
CP 205, Guaratinguetá, São Paulo, Brazil
dalmazi@grt000.uesp.ansp.br

Abstract

We show that the partition function of the super eigenvalue model satisfies an infinite set of constraints with even spins $s = 4, 6, \cdots, \infty$. These constraints are associated with half of the bosonic generators of the super $\left(W_{\frac{\infty}{2}} \oplus W_{\frac{1+\infty}{2}}\right)$ algebra. The simplest constraint ($s = 4$) is shown to be reducible to the super Virasoro constraints, previously used to construct the model. All results hold for finite N.

1 Introduction

Some time ago, Kazakov showed [1] that the discrete hermitian one-matrix model exhibits a transition to a massless phase. In the continuum limit, it describes the $(p, q) = (2, 2k - 1)$, $k = 2, 3, 4, \ldots$, minimal models conformally coupled to $2D$-gravity. One of the basic features of this model is the presence of the Virasoro constraints satisfied by its partition function. These constraints can be derived by various methods [2, 3, 4]. Indeed, they hold even before the phase transition takes place (see for instance [3, 4]). In [4], the Virasoro constraints were shown to be a consequence of a set of Schwinger-Dyson (S-D) equations associated with the differential operators $l_n = -\sum_{i=1}^{N} x_i^{n+1}\partial_i$ $(n \geq -1)$, where x_i $(i = 1, \ldots, N)$ are the eigenvalues of the hermitian $N \times N$ matrix. This fact rises the following question: what is the rôle of the S-D equations associated with the higher order (or higher spin) differential operators $W_n^s = \sum_{i=1}^{N} x_i^n \partial_i^{s-1}$ $(n \geq 0, s \geq 2)$? This set contains the Virasoro generators l_n and forms a $W_{1+\infty}$ algebra. As shown in [5], each operator W_n^s gives rise to a S-D equation which, on its turn, originates a constraint on the partition function. However, such constraints can be reduced to the Virasoro constraints, at least for the spins $s = 3, 4$.

The main purpose of this work is to address the issue of higher spins constraints in the super eigenvalue model. This supersymmetric discrete model was proposed in [6] as a way to describe some minimal models coupled to $2D$-supergravity. It is supposed to be a supersymmetric extension of the effective bosonic eigenvalue model. We will show that there actually is an infinite set of differential operators which give rise to S-D equations and corresponding higher spin constraints on the super eigenvalue partition function. However, opposing the bosonic theory, we only find even spin constraints ($s = 4, 6, 8, \ldots$). Furthermore, the corresponding differential operators seem to be (at least for $s = 4, 6$) linear combinations of half of the bosonic generators of the super algebra $\left(W_{\frac{\infty}{2}} \oplus W_{\frac{1+\infty}{2}}\right)$ [7, 8]. This algebra contains the $N = 1$ superconformal algebra and forms a natural $N = 1$ supersymmetric extension of the W_n^s operators. The simplest constraint, with spin $s = 4$, is worked out explicitly. As in the bosonic model, it can be reduced to the super Virasoro constraints.

This paper is organized as follows. In section II, we review the results on the bosonic theory, based on reference [5]. In section III, we obtain the higher spin constraints in the super eigenvalue model and prove that the constraint $s = 4$ is reducible. In section IV, we relate the super $\left(W_{\frac{\infty}{2}} \oplus W_{\frac{1+\infty}{2}}\right)$ algebra to the higher spins constraints. Section V contains a brief summary of the results and comments on the reducibility of the constraints with $s > 4$.

2 Higher spin constraints in the hermitian one-matrix model

The partition function of the hermitian one-matrix model is given by:

$$Z = \int \mathcal{D}M \exp\left(-N \sum_{k=0}^{\infty} g_k \, Tr\, M^k\right) = \int \left(\prod_{i=1}^{N} dx_i e^{V(x_i)}\right) \Delta_N^2 \quad , \tag{1}$$

where x_i $(i = 1, \ldots, N)$ are the eigenvalues of hermitian $N \times N$ matrix M; $\mathcal{D}M$ is the flat measure; $\Delta_N = \prod_{i<j=1}^{N}(x_i - x_j)$ is the van der Monde determinant and

$$V(x_i) = -N \sum_{k=0}^{\infty} g_k x_i^k \tag{2}$$

is the potential, which depends on the coupling constants g_k. By making infinitesimal non-singular conformal transformations, $\delta x_i = [x_i, \epsilon_n l_n] = \epsilon_n x_i^{n+1}$, which are generated by the differential operators $l_n = -\sum_{i=1}^{N} x_i^{n+1}\partial_i$, one derives the Virasoro constraints:

$$\hat{L}_n Z = 0 \quad , \quad n \geq -1 \quad , \tag{3}$$

where

$$\hat{L}_n = \sum_{k \geq 0} k g_k \frac{\partial}{\partial g_{n+k}} + \frac{1}{N^2} \sum_{\mu=0}^{n} \frac{\partial^2}{\partial g_\mu \partial g_{n-\mu}} \quad . \tag{4}$$

The operators $\hat{L}_n$ and l_n satisfy the same algebra, $[\hat{L}_n, \hat{L}_m] = (n - m)\hat{L}_{n+m}$. The Virasoro constraints (3) will be henceforth called spin two ($s = 2$) constraints.

In [5], the authors investigated the higher spin ($s > 2$) constraints associated with the operators $W_n^s = \sum_{i=1}^{N} x_i^{n+1}\partial_i^{s-1}$ $\quad (n \geq -1 \ , \ s \geq 1)$, which generate the $W_{1+\infty}$ algebra.

The infinitesimal transformations associated with W_n^s for $s > 2$ cannot be written in a local form in x_i configuration space. Therefore it is convenient to derive the constraints from the S-D equations. These equations follow from integrals of total derivatives, which must be suitably chosen because the action of the operators W_n^s on $\Delta_N^2 e^{\sum_{i=1}^N V(x_i)}$ is, in general, rather complicated. The solution to this puzzle comes from the following property of the van der Monde determinant:

$$\sum_{i=1}^N \partial_i^s \Delta_N = 0 \quad , \tag{5}$$

where $s \geq 1$ is an integer. As shown in appendix 1, this property implies[1]

$$\left(\sum_{i=1}^N \frac{1}{p - x_i} \partial_i^{s-1}\right) \Delta_N = \Delta_N \frac{(\partial + w(p))^s \cdot 1}{s} \quad . \tag{6}$$

We have introduced the loop variable $w(p) = \sum_{i=1}^N \frac{1}{p-x_i} = \sum_{i=1}^N \sum_{n\geq 0} \frac{x_i^n}{p^{n+1}}$ and the notation $\partial \equiv \partial/\partial p$. Equation (6) can be generalized [5] as follows,

$$W_s(p)\left(\Delta_N e^{\beta V}\right) = \Delta_N e^{\beta V} \frac{[(\partial + w(p) + \beta V'(p))^s \cdot 1]_-}{s} \quad ,$$

$$W_s^\dagger(p)\left(\Delta_N e^{\beta V}\right) = -(\Delta_N e^{\beta V}) \frac{[(\partial - (w(p) + \beta V'(p)))^s \cdot 1]_-}{s} \quad , \tag{7}$$

where $[f(p)]_-$ means only negative powers of p; β is a real constant and $W_s(p)$ ($W_s^\dagger(p)$) is the resolvent operator for the differential operators W_n^s ($W_n^{\dagger\, s}$):

$$W_s(p) = \sum_{i=1}^N \frac{1}{p - x_i} \partial_i^{s-1} = \sum_{n\geq 0} \frac{W_{n-1}^s}{p^{n+1}} \quad ,$$

$$W_s^\dagger(p) = (-1)^{s-1} \sum_{i=1}^N \partial_i^{s-1} \frac{1}{p - x_i} \quad . \tag{8}$$

The simplicity of the r.h.s. of equation (7) suggests [5] that we take the identities

$$\begin{aligned} &\int \prod_{i=1}^N dx_i \left[\Delta_N e^{(1-\alpha)\sum_i V(x_i)} W_s\left(\Delta_N e^{\alpha \sum_i V(x_i)}\right) - \Delta_N e^{\alpha \sum_i V(x_i)} W_s^\dagger\left(\Delta_N e^{(1-\alpha)\sum_i V(x_i)}\right)\right] \\ &= 0 \quad , \end{aligned} \tag{9}$$

where α is an arbitrary real constant. Using (7) we have the S-D equations for $s = 2, 3, 4$, respectively,

$$\left\langle \left[(\partial\phi)^2\right]_- \right\rangle = 0 \quad , \tag{10}$$

$$\left\langle \left[(\partial + (2\alpha - 1)V')(\partial\phi)^2\right]_- \right\rangle = 0 \quad , \tag{11}$$

[1] The opposite is also true: from (6) one can prove (5) by induction.

$$\begin{aligned} &\left\langle \left[(\partial\phi)^4 - \left(\partial^2\phi\right)^2 + 2\partial^2\left(\partial\phi\right)^2\right]_- \right\rangle + 3(2\alpha - 1)\left\langle \left[\partial\left(V'\left(\partial\phi\right)^2\right)\right]_- \right\rangle \\ &\qquad + \frac{3}{2}(2\alpha - 1)^2 \left\langle \left[V'^2\left(\partial\phi\right)^2\right]_- \right\rangle = 0 \quad . \end{aligned} \tag{12}$$

where $V'(p) = -N\sum_k k g_k p^{k-1}$ and $\partial\phi \equiv w(p) + \frac{1}{2}V'(p)$ behaves like a spin one current. Equation (10) is the so called loop equation which can be solved perturbatively in $1/N$. Notice that, if we choose $\alpha = 1/2$, all equations will be written in terms of $\partial\phi$ and its derivatives only.

We stress that the above equations also hold for the reduced models, i.e. when $g_k = 0$ for some $k > m$. However, only in the general case ($g_k \neq 0$ for any k), we can rewrite them as constraints on the partition function Z. Using the property

$$-\frac{1}{N}\frac{\partial}{\partial g_n} e^V = \sum_{i=1}^N x_i^n e^V \quad , \tag{13}$$

and introducing the loop operator

$$\hat{w} = -\frac{1}{N}\sum_{n\geq 0} \frac{1}{p^{n+1}} \frac{\partial}{\partial g_n} \quad , \tag{14}$$

equation (10) becomes

$$(\hat{w}^2 + V'\hat{w})_- Z = \sum_{n\geq -1} \frac{\hat{L}_n}{p^{n+2}} Z \quad , \tag{15}$$

where $\hat{L}_n$ was given in (4). Therefore we recover the constraints (3). Analogously, equations (11) and (12) give rise to further constraints on the partition function,

$$\hat{W}_\mu^3 Z = 0 \quad , \quad \mu \geq -2 \quad ;$$

$$\hat{W}_v^4 Z = 0 \quad , \quad v \geq -3 \quad ;$$

where the operators $\hat{W}_\mu^3$ and $\hat{W}_v^4$ can be written as

$$\hat{W}_\mu^{(3)} = (\mu + 2)\hat{L}_\mu + N(2\alpha - 1)\sum_{k\geq 0} k g_k \hat{L}_{\mu+k} \quad , \tag{16}$$

$$\begin{aligned} \hat{W}_v^{(4)} \;=\; &\frac{3}{2}(v+2)(v+3)\hat{L}_v + 3N(2\alpha - 1)(v+3)\sum_{k\geq 0} k g_k \hat{L}_{v+k} + \sum_{n=-1}^{v+1} \hat{L}_n \hat{L}_{v-n} \\ &+ N^2(2 + 6\alpha(\alpha - 1))\sum_{k,k'} k k' g_k g_{k'} \hat{L}_{v+k+k'} - 2N\sum_{k\geq 0}\sum_{n=0}^{v+k+1} k g_k \frac{\partial}{\partial g_n}\hat{L}_{v+k-n} \end{aligned} \tag{17}$$

As stressed in [5], the $s = 3, 4$ constraints are reducible to the $s = 2$ Virasoro constraints and therefore impose no further restrictions on Z. It has been conjectured in [5] that this should also hold when $s > 4$, although no proof is available. Now a comment is in order: if the algebra of the $\hat{W}^{(s)}_{\mu}$ constraints were isomorphic to the algebra of the differential operators $W^s_\mu = \sum_i x_i^{\mu+1}\partial_i^{s-1}$, from which they indirectly come, then it should be obvious that $\hat{W}^{(s)}_\mu Z = 0$ for $s > 4$, because any operator $W^{(s)}_n$ can be obtained from $W^{(2)}_n$ and $W^{(3)}_n$ via commutators. However, the operators $s = 2$ and 3 obey the commutation relation

$$[\hat{L}_m, \hat{W}^3_\mu] = -2m(m+1)\hat{L}_{\mu+m} + (2m-\mu)\hat{W}^{(3)}_{\mu+m} + \frac{2(2\alpha-1)}{N}\sum_{k=0}^{m} k\frac{\partial}{\partial g_{m-k}}\hat{L}_{\mu+k} \qquad (18)$$

The first two terms on the r.h.s. of (18) correspond to $[-\sum_i x_i^{m+1}\partial_i, 2\sum_i x_i^{\mu+2}\partial_i^2]$, but the last one breaks the isomorphism. These commutators may be isomorphic only for $\alpha = 1/2$, when $\hat{W}^3_\mu(\alpha = 1/2) = (\mu+2)\hat{L}_\mu$. However, after calculating the commutation relations between higher spin operators, we concluded that there is no value of α for which the algebras (of constraints and differential generators) are isomorphic.

3 Higher spin constraints in the supereigenvalue model

The partition function for the super eigenvalue model (Z^S) was introduced in [6] and reads:

$$Z^S = \int \mathcal{D}\mu\ \Delta^S_N e^{\sum_{i=1}^N (V(x_i)+\psi(x_i)\theta_i)} \quad , \qquad (19)$$

where

$$\mathcal{D}\mu = \prod_{i=1}^N dx_i d\theta_i \quad , \quad \Delta^S_N = \prod_{i<j=1}^N (x_i - x_j - \theta_i\theta_j) \quad ,$$

$$V(x_i) = -N\sum_{k=0}^\infty g_k x_i^k \quad , \quad \psi(x_i) = -N\sum_{k=0}^\infty \psi_k x_i^k \quad , \qquad (20)$$

If one makes infinitesimal non-singular superconformal transformations

$$\delta x_i = [x_i, \epsilon_n g_{n+1/2} + \alpha_n l_n] = -\epsilon_n\theta_i x_i^{n+1} + \alpha_n x_i^{n+1} \quad ,$$

$$\delta\theta_i = [\theta_i, \epsilon_n g_{n+1/2} + \alpha_n l_n] = \epsilon_n x_i^{n+1} + \frac{(n+1)}{2}\alpha_n\theta_i x_i^n \quad ,$$

where ϵ_n are grassmann-odd parameters, one arrives at the following constraints:

$$\hat{G}_{n+1/2}Z^S = 0 = \hat{L}^S_n Z^S \quad , \quad n \geq -1 \quad . \qquad (21)$$

The operators $G_{n+1/2} = G_{n+1/2}(g_k, \frac{\partial}{\partial g_k}, \psi_k, \frac{\partial}{\partial\psi_k})$ and $\hat{L}^S_n = \hat{L}^S_n(g_k, \frac{\partial}{\partial g_k}, \psi_k, \frac{\partial}{\partial\psi_k})$ can be found in the literature (see page 156 of [9]). They satisfy a subalgebra of the $N = 1$ superconformal algebra, which is isomorphic to the algebra of the differential operators:

$$g_{n+1/2} = \sum_{i=1}^N x_i^{n+1}(\theta_i\partial_i - \Pi_i) \quad , \quad \Pi_i = \frac{\partial}{\partial\theta_i} \quad , \qquad (22)$$

$$l_n = -\sum_{i=1}^N\left(x_i^{n+1}\partial_i + \frac{(n+1)}{2}x_i^n\theta_i\Pi_i\right) \quad , \qquad (23)$$

namely,

$$\{g_{n+1/2}, g_{m+1/2}\} = 2l_{n+m+1} \quad , \quad \{l_n, l_m\} = (n-m)l_{n+m} \quad ,$$

$$\{l_n, g_{m+1/2}\} = \frac{(n-1-2m)}{2}g_{n+m+1/2} \quad . \qquad (24)$$

The constraints $\hat{G}_{n+1/2}Z = 0$ and $\hat{L}^s_n Z = 0$ correspond to spins $s = 3/2$ and 2 respectively.

Inspired by the results of the previous section, it is possible to obtain the $s = 3/2, 2$ constraints from the following identity,

$$\int \mathcal{D}\mu\ \left[e^U(W^s(p)\Delta^S_N) - \Delta^S_N(W^{\dagger\, s}(p)e^U)\right] = 0 \quad , \qquad (25)$$

which is simply an integral of a total derivative. Above, $U = \sum_{i=1}^N(V(x_i)+\psi(x_i)\theta_i)$; $W^s(p) = \sum_{n\geq 0}\frac{1}{p^{n+1}}W^s_n$, where W^s_n is the first order differential operators for $s = 3/2, 2$ given in (22) and (23) respectively. These identities can be written as S-D equations,

$$\left\langle [T_{3/2}]_- \right\rangle \equiv \langle [(\partial\Phi)\Psi]_- \rangle = 0 \quad , \qquad (26)$$

$$\langle [T]_- \rangle \equiv \left\langle [(\partial\Phi)^2 + (\partial\Psi)\Psi]_- \right\rangle = 0 \quad , \qquad (27)$$

where we have introduced the notation $\partial\Phi(p) = w(p) + V'(p)$ and $\Psi(p) = \nu(p) + \psi(p)$; $w(p) = \sum_{i=1}^N \frac{1}{p-x_i}$ and $\nu(p) = \sum_{i=1}^N \frac{\theta_i}{p-x_i}$ are the super-loop variables and (26), (27) are called super-loop equations [2] which give rise to the constraints (21)

$$:(\hat{T}_{3/2})_- : Z^S = [(\partial\hat{\Phi}\hat{\Psi})_-]Z^S = 0 \quad ,$$

$$:[\hat{T}]_- : Z^S =: [(\partial\hat{\Phi})^2 + (\partial\hat{\Psi})\hat{\Psi}]_- : Z^S = 0 \quad . \qquad (28)$$

The operator $\hat{w}$ was given in (14) and $\hat{\nu} = -\frac{1}{N}\sum_{n\geq 0}\frac{1}{p^{n+1}}\frac{\partial}{\partial\psi_n}$.

Now we turn to the higher order differential operators ($s > 2$). Since it seems that there is no supersymmetric analog of the property (5) for Δ^S_N, we found convenient to factorize the bosonic van der Monde determinant Δ_N from Δ^S_N, by writing

$$\Delta^S_N(x_i - x_j - \theta_i\theta_j) = \Delta_N(x_i - x_j)e^F \quad , \qquad (29)$$

where we define the following function:

$$F = -\frac{1}{2}\sum_{i\neq j}\frac{\theta_i\theta_j}{x_i - x_j} \quad . \qquad (30)$$

For even spin s, there is a remarkably simple formula for the action of some differential operator of spin s ($\mathcal{O}^{(s)}$) on the fermionic part of Δ^S_N. It is given by the equation

$$\mathcal{O}^{(s)}e^F \equiv \sum_{i=1}^{N}\left(\frac{1}{p-x_i}\partial_i^{s-1}+\partial_i^{s-1}\frac{1}{p-x_i}-\partial_p^{s-1}\frac{\theta_i}{p-x_i}\Pi_i\right)e^F=\left[(\partial_p^{s-1}\nu)\nu-\partial_p^s w\right]e^F \quad (31)$$

which is demonstrated in appendix 2. Higher spin constraints are thus obtained from the identity

$$\int \mathcal{D}\mu\left[e^U\Delta_N(\mathcal{O}^{\dagger(s)}(p)e^F)\right]-\int\mathcal{D}\mu\left[\mathcal{O}^{(s)}(p)(e^U\Delta_N)e^F\right]=0 \quad , \quad (32)$$

where $\mathcal{O}^{\dagger(s)} = -\mathcal{O}^{(s)} - (\partial_p^{s-1}w)$. The second integral in (32) can be written as a local (although rather complicated) function of w, ν and its derivatives (see appendix 2). Finally, we obtain an infinite set of S-D equations associated with even spin differential operators. For $s=2$, we recover from (32) the bosonic loop equation (27), which is associated with the Virasoro constraints. For the next spin, $s=4$, we obtain:

$$\left\langle\left[\frac{1}{2}(\partial\Phi)^4+2\partial^3\Phi\partial\Phi+\frac{3}{2}(\partial^2\Phi)^2+\partial^3\Psi\Psi\right]_-\right\rangle=0 \quad . \quad (33)$$

The above S-D equation can be rewritten as a constraint,

$$\left[:\frac{\hat{T}^2}{2}+\hat{T}_{3/2}\partial\hat{T}_{3/2}+\partial^2\hat{T}-\left(\frac{(\partial^2\hat{\Phi})^2}{2}+\hat{\Psi}''\hat{\Psi}'\right):\right]_- Z^S=0 \quad . \quad (34)$$

In order to relate it to the constraints (28), we split $\hat{T}$ and $\hat{T}_{3/2}$ in parts with negative and non-negative powers of p, that is $\hat{T}=\hat{T}_-+\hat{T}_+$, $\hat{T}_{3/2}=\hat{T}_{3/2(-)}+\hat{T}_{3/2(+)}$. We end up with the equation

$$\left[\frac{:\hat{T}_-::\hat{T}_-:}{2}+:\hat{T}_+::\hat{T}_-:+:\hat{T}^+_{3/2}::\hat{T}'^-_{3/2}:-:\hat{T}'^+_{3/2}::\hat{T}^-_{3/2}:+:\hat{T}^-_{3/2}::\hat{T}'^-_{3/2}:+:\partial^2\hat{T}_-:\right.$$
$$\left.-:\left(\frac{(\partial^2\Phi)^2}{2}+\Psi''\Psi'\right):+\ 21\ \text{commutators}\right]_- Z^S=0 \quad . \quad (35)$$

We stress that $\Psi(p)$ and $\partial\Phi(p)$ behave like a two dimensional free fermion and a spin one currents, respectively. The commutators between these quantities, calculated at the same point are ill defined in general. However, due to the projections on negative and non-negative frequencies and because we only care for commutators acting on the partition function, the calculations can be done without ambiguities. In appendix 3 we work out a sample calculation explicitly.

After collecting the results, we find:

$$\left[21\ \text{commutators}-:\left(\frac{(\partial^2\Phi)^2}{2}+\Psi''\Psi'\right):\right]_- Z^S=\left(\frac{\partial^2\hat{T}}{4}\right)_- Z^S \quad . \quad (36)$$

Therefore, from equations (35) and (36), we conclude that the $s=4$ constraint on Z^S is automatically satisfied as long as Z^S already obeys the $s=3/2,2$ constraints (super-loop equations). We do not have a proof of reducibility for the $s>4$ constraints.

4 Super $\left(W_{\frac{\infty}{2}}\oplus W_{\frac{1+\infty}{2}}\right)$ algebra and the higher spin constraints

Instead of looking for identities like (31), we could have asked ourselves what are the differential operators that extend the $N=1$ superconformal operators $g_{n+1/2}$ and l_n to infinitely higher spins, in the same way the operators $W_n^s=\sum_{i=1}^N x_i^{n+1}\partial_i^{s-1}$ extend the Virasoro generators l_n. In other words, what is the $N=1$ supersymmetric analog of the algebra generated by W_n^s $(s\geq 2)$? Interesting enough, the answer to this question seems to be unique [10]. Namely, if we define the spin of a fermionic (bosonic) differential generator as 1/2 (1) plus the highest power of the operator ∂ present in the generator, we have the following ansatz for the generator $W_n^{5/2}$:

$$W_{m-1}^{5/2}=\sum_{i=1}^N\left[x_i^m\theta_i\partial_i^2+c_m x_i^m\partial_i\Pi_i+d_m x_i^{m-1}\Pi_i+e_m x_i^{m-2}\theta_i\right] \quad . \quad (37)$$

It is the most general ansatz [10] compatible with the spin composition rule $[W_m^s,W_n^{s'}]_+ = f_{mn}^{ss'}W_{n+m}^{s+s'-1}+$ *lower spins*, which is obeyed by super W-algebras. If we require a closed algebra with $g_{n+1/2}$ and l_n, the coefficients in (37) get fixed: $c_m=1$, $d_m=m$ and $e_m=0$. Simultaneously, a spin 2 generator $(\widetilde{W}_m^2)$ must be defined as:

$$\widetilde{W}_m^2=\sum_{i=1}^N x_i^{m+1}(1-\theta_i\Pi_i)\partial_i \quad . \quad (38)$$

No further ansatz or definitions are needed and the higher spin generators can be obtained by the algebra of (anti)commutators of the generators W_m^s $s\leq 5/2$, already defined. Curiously, no odd spin generators come out and there is a doubling of even spin generators at each spin level (see [10] for details). For instance, at $s=4$ we have

$$W_n^4=\sum_{i=1}^N\left[2x_i^{n+2}\partial_i^3+3(n+2)x_i^{n+1}\partial_i^2+(n+1)(n+2)x_i^n\partial_i-(n+2)x_i^{n+1}(1-\theta_i\Pi_i)\partial_i^2\right]$$

$$\widetilde{W}_n^4=\sum_{i=1}^N(1-\theta_i\Pi_i)\left(x_i^{n+1}\partial_i^3+(n+1)x_i^n\partial_i^2\right) \quad . \quad (39)$$

Apparently, the super algebra so obtained was first discovered by the authors of reference [7] (see also [8]), where it was called [2] Super $W_{\frac{\infty}{2}}$. We believe that this algebra corresponds to the $N=1$ analog of the algebra of W_n^s $(s\geq 2)$ generators. We have tried, in vain, to derive fermionic constraints on Z^S from the fermionic differential operators W_n^s. However, the situation for the even spin bosonic generators looks much better. We found a sharp connection between the differential operators $\mathcal{O}^{(s)}$ of the last section and the bosonic W_n^s operators. By using $\frac{1}{(p-x_i)}=\sum_{n\geq 0}x_i^n/p^{n+1}$ we have, from (31) and (39),

[2] We prefer to call it super $\left(W_{\frac{\infty}{2}}\oplus W_{\frac{1+\infty}{2}}\right)$ algebra, since it is possible to redefine the bosonic generators W_n^s and $\widetilde{W}_n^s$ such that they split in two decoupled algebras, $W_{\frac{\infty}{2}}$ and $W_{\frac{1+\infty}{2}}$, which correspond to truncations into even spins of the W_∞ and $W_{1+\infty}$ algebras.

$$\mathcal{O}^{(2)}(p)e^F = \sum_{n\geq 0} \frac{W^2_{n-1}}{p^{n+1}} e^F \quad ,$$

$$\mathcal{O}^{(4)}(p)e^F = \sum_{n\geq 0} \frac{1}{p^{n+1}} \left[W^4_{n-2} + 2n(n-1)W^2_{n-3}\right] e^F \quad , \tag{40}$$

$$\mathcal{O}^{(6)}(p)e^F = \sum_{n\geq 0} \frac{1}{p^{n+1}} \left[W^6_{n-3} + 3n(n-1)W^4_{n-4} + 2n(n-1)(n-2)(n-3)W^2_{n-5}\right] e^F.$$

Therefore, the differential operators, which give rise to the higher even-spin constraints on Z^S, are simple linear combinations of even spin W^s_n generators of the super $\left(W_{\frac{\infty}{2}} \oplus W_{\frac{1+\infty}{2}}\right)$ algebra. It must be stressed that the W^s_n generators have very specific numerical factors in their definitions (see (39)) and the above identification is a rather non-trivial result, since $\mathcal{O}^{(s)}$ and W^s_n were found by completely different methods.

Some comments are in order. First, we have used the identity $\theta_i\partial_i e^F = 0$ to write (40). Therefore, we can say that the differential operators W^s_n differ from the differential operators contained in $\mathcal{O}^{(s)}(p)$ by terms proportional to $\theta_i\partial^m_i$ $(m \geq 1)$. It would thus be impossible to guess W^s_n from $\mathcal{O}^{(s)}$. However, the formula (31) could have been obtained from $W^s_n e^F$. Furthermore, since we have a doubling of even spin differential operator (W^s_n and $\widetilde{W}^s_n$) one might also expect a doubling of even spins constraints at each spins s level. However, we have not been able to get any constraint or S-D equation associated with the $\widetilde{W}^s_n$ operators for $s > 2$. For $s = 2$ the operators $\widetilde{W}^2_n$, which obey the same algebra of the Virasoro generators W^2_n, lead to the same constraint (28).

5 Summary and final remarks

We have derived an infinite set of even spin ($s = 2r = 2, 4, 6, ...$) constraints on the partition function of the supereigenvalue model which include the Virasoro constraints ($s = 2$) previously found in [6]. All constraints can be written in terms of the superloop variables $w(p)$ and $\nu(p)$ and the potentials $\psi(p)$, $V(p)$. We have shown that the constraints at $s = 4$ are reducible to the $s = 3/2$ and $s = 2$ super Virasoro constraints (super-loop equations), but we do not have a proof of the reducibility for higher spins. The situation is very similar to the bosonic eigenvalue model. We were naturally led to define super differential operators which are the $N = 1$ supersymmetric version of $x^n_i\partial^{s-1}_i$ $(s \geq 2,\ n \geq 0)$. Those operators satisfy the super $\left(W_{\frac{\infty}{2}} \oplus W_{\frac{1+\infty}{2}}\right)$ algebra, whose bosonic sector possesses a doubling of even spin operators (W^s_n and $\widetilde{W}^s_n$) at each spin s level. The constraints thus obtained are associated with linear combinations of the bosonic operators W^s_n. The algebraic meaning of such combinations is unclear.

We have two important remarks. First, all results the we have obtained are non perturbative and hold for finite N. Second, although we have not been able to derive any constraints associated with the remaining bosonic differential operators $\widetilde{W}^s_n$ (except for $\widetilde{W}^2_n$ which also gives rise to the Virasoro constraints) and the fermionic ones W^s_n $(s = 5/2, 7/2, \cdots)$, we conjecture that such constraints do exist and are reducible. For the same reasons we suspect that the even spin constraints for $s > 4$ are also reducible. Our conjecture is based on the super $\left(W_{\frac{\infty}{2}} \oplus W_{\frac{1+\infty}{2}}\right)$ algebra and the association between the constraints $\hat{G}_{n+1/2}Z^S = 0$, $\hat{\mathcal{O}}^s_n Z^S = 0$ and the differential operators $W^{3/2}_n \equiv g_{n+1/2}$ and W^s_n $(s = 2, 4, 6, \cdots)$. From (anti)commutators between $g_{n+1/2}$ and W^s_n $(s = 2, 4)$ we can generate the remaining differential operators and thus, we expect that the remaining constraints can be obtained via (anti) commutators between $\hat{G}_{n+1/2}$, $\hat{\mathcal{O}}^s_n$. However, we do not expect an isomorphism between the algebras of constraint and differential operators. The constraints thus obtained should be reducible to the $s = 3/2, 2$ super Virasoro constraints.

It must be emphasized that the complete set of couplings (g_k, ψ_k) is necessary to rewrite the S-D equations in the form of constraints. Nevertheless, the S-D equations clearly hold for any finite-degree polynomial potential. It is therefore natural to ask whether the reducibility of the constraints can be carried through the corresponding S-D equation for the reduced models. We do not have a non-perturbative answer for this question, but at leading order in $1/N$ it is easy to show, by using the $1/N$ factorization of observables ($\langle AB\rangle = \langle A\rangle\langle B\rangle + O(1/N)$) that the $s = 4$ S-D equations are reducible to the $s = 2$ and $s = 3/2$ equations. The same situation appears in the bosonic hermitian matrix model.

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Appendices

A 1. Property (6)

Here we demonstrate the equation (6) starting from the property (5). We extend Δ_N to $\Delta_{N+1} = \prod_{I<J}(x_I - x_J)$ by introducing an auxiliary eigenvalue $x_o = p$. The extended determinant can be written as

$$\Delta_{N+1} = e^\phi \Delta_N(x_i - x_j) \quad , \tag{41}$$

where $\phi = \ln \prod^N_{i=1}(p - x_i)$, and using $\partial^k_i e^\phi = 0$, $k \geq 2$, eq. (5) implies

$$\partial^s_p e^\phi \Delta_N = -s\sum^N_{i=1}(\partial_i e^\phi)(\partial^{s-1}_i \Delta_N) - \sum^N_{i=1} e^\phi \partial^m_i \Delta_N \quad . \tag{42}$$

Using once more the property (5), for Δ_N, and the expression $\partial_i e^\phi = e^\phi(-1/(p - x_i))$, equation (42) finally implies

$$\sum_{i=1}^{N}\frac{1}{p-x_i}\partial_i^{s-1}\Delta_N = \frac{\Delta_N}{s}(e^{-\phi}\partial_p e^{\phi})^s = \Delta_N\frac{(\partial_p + w(p))^s\cdot 1}{s} \quad . \tag{43}$$

A 2. On the even spin constraints

In order to obtain the expression (31), we start from the equations

$$\frac{\partial}{\partial\theta_k}e^F = -e^F\sum_{j\neq k=1}^{N}\frac{\theta_j}{x_k - x_j} \quad , \tag{44}$$

$$\sum_{k=1}^{N}\frac{\partial_k^m}{p-x_k}e^F = -e^F\sum_{j\neq k}^{N}\frac{(-1)^m m!\theta_k\theta_j}{(x_k-x_j)^{m+1}(p-x_k)} \quad . \tag{45}$$

For odd m, the factor $\frac{\theta_k\theta_j}{(x_k-x_j)^{m+1}}$ is anti-symmetric, and we may rewrite (45) as

$$\sum_{k=1}^{N}\frac{1}{(p-x_k)}\partial_k^m e^F = \frac{m!}{2}e^F\sum_{k\neq j}\frac{\theta_k\theta_j}{(x_k-x_j)^{m+1}(p-x_k)(p-x_j)} = \tag{46}$$
$$\frac{m!}{2}e^F\left[\sum_{n=0}^{m-2}\sum_{j\neq k}\frac{(-1)^n\theta_k\theta_j}{(x_k-x_j)^{m-n}(p-x_k)^{2+n}} + \sum_{j\neq k}\frac{\theta_k\theta_j}{x_{kj}(p-x_k)^{m+1}} - \sum_{j\neq k}\frac{\theta_k\theta_j}{(p-x_k)^{m+1}(p-x_j)}\right]$$

where $x_{ij} = x_i - x_j$. Above, we have repeatedly used the identity:

$$\frac{1}{p-x_j} = \frac{1}{p-x_k} - \frac{x_{kj}}{(p-x_j)(p-x_k)} \quad . \tag{47}$$

The last term in (46) corresponds to $(\partial^m\nu)\nu/m!$. Using (44) and (45) we find

$$\sum_{k=1}^{N}\left[\frac{2}{p-x_k}\partial_k^m + \sum_{n=0}^{m-2}\frac{m!}{(m-(n+1))!(p-x_k)^{n+2}}\partial_k^{m-(n+1)} + \frac{m!}{(p-x_k)^{m+1}}\theta_k\frac{\partial}{\partial\theta_k}\right]e^F$$
$$= (\partial^m\nu)\nu e^F \tag{48}$$

The expression (31) follows immediately from (48), for $m = s-1$. We mention that eq. (31) can be further simplified (compare with (6):

$$\sum_{k=1}^{N}\left[\frac{1}{p-x_k}\partial_k^m + \partial_k^m\frac{\theta_k}{(p-x_k)}\frac{\partial}{\partial\theta_k}\right]e^F = (\partial^m\nu\nu)e^F \quad . \tag{49}$$

To derive the constraints from (32), we also need to calculate $\mathcal{O}^{(s)}(e^U\Delta_N)$. It is easy to derive

$$\mathcal{O}^{\dagger(s)}(e^U\Delta_N) = -(\mathcal{O}^{(s)} + \partial_p^{s-1}w)(e^U\Delta_N) \quad . \tag{50}$$

To calculate $\mathcal{O}^{(s)}(e^U\Delta_N)$, it is sufficient to determine the expressions

$$\tilde{\mathcal{L}}^{(s)}(e^U\Delta_N) \equiv \partial_p^{s-1}\sum_{i=1}^{N}\frac{\theta_i}{p-x_i}\frac{\partial}{\partial\theta_i}(e^U\Delta_N) \quad , \tag{51}$$

$$\mathcal{L}^{(m)}(e^U\Delta_N) \equiv \sum_{i=1}^{N}\frac{1}{p-x_i}\partial_i^m(e^U\Delta_N) \quad . \tag{52}$$

As for (51), we notice that

$$\tilde{\mathcal{L}}^{(s)}(e^U\Delta_N) = \Delta_N e^U\sum_{k=1}^{N}\sum_{l\leq 0}\frac{\theta_k}{p-x_k}\psi_l x_k^l = -\Delta_N e^U\partial_p^{s-1}(\nu(p)\psi(p))_- \quad . \tag{53}$$

After similar manipulations and using formula (7), we get from (53) the following result:

$$\mathcal{L}^{(m)}(\Delta_N e^U) = \frac{\Delta_N e^U}{(m+1)}[(\partial_p + w + V')^{m+1}\cdot 1]_- + e^U\sum_{l=0}^{m-1}\binom{m}{l}\left[G_{n-l}(V(p),\psi(p))\mathcal{L}_F^{(l)}\Delta\right]_- \tag{54}$$

where

$$G_{m-l} = \sum_{t=1}^{m-l}\binom{m-l}{t}(\partial^t\psi(p)).(\partial + V')^{m-l-t}\cdot 1 \quad , \tag{55}$$

and

$$\mathcal{L}_F^{(l)}(p)\Delta_N = \left(\sum_i\frac{\theta_i}{p-x_i}\partial_i^l\right)\Delta_N \quad . \tag{56}$$

We have not been able to write the r.h.s. of (56) as a function of $w(p)$ and $\nu(p)$, but using

$$\int\mathcal{D}\mu\; e^{U+F}(\mathcal{L}_F^{(l)}\Delta_N) = \int\mathcal{D}\mu\;\Delta(\mathcal{L}_F^{\dagger(l)}e^{U+F}) \tag{57}$$

and

$$\mathcal{L}_F^{\dagger(l)}e^{U+F} = e^{U+F}\sum_{t=0}^{l}(-1)^t\binom{l}{t}\partial_p^{l-t}(\nu(p)(\partial + V')^t\cdot 1)_- \quad , \tag{58}$$

we finally obtain

$$\begin{aligned}\int\mathcal{D}\mu\; e^F\mathcal{L}^{(m)}(\Delta_N e^U) \;&=\; \int\mathcal{D}\mu\; e^U\Delta_N\left[\frac{(\partial_p + w + V')^{m+1}\cdot 1}{m+1}\right.\\ &+ \left.\sum_{l=0}^{m-1}\sum_{t=0}^{l}(-1)^t\binom{m}{l}\binom{l}{t}G_{m-l}(p)\partial_p^{l-t}(\nu(\partial + V')^t\cdot 1)_-\right]_-\end{aligned} \tag{59}$$

From (50), (53), (59) and (31) we can find a closed expression for the second integral of (32) as a function of $\nu(p)$, $w(p)$, $V'(p)$, $\psi(p)$, which completes the derivation of the even spins constraints.

A 3. Commutators

Here we present an example of the calculation of one of the commutators in (35). We will take $\hat{C} \equiv \partial\Psi^{(+)}[(\partial\Psi^{(+)}\hat{\Psi}^{(-)})_-, \hat{\Psi}^{(-)}]$, where

$$\Psi^{(+)}(p) = -N\sum_{k\geq 0}\psi_k p^k \quad , \quad \hat{\Psi}^{(-)}(p) = -\frac{1}{N}\sum_{n\geq 0}\frac{1}{p^{n+1}}\frac{\partial}{\partial\psi_n} \quad . \tag{60}$$

From such definitions, we have

$$(\partial\Psi^{(+)}\hat{\Psi}^{(-)})_- = \sum_{k\geq 0}\sum_{\mu\geq -1}\frac{k\psi_k}{p^{\mu+2}}\frac{\partial}{\partial\psi_{\mu+k}} \quad , \quad (\partial\Psi^{(+)}\hat{\Psi}^{(-)})_+ = \sum_{k\geq 0}\sum_{\mu\geq 2}k\psi_k p^{\mu-2}\frac{\partial}{\partial\psi_{k-\mu}} \quad . \tag{61}$$

Considering the action of these operators on the partition function Z^S, we obtain:

$$\frac{1}{Z^S}(\hat{C}Z^S) = \sum_{k\geq 0}\sum_{\mu\geq 2}\sum_{\hat{\mu}\geq -1}\sum_{i=1}^{N}\frac{k(k-\mu)N\psi_k}{p^{\hat{\mu}-\mu+4}}\left\langle x_i^{\hat{\mu}-\mu+k}\theta_i\right\rangle \tag{62}$$

$$= \sum_{i=1}^{N}\left\langle\sum_{\hat{\mu}\geq -1}\frac{x_i^{\hat{\mu}+1}\theta_i}{p^{\hat{\mu}+2}}\sum_{k\geq 0}\sum_{\mu\geq 2}(-Nkp^{k-1}\psi_k)\frac{(k-\mu)x_i^{k-\mu-1}}{p^{k-\mu+1}}\right\rangle = \sum_{i=1}^{N}\left\langle\frac{\theta_i}{p-x_i}\left(\frac{\partial\Psi^{(+)}(p)}{(p-x_i)^2}\right)_+\right\rangle$$

Now, from the properties

$$(f(p))_+ = \frac{1}{2\pi i}\oint_{w>p}\frac{dw}{w-p}f(w) \quad , \quad (f(p))_- = -\frac{1}{2\pi i}\oint_{w<p}\frac{dw}{w-p}f(w) \quad , \tag{63}$$

we find

$$\begin{aligned}\hat{C}Z^S &= Z^S\left\langle\frac{\partial^2\Psi^{(-)}\Psi^{(+)}}{2} - \frac{\partial^2}{2}(\Psi^{(-)}\partial\Psi^{(+)})_- - \partial[\Psi^{(-)}\partial^2\Psi^{(+)}]_-\right\rangle \\ &= \frac{Z^S}{2}\left\langle(\partial^2\Psi^{(-)}\partial\Psi^{(+)})_+ + (\Psi^{(-)}\partial^3\Psi^{(+)})_-\right\rangle \quad .\end{aligned} \tag{64}$$

Therefore, while acting on the partition function Z^S, the operator $\hat{C}$ reads

$$\hat{C} = -\frac{1}{2}(\partial\Psi^{(+)}\partial^2\hat{\Psi}^{(-)})_+ - \frac{1}{2}(\partial^3\Psi^{(+)}\hat{\Psi}^{(-)})_- \quad . \tag{65}$$

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