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DYNAMICS OF A TWO GRADED-RESPONSE NEURON NETWORK WITH DELAY: NEGATIVE FEEDBACK

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Abstract: The dynamics of two graded response neurons interconnected through delayed connections is studied. We suppose that the interneuron connection is inhibitory in one direction and excitatory in the opposite direction. This network forms a negative feedback loop. We show the presence of a unique periodic solution which attracts all the trajectories of all non-oscillating, non-zero initial conditions. This work complements previous works establishing local existence of periodic solutions in such systems.

1 The model

The dynamics of two graded response neurons (GRNs) [1] connected to each other by weights w and w' and delays $A_1 > 0$ and $A_2 > 0$, are determined by the following delay differential equations (DDEs):

$$\begin{cases} \frac{da}{dt}(t) = -\gamma a(t) + w\sigma_\alpha(b(t - A_1)) \\ \frac{db}{dt}(t) = -\gamma' b(t) + w'\sigma_{\alpha'}(a(t - A_2)) , \end{cases} \tag{1}$$

where γ, γ' are strictly positive constants and:

$$\sigma_\alpha(a) = \tanh(\alpha a) = \frac{e^{\alpha a} - e^{-\alpha a}}{e^{\alpha a} + e^{-\alpha a}} \quad \text{for } \alpha > 0 , \tag{2}$$

$$\sigma_\infty(a) = \begin{cases} 1 & \text{for } a > 0 , \\ -1 & \text{for } a \le 0 . \end{cases} \tag{3}$$

After changing the variables: $x(t) = a(At)$ and $y(t) = b(At - A_1)$ [2], where $A = A_1 + A_2$ and renaming the parameters: $\epsilon = \frac{1}{\gamma A}$, $\epsilon' = \frac{1}{\gamma' A}$, $W = w/\gamma$ and $W' = w'/\gamma'$, system (1) is transformed into:

$$
\begin{cases}
\epsilon \frac{dx}{dt}(t) = -x(t) + W\sigma_\alpha(y(t)) \\
\epsilon' \frac{dy}{dt}(t) = -y(t) + W'\sigma_{\alpha'}(x(t-1))
\end{cases} \tag{4}
$$

The phase space of system (4) is the product $S = C([-1,0],\mathbb{R}) \times \mathbb{R}$. For any initial condition $\phi \in S$, DDE (4) has a unique solution $z(t,\phi) = (x(t,\phi), y(t,\phi))$ defined for all $t \geq 0$. We denote by $z_t(\phi) = (x_t(\phi), y_t(\phi))$ the associated semi-flow [3], that is $z_t(\phi) \in S$, and $y_t(\phi) = y(t,\phi)$, $x_t(\phi)(\theta) = x(t+\theta,\phi)$ for all $-1 \leq \theta \leq 0$.

System (4) is invariant under the transformation $x \to -x$ and $y \to -y$. Thus we need only to study the two cases of $W > 0$, $W' < 0$ (negative feedback) and $W > 0$, $W' > 0$ (positive feedback).

2 Case $\alpha = \alpha' = \infty$

We shall describe the dynamics of system (4) in the case of $\alpha = \alpha' = \infty$, for which the solutions of system (4) are given by:

$$
\begin{cases}
x(t) = (x(\tau) - \eta W)\mathrm{e}^{(\tau-t)/\epsilon} + \eta W \\
y(t) = (y(\tau) - \eta' W')\mathrm{e}^{(\tau-t)/\epsilon'} + \eta' W'
\end{cases} \tag{5}
$$

for $\tau > 0$, η and η' such that $|\eta| = |\eta'| = 1$, $\eta' x(\theta') \geq 0$ for all $\theta' \in [\tau-1, t-1]$, and $\eta y(\theta) \geq 0$ for all $\theta \in [\tau, t]$.

We study the orbits of initial conditions $\phi = (\phi_1, \phi_2) \in S$ such that $\phi_2 \neq 0$ and ϕ_1 does not change sign, that is:

$$
\phi_1(\theta) \leq 0 \quad \text{or} \quad \phi_1(\theta) > 0 \quad \text{for all } \theta \in [-1,0]
$$

For such an initial condition ϕ we have $z_t(\phi) = z_t(r)$, for all $t \geq 1$, where $r = (r_1, r_2) \in S$ is defined by: $r_1(\theta) = \phi_1(0)$, for all $\theta \in [-1,0]$ and $r_2 = \phi_2$. Consequently we limit the study to the orbits of "constant" initial conditions, that we identify with their image in $\mathbb{R}^2$.

We shall study the dynamics of a network with negative feedback, that is, the weights have opposite sign. As already mentioned, due to the reflection symmetry of system (4), we need to consider only the case $W > 0$ and $W' < 0$.

3 Negative feedback results

In this situation, we show that there is a unique periodic solution which attracts the solutions of all constant initial conditions except the origin $(0,0)$. In order to prove this result we associate to DDE (4), a one-dimensional map, denoted by f, similar to a Poincaré return map. We show that this map has a unique stable equilibrium point which attracts the orbits of all initial conditions except the origin. Similar methods have been applied for the study of scalar DDEs with piecewise constant nonlinearities (see references in [4, 5]).

As for the Poincaré return map associated to an ordinary differential equation [6], the iterates of the map f characterize the asymptotic behavior of the solutions of DDE (4).

In the (x,y) plane, the trajectories of solutions of constant initial conditions (except the origin), turn clockwise around the origin, crossing successively the two axes. For a given initial condition $r = (x_0, y_0) \in \mathbb{R}^2 - \{(0,0)\}$, we denote by $Y = (0,y)$ the first crossing of the positive half y-axis by $z(t,r)$. Thus, the asymptotic dynamics of $z(t,r)$ are derived from the behavior of $z(t, Y)$.

For $Y = (0, y)$ on the positive half y-axis, we define the sequence $Y_n = (0, y_n) = (0, f^n(y_1))$ (where $f^n = f \circ \cdots \circ f$, n times) as the successive intersections of the trajectory of $z(t, Y)$ with the positive half y-axis. Thus the map f defines the point of first return to the positive half y-axis. A solution $z(t, Y)$ is periodic if and only if $\{y_n\}_{n \geq 0}$ is periodic. In this case, the periodic solution is stable if and only if the corresponding sequence is also stable. The construction of f is schematically represented in Fig. 1.

The return-map f can be explicitly evaluated (Eq. (6) in Appendix A). The geometrical characteristics of f can be derived from the properties of the solutions of DDE (4). f is strictly increasing, concave, and $0 < f(0^+) < f(y) < -W'$ for all $y > 0$. Thus f has a unique fixed point, $y^* = f(y^*)$, and its slope f' satisfies $0 < f'(y^*) < 1$. Hence for all $y > 0$, $f^n(y) \to y^*$ as $n \to \infty$. Thus we have:

Asymptotic behavior. *For all initial conditions $r \in I\!R^2 - \{(0,0)\}$, the solution $z(t, r)$ tends to the periodic solution $z(u, Y^*)$ as $t \to \infty$, where $Y^* = (0, y^*)$.*

An example of the closed trajectory representing a periodic solution in the x, y plane is shown in Fig. 2, with the corresponding temporal evolutions of $x(t)$ and $y(t)$ (respectively solid and dashed lines in Fig. 3) and the associated return map f (Fig. 4).

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A The return-map f

$$-W'\left(1-\left(2-\frac{1}{\left(2\exp(1/\epsilon')-\frac{1}{\left(2-\frac{1}{\left(2\exp(1/\epsilon')-1-\frac{y}{W'}\right)^{\frac{\epsilon'}{\epsilon}}}\right)^{\frac{\epsilon}{\epsilon'}}}\right)^{\frac{\epsilon'}{\epsilon}}}\right)^{-\frac{\epsilon}{\epsilon'}}\right) \qquad (6)$$