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TRANSITION LAYER EQUATIONS FOR POSITIVE-FEEDBACK DELAYED EQUATIONS

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Abstract

We consider the scalar delayed differential equation $\epsilon\dot{x}(t) = -x(t) + f(x(t-1))$, where $\epsilon > 0$ and f verifies $df/dx > 0$ and some other conditions. This equation has three equilibria $-\gamma_1$, 0, and γ_3. In the study of the singular limit $\epsilon \to 0$ a crucial role is played by the so called transition layer equation related to the above equation. In this case the transition layer equation is given by $\dot{y}(t) = -y(t) + f(y(t+r))$, where $r > 0$. We prove that there is a special value of r for which the transition layer equation has a solution such that $y(t) \to -\gamma_1$ as $t \to -\infty$, and $y(t) \to \gamma_3$ as $t \to \infty$.

Key words: delayed differential equation, singular perturbation, transition layer.

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1 A Theorem

Let us consider the following family of scalar advanced differential equations

$$\dot{y}(t) = -y(t) + f(y(t+r)), \tag{1}$$

where r is a positive parameter. We assume that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and verifies the following hypotheses:

(H1) $f(0) = 0$, $f(-\gamma_1) = -\gamma_1$, $f(\gamma_2) = \gamma_2$, where $\gamma_1 > 0$, $\gamma_2 > 0$, and $f(x) \neq 0$ for $x \in (-\gamma_1, 0) \cup (0, \gamma_2)$;

(H2)

$$\frac{df}{dx}(x) \geq 0, \quad \text{and} \quad \frac{df}{dx}(0) > 1.$$

Our goal in this paper is to prove the following theorem.

Theorem 1 *There exists $r_* > 0$ such that equation (1) with $r = r_*$ has a solution $\phi : \mathbb{R} \to \mathbb{R}$ with the following properties:*

$$\frac{d\phi}{dt}(t) \geq 0, \quad for \quad t \in \mathbb{R}, \quad \phi(0) = 0,$$
$$\lim_{t\to-\infty} \phi(t) \to -\gamma_1, \quad \lim_{t\to\infty} \phi(t) \to \gamma_2.$$

There also exists $r_{**} > 0$ such that equation (1) with $r = r_{**}$ has a solution $\chi : \mathbb{R} \to \mathbb{R}$ with the following properties:

$$\frac{d\chi}{dt}(t) \leq 0, \quad for \quad t \in \mathbb{R}, \quad \chi(0) = 0,$$
$$\lim_{t\to-\infty} \chi(t) \to \gamma_2, \quad \lim_{t\to\infty} \chi(t) \to -\gamma_1.$$

Moreover, if f is an odd function then $r_* = r_{**}$ and $\phi(t) = -\chi(t)$.

In order to prove theorem 1 it is sufficient to show the existence of r_* and ϕ. The existence of r_{**} and χ is a consequence of this result applied to equation (1) after the change of variables $y \to -y$. If f is odd then $\phi(t) = -\chi(t)$ is a consequence of the symmetry of equation (1) with respect to the change of variables $y \to -y$.

The proof of the above theorem will be made in several steps. In section 2 we consider a family of auxiliary problems defined in compact sets $[-L, L]$ of the real line. We show that these problems have solutions ϕ_L, r_L for all L. In section 3 we show that r_L is uniformly bounded with respect to L from above and below and that there is a sequence L_n, $n = 1, 2, \ldots$, of values of L such that $\phi_{L_n}, r_{L_n} \to \phi, r_*$, in compact subsets of $\mathbb{R}$, as $n \to \infty$. In section 4 we show that the function ϕ obtained in section 3 has the properties in theorem 1.

2 A family of approximating problems

We start this section with some definitions. For $L > 0$, let C_L be the Banach space defined by

$$C_L \stackrel{def}{=} \{z : [-L, L] \to \mathbb{R} \mid z \quad \text{continuous}\}, \qquad ||z||_L = \sup_{|t|\leq L} |z(t)| \quad .$$

Let Λ_L be the following subset of C_L (endowed with the induced topology)

$$\Lambda_L \stackrel{def}{=} \{z \in C_L \mid z(0) = 0,\ t \leq t' \Rightarrow z(t) \leq z(t'),\ -\gamma_1 \leq z(t) \leq \gamma_2\} \quad .$$

Proposition 1 *The set Λ_L has the following properties:*

- *i) it is bounded,*
- *ii) it is closed,*
- *iii) it is convex.*

These properties can be easily verified.

Let X be the set of functions given by

$$\begin{aligned} X \stackrel{def}{=} & \{z : \mathbb{R} \to \mathbb{R} \mid z \quad \text{continuous for } t \in \mathbb{R}, \text{nondecreasing for } \ t < 0, \\ & \text{strictly increasing for } t > 0, z(0) < 0, \\ & \lim_{t\to-\infty} z(t) = -\gamma_1, \text{ and } \lim_{t\to\infty} z(t) = \gamma_2\} \quad . \end{aligned}$$

We endow X with the metric $d(x, z) = \sup_{t\in\mathbb{R}}|x(t) - z(t)|$. Denoting the restriction of a function $z : \mathbb{R} \to \mathbb{R}$ to the interval $[-L, L]$ by $z|_L$ we define the set X_L as

$$X_L \stackrel{def}{=} \{z : \mathbb{R} \to \mathbb{R} \mid z|_L \in \Lambda_L, \quad z(t) = -\gamma_1,\ t < -L, \quad z(t) = \gamma_2,\ t > L\} \quad .$$

We endow X_L with the the metric $d(x, z) = \sup_{|t|\leq L}|x(t) - z(t)|$. Notice that every function in X_L is an extension to $\mathbb{R}$ of a function in Λ_L, originally defined on the interval $[-L, L]$. We denote this extension mapping by $\overline{\Gamma} : \Lambda_L \to X_L$. We define a mapping $\underline{A}_L : X_L \to X$ by

$$\underline{A}_L z(t) \stackrel{def}{=} e^{-t} \int_{-\infty}^{t} e^s f(z(s))ds = \int_{-\infty}^{0} e^s f(z(s + t))ds \quad .$$

It is not hard to verify that $\underline{A}_L z$ indeed belongs to X. For each $z \in X$ there exists a unique $r(z) \in \mathbb{R}$, $r(z) > 0$, such that $z(r(z)) = 0$. For a fixed L, the composed function $r \circ \underline{A}_L : X_L \to \mathbb{R}_+$ satisfies the following bounds, independently of z.

Proposition 2 *For a given L and any $z \in X_L$ we have*

$$\frac{e^{-L}\gamma_1}{\gamma_2} + 1 \leq e^{r\circ\underline{A}_L(z)} \leq \frac{\gamma_1}{\gamma_2} + e^L .$$

Proof. In the following, in order to simplify the notation, we will write $r \circ \underline{A}_L(z)$ just as r. The definition of $\underline{A}_L$ implies that

$$\int_{-\infty}^{r} e^s f(z(s))ds = 0 \tag{2}$$

If $r \leq L$ then the upper bound for r is trivial. So, let us assume that $r > L$. Using that $-f(z(s)) \leq \gamma_1$ for $s \leq 0$ and that $f(z(s)) \geq 0$ for $s \geq 0$, equation (2) implies

$$\gamma_1 = \gamma_1 \int_{-\infty}^{0} e^s ds \geq -\int_{-\infty}^{0} e^s f(z(s))ds = \int_{0}^{r} e^s f(z(s))ds \geq \int_{L}^{r} e^s f(z(s))ds = \gamma_2[e^r - e^L] \quad .$$

This inequality implies the upper bound for r. Equation (2) implies that

$$e^{-L}\gamma_1 - \int_{-L}^{0} e^s f(z(s))ds = \int_{0}^{r} e^s f(z(s))ds \quad . \tag{3}$$

The lower bound for r comes from the following inequality obtained from equation (3)

$$e^{-L}\gamma_1 \leq \int_{0}^{r} e^s f(z(s))ds \leq \gamma_2(e^r - 1) \quad .$$

□

We define the set X_* as

$$\begin{aligned} X_* \stackrel{def}{=} & \{z : \mathbb{R} \to \mathbb{R} \mid z \quad \text{continuous for } t \in \mathbb{R}\ , \text{nondecreasing for } \ t < 0, \\ & \text{strictly increasing for } t > 0, \\ & \lim_{t\to-\infty} z(t) = -\gamma_1, \text{ and } \lim_{t\to\infty} z(t) = \gamma_2\} \quad . \end{aligned}$$

We endow X_* with the metric $d(x, z) = \sup_{t\in\mathbb{R}}|x(t) - z(t)|$. We define the mapping $T_r : X \to X_*$ as $T_r z(t) = z(t+r(z))$ and the restriction mapping $\underline{\Gamma} : X_* \to \Lambda_L$. Finally, we define a mapping $A_L : \Lambda_L \to \Lambda_L$ as

$$A_L = \underline{\Gamma} \circ T_r \circ \underline{A}_L \circ \overline{\Gamma} \quad . \tag{4}$$

It is easy to check that $A_L z : [-L, L] \to \mathbb{R}$ is: continuous, bounded, nondecreasing, and satisfies $A_L z(0) = 0$. So, $A_L z$ indeed belongs to Λ_L. A more explicit way to write A_L is

$$A_L z(t) \stackrel{def}{=} e^{-t-r} \int_{-\infty}^{t+r} e^s f_L(z(s))ds \quad , \tag{5}$$

where

$$\begin{aligned} & f_L(z(s)) = -\gamma_1 && \textit{for} \quad s < -L, \\ & f_L(z(s)) = \gamma_2 && \textit{for} \quad s > L, \\ & f_L(z(s)) = f(z(s)) && \textit{for} \quad |s| \leq L \\ & r \quad \text{is an abbreviation of} && r \circ \underline{A}_L(z). \end{aligned}$$

Proposition 3 *The mapping $A_L : \Lambda_L \to \Lambda_L$ is continuous.*

Proof. In order to prove the theorem we have to show that the four mappings in the definition (4) of A_L are continuous. It is easy to show that $\overline{\Gamma}$ and $\underline{\Gamma}$ are continuous. The continuity of $\underline{A}_L$ is proved in the following way. As f is continuously differentiable there exists a constant μ such that

$$|f(x) - f(y)| \leq \mu |x - y| \qquad \text{for} \qquad -\gamma_1 \leq x \leq \gamma_2, \quad -\gamma_1 \leq y \leq \gamma_2 \quad .$$

Thus, if $x \in X_L$, $z \in X_L$, satisfy $d(x, z) < \delta$ then

$$\begin{aligned} |\underline{A}_L x(t) - \underline{A}_L z(t)| &= \left| \int_0^\infty e^{-s} [f(x(t-s)) - f(z(t-s))] ds \right| \\ &\leq \mu \delta \int_0^\infty e^{-s} ds = \mu \delta \quad . \end{aligned}$$

This inequality implies the continuity of $\underline{A}_L$. The continuity of T_r is a more difficult point, because r itself is a function of the point $z \in X_L$ to which we apply T_r. Let us denote by z, z' two points in X and by r and r' their respective zeroes ($z(r) = 0, z'(r') = 0$), or, equivalently, the values of the function r at z and z' ($r(z) = r$, $r(z') = r'$). We want to show that for any given $z \in X$ and $\epsilon > 0$ there exists a $\delta > 0$ such that $d(z, z') < \delta$ implies that

$$d(T_{r'} z', T_r z) = \sup_{t \in \mathbb{R}} |T_{r'} z'(t) - T_r z(t)| < \epsilon \quad ,$$

where we used the notation $T_r z(t) = z(t + r)$ and $T_{r'} z'(t) = z'(t + r')$. Let $\delta < \epsilon/2$. Since

$$\begin{aligned} |T_{r'} z'(t) - T_r z(t)| &= |T_{r'} z'(t) - T_{r'} z(t) + T_{r'} z(t) - T_r z(t)| \\ &\leq |T_{r'} z'(t) - T_{r'} z(t)| + |T_{r'} z(t) - T_r z(t)|, \end{aligned}$$

and $|T_{r'} z'(t) - T_{r'} z(t)| = |z'(t + r') - z(t + r')| < \delta < \epsilon/2$ for any $t \in \mathbb{R}$, we just have to show that it is possible to further decrease $\delta > 0$ such that the following inequality becomes true

$$\sup_{t \in \mathbb{R}} |T_{r'} z(t) - T_r z(t)| = \sup_{t \in \mathbb{R}} |z(t + r') - z(t + r)| = \sup_{t \in \mathbb{R}} |z(t + r' - r) - z(t)| < \epsilon/2.$$

The continuity, monotonicity, and boundness of z imply that z is uniformly continuous. So, it is possible to find the desired δ if we show that the function $z \to r(z)$ is continuous, namely, that for any given $z \in X$ and $\overline{\epsilon} > 0$ there exists a $\overline{\delta} > 0$ such that $d(z', z) < \overline{\delta}$ implies $|r' - r| < \overline{\epsilon}$. In order to prove this we set $\epsilon_1 \stackrel{def}{=} \min\{\overline{\epsilon}, r/2\}$. Notice that z is strictly increasing in the interval $(r - 2\epsilon_1, r + 2\epsilon_1)$, because z is strictly increasing in $(0, \infty)$. Now, we define $\overline{\delta} \stackrel{def}{=} \min\{|z(r - \epsilon_1)|, |z(r + \epsilon_1)\} > 0$. The definitions of ϵ_1 and $\overline{\delta}$ imply that: if $|r - r'| \geq \epsilon_1$ then $d(z, z') \geq |z(r') - z'(r')| = |z(r')| \geq \overline{\delta}$. So, if $d(z, z') < \overline{\delta}$ then $|r - r'| < \epsilon_1 \leq \overline{\epsilon}$, which proves that $z \to r(z)$ is continuous and ends the proof of the proposition. □

Proposition 4 *The mapping A_L is completely continuous, namely, A_L is continuous and maps bounded sets to compact sets (see [2] section 2.2).*

Proof. Since $\Lambda_L \subset C_L$ is bounded and $A_L : \Lambda_L \to \Lambda_L$ is continuous by proposition 3, then, in order to prove that A_L is completely continuous, it is enough to show that the range of

A_L is compact. This is a consequence of the Arzela-Ascoli's theorem if we show that there exists a constant K', independent of $z \in \Lambda_L$, such that

$$|A_L z(t) - A_L z(t')| \leq K' |t - t'| \quad \text{for all} \quad |t| \leq L, |t'| \leq L.$$

The definition of A_L and the fact that $r(z) > 0$ imply that the above inequality is true if there exists a constant K, independent of $z \in X_L$, such that

$$|\underline{A}_L z(t) - \underline{A}_L z(t')| \leq K |t - t'| \quad \text{for all} \quad t > -L, t' > -L. \tag{6}$$

For $|t| < L$, $\underline{A}_L z$ is differentiable and

$$\frac{d}{dt} \underline{A}_L z(t) = -\underline{A}_L z(t) + f(z(t)),$$

which implies

$$\left| \frac{d}{dt} \underline{A}_L z(t) \right| \leq |\underline{A}_L z(t)| + |f(z(t))| \leq 2 \max\{\gamma_1, \gamma_2\} \quad . \tag{7}$$

For $t > L$, $\underline{A}_L z$ is explicitly given by

$$\underline{A}_L(t) z = e^{-t} \left\{ -e^{-L} \gamma_1 + \int_{-L}^{L} e^s f(z(s)) ds + \gamma_2 (e^t - e^L) \right\} = e^{-t} \left\{ \underline{A}_L z(L) + \gamma_2 (e^t - e^L) \right\},$$

which implies that $\underline{A}_L z$ is differentiable and

$$\left| \frac{d}{dt} \underline{A}_L z(t) \right| \leq e^{-t} |\underline{A}_L z(L)| + \gamma_2 \leq 2\gamma_2 \quad . \tag{8}$$

Inequalities (7) and (8) and the continuity of $\underline{A}_L z$ at $t = L$ imply that inequality (6) is true. □

The following proposition is an immediate consequence of the definition of Λ_L.

Proposition 5 *The null function $\underline{0} \in \Lambda_L$ is not a fixed point of A_L.*

Finally, propositions 1, 4 and 5, and the Schauder fixed point theorem (see for instance [2], section 2.2), imply the following lemma.

Lemma 1 *The mapping $A_L : \Lambda_L \to \Lambda_L$ has a fixed point ϕ_L different from $\underline{0}$.*

3 Uniform bounds

Let us denote by r_L the shift that appears in the definition of A_L (5) and that is related to the fixed point ϕ_L given by lemma 1. Our goal in this section is to find bounds, independently of L, for r_L and for the derivative of ϕ_L.

From the definition of A_L (equation (5)), for $|t| \leq L$, we have

$$\phi_L(t) = e^{-t-r_L} \int_{-\infty}^{t+r_L} e^s f(\phi_{L*}(s)) ds \quad , \tag{9}$$

where:

$$\phi_{L*}(s) = \phi_L(s) \qquad for \quad |s| \le L,$$
$$\phi_{L*}(s) = -\gamma_1 \qquad for \quad s < -L,$$
$$\phi_{L*}(s) = \gamma_2 \qquad for \quad s > L.$$

Using that $\phi_L(0) = 0$, we can rewrite (9) as

$$\phi_L(t) = e^{-t-r_L} \int_{r_L}^{t+r_L} e^s f(\phi_{L*}(s)) ds \quad . \tag{10}$$

We shall find an upper bound for r_L in several steps.

Proposition 6 *There exists $M_1 > 0$ such that if $L > M_1$ then $r_L < L$.*

Proof. Let us assume that $r_L \ge L$. Then, from (10), we obtain that for $t \in [0, L]$

$$\phi_L(t) = e^{-t-r_L} \gamma_2 \int_{r_L}^{t+r_L} e^s ds = \gamma_2(1 - e^{-t}) \quad . \tag{11}$$

Now, using (11), the facts that $|f(z)| \ge |z|$ for $-\gamma_1 \le z \le \gamma_2$, and $\phi_L(0) = 0$, we get

$$\begin{aligned} \gamma_1 \ \ge\ & -\int_{-\infty}^{0} e^s f(\phi_{L*}(s)) ds = \int_0^{r_L} e^s f(\phi_{L*}(s)) ds \\ \ge\ & \int_0^{r_L} e^s \phi_{L*}(s) ds \ge \int_0^L e^s \gamma_2 (1 - e^{-s}) ds = \gamma_2 [e^L - 1) - L]. \end{aligned}$$

This inequality holds if, and only if, $L \le M_1$, where M_1 is the positive root of

$$\frac{\gamma_1}{\gamma_2} + 1 = e^{M_1} - M_1 \quad .$$

Therefore, if $L > M_1$ then $r_L < L$. □

Proposition 7 *For $L > M_1$ the following two inequalities are true:*

$$\frac{\phi_L(r_L)}{f(\phi_L(r_L))} \ge 1 - e^{-r_L} \quad , \tag{12}$$
$$\frac{\gamma_1}{g(r_L)} \ge \phi_L(r_L) \quad , \tag{13}$$

where $g(r_L) = e^{r_L} - 1 - r_L$.

Proof. From (10), proposition 6 and $0 \le t \le r_L$ we obtain

$$\begin{aligned} \phi_L(t) \ =\ & e^{-t-r_L} \int_{r_L}^{t+r_L} e^s f(\phi_{L*}(s)) ds \\ \ge\ & e^{-t-r_L} \int_{r_L}^{t+r_L} e^s f(\phi_{L*}(r_L)) ds = f(\phi_L(r_L))[1 - e^{-t}] \quad . \end{aligned} \tag{14}$$

For $t = r_L$ this inequality gives (12). From inequality (14), proposition 6, and $\phi_L(0) = 0$, we obtain

$$\begin{aligned} \gamma_1 \ \ge\ & -\int_{-\infty}^{0} e^s f(\phi_{L*}(s)) ds = \int_0^{r_L} e^s f(\phi_L(s)) ds \\ \ge\ & \int_0^{r_L} e^s \phi_L(s) ds \ge \int_0^{r_L} e^s f(\phi_L(r_L))(1 - e^{-s}) ds \\ =\ & f(\phi_L(r_L))[e^{r_L} - 1 - r_L] \ge \phi_L(r_L) g(r_L) \quad . \end{aligned}$$

□

The fact that f is continuously differentiable, $f(0) = 0$, and $\frac{df}{dz}(0) = \nu > 1$, imply that there exists $b > 0$ such that

$$\frac{f(z)}{z} > \frac{\nu + 1}{2} \qquad \text{for} \quad 0 \le z \le b \quad . \tag{15}$$

The function g appearing in proposition 7 has the following properties:

$$g(0) = 0, \qquad \frac{dg}{dr}(r) > 0 \quad \text{for} \quad r > 0, \quad \lim_{r \to \infty} g(r) = \infty \quad .$$

Therefore, there exists a unique r_* such that $g(r_*) = \gamma_1 / b$ and $g(r_L) > \gamma_1 / b$, for $r_L > r_*$. This and inequality (13) imply that

$$\phi_L(r_L) \le \frac{\gamma_1}{g(r_L)} < b, \qquad \text{if} \quad r_L > r_* \quad . \tag{16}$$

Now, let r_{**} be the only positive root of

$$\frac{2}{\nu + 1} = 1 - e^{-r_{**}} \quad .$$

This implies that

$$\frac{2}{\nu + 1} < 1 - e^{-r} \qquad \text{if} \quad r_L > r_{**} \quad . \tag{17}$$

Lemma 2 *Let $\overline{r} = \max\{r_*, r_{**}\}$ and $L > M_1$. Then $r_L \le \overline{r}$ independently of L.*

Proof. Let us argue by contradiction assuming that $r_L > \overline{r}$. This and inequality (16) imply that $\phi_L(r_L) < b$. Using (12) and (17) (since $r_L > \overline{r}$) we obtain

$$\frac{\phi_L(r_L)}{f(\phi_L(r_L))} \ge 1 - e^{-r_L} > \frac{2}{\nu + 1} \quad .$$

But this inequality, and the fact that $\phi_L(r_L) < b$, contradict inequality (15). Therefore $r_L \le \overline{r}$. □

Lemma 3 *Let $L > M_1$ and*

$$\underline{r} \overset{def}{=} \frac{\gamma_1 e^{-\overline{r}}}{\gamma_1 + \gamma_2} > 0 \quad .$$

Then $\underline{r} \le r_L$, independently of L.

Proof. Since $r_L < L$ (proposition 6) the function ϕ_L is differentiable for $t \in [-L, 0]$. Differentiating expression (9) and using that $f(z) \leq z$ for $z \in [-\gamma_1, 0]$ we obtain that, for $t \in [-L, 0]$,

$$\begin{aligned} \dot{\phi}_L(t) &= -\phi_L(t) + f(\phi_L(t + r_L)) \\ &\leq -f(\phi_L(t)) + f(\phi_L(t + r_L)) = \frac{d}{dt}\int_t^{t+r_L} f(\phi_L(s))ds \quad . \end{aligned}$$

Integrating this inequality in the interval $[-L, 0]$, we obtain

$$-\phi_L(-L) \leq \int_0^{r_L} f(\phi_L(s))ds - \int_{-L}^{-L+r_L} f(\phi_L(s))ds \leq (\gamma_1 + \gamma_2)r_L \quad . \tag{18}$$

Equation (9), the fact that $\phi_L(s) < 0$ for $s < 0$, that $r_L < L$, and lemma 2 imply that

$$\phi_L(-L) = e^{L-r_L}\{-\gamma_1 e^{-L} + \int_{-L}^{-L+r_L} e^s f(\phi_L(s))ds\} \leq -e^{-r_L}\gamma_1 \leq -e^{-\bar{r}}\gamma_1 \quad .$$

This and inequality (18) imply the inequality in the lemma. □

Lemma 4 *There exist infinite sequences L_n, r_n, ϕ_n, $n = 1, 2, \ldots$, with $L_n \to \infty$ as $n \to \infty$, such that the limits*

$$r_n \to r > 0, \quad \textit{and} \quad \phi_n \to \phi \quad \textit{as} \quad n \to \infty$$

converge. Moreover, ϕ_n converges uniformly, on compact intervals, to a function ϕ having the following properties:

- *it is continuously differentiable and nondecreasing;*
- *$\phi(0) = 0$;*
- *$-\gamma_1 \leq \phi(t) \leq \gamma_2$ for $t \in \mathbb{R}$;*
- *it is a solution of the transition layer equation (1).*

Also, $\dot{\phi}_n$ converges to $\dot{\phi}$ uniformly on compact intervals.

Proof. Let $L = L_1, L_2, L_3, \ldots$ be an infinite sequence of values of L and r_{L_k}, ϕ_{L_k}, $k = 1, 2, \ldots$ be their corresponding sequences of r_L and ϕ_L. Propositions 2 and 3 imply that the sequence r_{L_k} is bounded from above and below by positive numbers. The sequence ϕ_{L_k} is bounded, $-\gamma_1 \leq \phi_{L_k}(t) \leq \gamma_2$, $|t| \leq k$, and it is equicontinuous (the equicontinuity is a consequence of estimates (7) and (8) that are independent of L and are also valid for $\dot{\phi}_L$). The remainder of the proof of this lemma involves standard limiting arguments for sub-sequences of ϕ_{L_k} and r_{L_k} using Arzela-Ascoli's theorem and the fact that ϕ_{L_k}, r_{L_k} satisfy the integral identity (9). □

4 The nontriviality of ϕ

Our goal in this section is to show that the function ϕ obtained in lemma 4 is nontrivial. This is a consequence of the following lemma.

Lemma 5 *There exists $M > 0$ such that at least one of the following inequalities hold:*

$$i) \quad \phi(M) > 0, \qquad ii) \quad \phi(-M) < 0 \quad .$$

We make the following claims:

Claim 1:if $\phi(-M) < 0$ then $\phi(r) > 0$.
Suppose this is false. Then $\phi(t) = 0$ for $t \in [0, r]$, because ϕ is nondecreasing. But this contradicts the fact that ϕ is a solution of equation (1) (lemma 4). Indeed, in this case the theorem of uniqueness of backward continuation of solutions of (1) would imply $\phi(t) = 0$ for all $t < 0$, which is false.

Claim 2: if $\phi(b) > 0$, for some $b > 0$, then $\phi(t) < 0$, for $t < 0$.
In order to show this, let $t_* = \sup\{t|\phi(t) = 0\} \geq 0$. Using that ϕ is a solution of equation (1) we get $\dot{\phi}(t_*) = f(\phi(t_* + r)) > 0$. Thus, $\phi(t) < 0$ for $t < t_*$, because ϕ is nondecreasing. This and the fact that $\phi(0) = 0$ imply our claim.

Before proving this lemma let us use it to finish the proof of theorem 1. A consequence of lemma 5 and the two claims above is that

$$\phi(-t)\phi(t) < 0 \qquad \text{for all} \quad t \in \mathbb{R}. \tag{19}$$

This, the bounds $-\gamma_1 \leq \phi(t) \leq \gamma_2$, $t \in \mathbb{R}$, and the integral equation satisfied by ϕ,

$$\phi(t) = \int_{-\infty}^{0} e^s f(\phi(s + t + r))ds \quad , \tag{20}$$

imply the limits in the statement of theorem 1, namely

$$\lim_{t\to-\infty} \phi(t) \to -\gamma_1, \quad \lim_{t\to\infty} \phi(t) \to \gamma_2 \quad .$$

Indeed, using that ϕ is nondecreasing we conclude that the limits $\lim_{t\to\pm\infty} |\phi(t)| \stackrel{def}{=} |\phi(\pm\infty)|$ exist and are bounded by $\max\{\gamma_1, \gamma_2\}$. So, we can take limits on both sides of equation (20) to conclude that $\phi(\pm\infty) = f(\phi(\pm\infty))$. This, inequalities $-\gamma \leq \phi(-\infty) < 0$ and $0 < \phi(\infty) \leq \gamma_2$, and hypothesis (H1) on f (see section 1) imply the above limits.

The only thing remaining in order to complete the proof of theorem 1 is to prove lemma 5. This proof is the content of the rest of the paper.

Let us assume that lemma 5 is false. Then for any $K > 0$

$$||\phi_n||_K = \sup_{|t|\leq K \leq L_n} |\phi_n(t)| \to 0 \qquad \text{as} \qquad n \to \infty \quad .$$

Let N_K be such that $L_n + r_n \geq K$ for $n > N_K$. For $n > N_K$ we define a sequence of functions $x_n : I_K \to \mathbb{R}$, $I_K \stackrel{def}{=} [-K, K]$, as

$$x_n(t) = \frac{\phi_n(t)}{||\phi_n||_K} \quad .$$

Notice that $||x_n||_K = 1$ and at least one of the identities $x_n(-K) = -1$ or $x_n(+K) = 1$ is true. The function ϕ_n is differentiable for $t \in (-L, L-r)$. Differentiating expression (9) we find that in this interval ϕ_n satisfies

$$\dot{\phi}_n(t) = -\phi_n(t) + f(\phi_n(t + r_n)) \quad .$$

This implies that $x_n : I_K \to \mathbb{R}$, $n > N_K$, are differentiable, satisfy $\dot{x}_n(t) \geq 0$, and also

$$\dot{x}_n(t) = -x_n(t) + \nu x_n(t + r_n) + R(||\phi_n||_K, x_n(t + r_n)), \tag{21}$$

where R is a continuous function such that $R(0, x) = 0$ and, for $\xi \neq 0$,

$$R(\xi, x) \stackrel{def}{=} -\nu x + \frac{f(\xi x)}{\xi} \qquad \text{with} \quad \frac{df}{dx}(0) = \nu > 1 \quad .$$

Equation (21), the definition of R, and $||x_n||_K = 1$ imply that $||\dot{x}_n||_K$ are uniformly bounded for $n > N_K$. This, $||x_n||_K = 1$, the uniform boundness of r_n, and Arzela-Ascoli's theorem imply that there exist sub-sequences r_{n_j}, x_{n_j} that converge to r and x, respectively, as $j \to \infty$. This, equation (21), and the fact that $R(||\phi_n||_K, x(t + r_n)) \to 0$ as $n \to \infty$ uniformly for $|t| \leq K$ (because $||\phi_n|| \to 0$ as $n \to \infty$), imply that x is continuously differentiable and satisfies the linear equation

$$\dot{x}(t) = -x(t) + \nu x(t + r), \qquad for \quad t \in [-K, K - r]. \tag{22}$$

The properties of x_n easily imply that $||x||_K = 1$, $\dot{x} \geq 0$, and $x(0) = 0$. These properties, the fact that x is a solution of equation (22), and an argument similar to the one that lead us to statement (19), imply that $x(-t)x(t) < 0$, for $t \neq 0$, and $\dot{x}(0) > 0$. Let us define the function

$$y(t) \stackrel{def}{=} -x(-t), \qquad t \in [-K, K] \quad .$$

This function satisfies the equation

$$\dot{y}(t) = +y(t) - \nu y(t - r), \qquad for \quad t \in [-K + r, K], \tag{23}$$

and has the following properties:

$$\dot{y} \geq 0, \tag{24}$$
$$y(0) = 0, \tag{25}$$
$$y(-t)y(t) < 0 \quad \text{for} \quad t \neq 0 \quad . \tag{26}$$

The following lemma contradicts our assumption that $K > 0$ can be chosen arbitrarily large, thus proving lemma 5.

Lemma 6 *There exists $M > 0$ such that if $K > M$ then any solution $y : [-K, K] \to IR$ of equation (23) cannot simultaneously satisfy properties (24), (25), and (26).*

In order to prove this lemma we need some definitions from the theory of linear delayed differential equations (see [2], [1]). The characteristic equation related to equation (23) is

$$P(\lambda) \stackrel{def}{=} \lambda - 1 + \nu e^{-r\lambda} = 0 \quad . \tag{27}$$

All the roots of the characteristic equation are on the left hand side of a vertical straight line (c) in the complex plane. The fundamental solution ξ of equation (23) is defined as the one that satisfies $\xi(t) = 0$ for $t < 0$, and $\xi(0) = 1$. For $0 \leq t \leq r$ it is explicitly given by $\xi(t) = e^t$. The Laplace transform of ξ can be easily written in terms of P as

$$\hat{\xi}(u) \stackrel{def}{=} \int_0^\infty e^{-ut}\xi(t)dt = \frac{1}{P(u)} \quad . \tag{28}$$

The function $\hat{\xi}$ is defined for u complex and is analytic on the left hand side of the line (c). Using the inverse integral for the Laplace transform (see [2], [1]) one can show that the fundamental solution has the following integral representation in terms of P

$$\xi(t) = \int_{(c)} \frac{1}{P(\lambda)} d\lambda \quad . \tag{29}$$

Let

$$\eta \stackrel{def}{=} \max\{Re\lambda | P(\lambda) = 0\} \quad . \tag{30}$$

There is at most one pair of complex conjugate roots $\lambda_1, \overline{\lambda_1}$ of (27) (and only one root when λ_1 is real) such that $\mathrm{Re}\lambda_1 = \eta$. In the case that λ_1 is not real, then $\lambda_1, \overline{\lambda_1}$ are simple roots of the characteristic equation (27). Let

$$\eta' \stackrel{def}{=} \max\{Re\lambda | P(\lambda) = 0, \quad \lambda \neq \lambda_1, \quad \lambda \neq \overline{\lambda_1}\} < \eta \quad . \tag{31}$$

Using (29) it can be shown [2], [1] that if $\eta < 0$ then there exist $0 < a < -\eta$, and $b > 0$, such that

$$|\xi(t)| < be^{-at}, \qquad t > 0 \quad . \tag{32}$$

If $\eta \geq 0$ and $\lambda_1 = \eta + \omega i$, $\omega \neq 0$, then there exist constants $a \neq 0$, $b > 0$, $c \in [0, 2\pi)$, and $d \in (\eta', \eta)$, such that

$$|\xi(t) - ae^{\eta t}\cos(\omega t + c)| \leq be^{td}, \qquad t \geq 0 \quad . \tag{33}$$

This estimate is a consequence of formula (29) and the residue theorem (see [1] p. 116, ex.1). For $-K + r \leq t' < t \leq K$ the following "variation of constants formula" (see [2], [1]) is valid

$$y(t) = y(t')\xi(t - t') - \nu \int_{-r}^{0} \xi(t - t' - s - r)y(t' + s)ds \quad . \tag{34}$$

Using this formula and the above properties of ξ we will prove lemma 6. In order to simplify the exposition we break the proof into three propositions.

Proposition 8 *Assume that η defined in (30) satisfies $\eta < 0$ and that equation (23) has a solution y satisfying (25) (26) and such that $\dot{y} \geq 0$ for $t \in [0, r]$. Then there is $M_1 > 0$ such that $y(K) < y(r)$ for all $K > M_1$. In particular y cannot satisfy (24) if $K > M_1$.*

Proof. The variation of constants formula (34) with $t' = r$ and inequality (32) imply

$$\begin{aligned} y(t) &\leq y(r)\left\{|\xi(t-r)| + \nu\int_{-r}^{0}|\xi(t-r-s-r)|ds\right\} \\ &\leq y(r)be^{-a(t-r)}\left\{1+\nu\int_{-r}^{0}e^{a(s+r)}ds\right\} \\ &= y(r)be^{-a(t-r)}\left\{1+\frac{\nu}{a}(e^{ar}-1)\right\}, \end{aligned}$$

where $y(r) > 0$. Now, there is M_1 such that

$$be^{-a(K-r)}\left\{1+\frac{\nu}{a}(e^{ar}-1)\right\} < be^{-a(M_1-r)}\left\{1+\frac{\nu}{a}(e^{ar}-1)\right\} = 1$$

for all $K > M_1$. This implies that $y(K) < y(r)$. This proves the proposition. □

Proposition 9 *Assume that η defined in (30) satisfies $\eta \geq 0$ and that $\lambda_1 = \eta + \omega i$ with $\omega > 0$. Moreover, assume that equation (23) has a solution y satisfying (25) and such that $y(t) < 0$ for $t \in [-K, 0)$. Then there is $M_2 > 0$ such that for all $K > M_2$ there is $t \in (0, K]$ such that $y(t) < 0$. In particular, y cannot satisfy (26) if $K > M_2$.*

Proof. In this case equation (33) implies that

$$|e^{-\eta t}\xi(t) - a\cos(\omega t + c)| \leq be^{-(\eta-d)t} \quad .$$

This equation, the fact that $\eta - d > 0$, and $\xi(t) > 0$ for $t \in [0, r]$, imply that there exists a $t = t_* > r$ such that $\xi(t_*) = 0$ and $\xi(t) > 0$ for $t \in [0, t_*)$. We claim that

$$\xi(t_*) = 0 \quad \Longrightarrow \quad \xi(t) < 0 \quad \text{for} \quad t \in (t_*, t_* + r) \quad . \tag{35}$$

Indeed, ξ satisfies equation (23) implying that $\dot{\xi}(t_*) = -\nu\xi(t_* - r) < 0$. Therefore, $\xi(t)$ is negative in some interval (t_*, δ). If $\xi(\delta) = 0$ and $\delta < t_* + r$ then $\dot{\xi}(\delta) = -\nu\xi(\delta - r) < 0$, which is absurd. So, $\delta \geq t_* + r$ and $\xi(t) < 0$ for $t \in (t_*, t_* + r)$.

Now, let us take $M_2 = t_* + r$ and $K > M_2$. The variation of constants formula (34) with $t' = 0$ and $t = t_* + r$ implies

$$y(t_* + r) = -\nu\int_{-r}^{0}\xi(t_* - s)y(s)ds \quad .$$

Using that $y(s) < 0$ for $s < 0$ and $\xi(t) < 0$ for $t \in (t_*, t_* + r)$, we obtain that $y(t_* + r) < 0$. □

Proposition 10 *Assume that η defined in (30) satisfies $\eta \geq 0$ and that $\lambda_1 = \eta$. Moreover, assume that equation (23) has a solution y satisfying (24) and (25). Then there is $M_3 > 0$ such that $y(-r) \geq 0$ for all $K > M_3$. In particular y cannot satisfy (26) if $K > M_3$.*

Proof.

Let $\zeta : [0, \infty) \to \mathbb{R}$ be the function defined as

$$\zeta(t) \stackrel{def}{=} \xi(t) - \nu\int_{-r}^{0}\xi(t-s-r)ds \tag{36}$$

Supose that there exists $\bar{t} > 0$ such that

$$\zeta(\bar{t}) \leq 0, \quad . \tag{37}$$

Let us take $M_3 = \bar{t} + 2r$ and $K > M_3$. The variation of constants formula (34) with $t' = -\bar{t} - r$ and $t = -r$ implies

$$y(-r) = y(-\bar{t}-r)\xi(\bar{t}) - \nu\int_{-r}^{0}\xi(\bar{t}-s-r)y(-\bar{t}-r+s)ds \quad . \tag{38}$$

Using that y is nondecreasing and that $y(0) = 0$ we obtain that $y(-\bar{t}-r+s) \leq y(-\bar{t}-r) \leq 0$ for $s \in [-r, 0]$. This, equation (38), and (37) imply that

$$y(-r) \geq y(-\bar{t}-r)\left\{\xi(\bar{t}) - \nu\int_{-r}^{0}\xi(\bar{t}-s-r)ds\right\} = y(-\bar{t}-r)\zeta(\bar{t}) \geq 0 \quad .$$

Therefore, in order to finish the proof of this proposition we just have to show that there exists $\bar{t}$ such that (37) is true. This is done in the following.

Let us define the function $\hat{\zeta} : (\eta, \infty) \to \mathbb{R}$ as

$$\hat{\zeta}(u) \stackrel{def}{=} \int_{0}^{\infty} e^{-ut}\zeta(t)dt \quad . \tag{39}$$

This definition and definition (36) of ζ imply that

$$\begin{aligned} \hat{\zeta}(u) &= \int_{0}^{\infty} e^{-ut}\zeta(t)dt \\ &= \int_{0}^{\infty} e^{-ut}\xi(t)dt - \nu\int_{0}^{\infty}\int_{-r}^{0} e^{-ut}\xi(t-s-r)dsdt \\ &= \hat{\xi}(u) - \nu\int_{-r}^{0}\int_{0}^{\infty} e^{-ut}\xi(t-s-r)dtds \\ &= \hat{\xi}(u) - \nu\int_{-r}^{0} e^{-u(s+r)}\int_{-s-r}^{\infty} e^{-ut'}\xi(t')dt'ds \\ &= \hat{\xi}(u)\left\{1 - \frac{\nu(1-e^{-ur})}{u}\right\} = \frac{\hat{\xi}(u)}{u}[u - \nu + \nu e^{-ur}], \end{aligned} \tag{40}$$

where $\hat{\xi}(u)$, $u \in (\eta, \infty)$, is the Laplace transform of ξ restricted to the infinite interval (η, ∞). Equations (28) and (40) imply that

$$\hat{\zeta}(u) = \frac{\hat{\xi}(u)}{u}[u - \nu + \nu e^{-ur}] = \frac{1}{uP(u)}[u - \nu + \nu e^{-ur}]. \tag{41}$$

Notice that

$$P(u) > 0 \qquad \text{for} \quad u > \eta \quad , \tag{42}$$

because η is the largest real root of $P(u) = 0$ and $P(u) \to \infty$ as $u \to \infty$. Using that $P(\eta) = 0$, P is continuous, and $\nu > 1$, we obtain that there is an $\epsilon > 0$ such that

$$u - \nu + \nu e^{-ur} = P(u) - \nu + 1 < 0 \qquad \text{for} \quad u \in (\eta, \eta + \epsilon] \quad . \tag{43}$$

Combining equations (41), (42), and (43) we obtain that $\hat{\zeta}(\eta + \epsilon) < 0$. This and the definition (39) of $\hat{\zeta}$ imply that $\zeta(t)$ must be negative on some interval, which implies the existence of $\bar{t}$ as stated in (37). □

Propositions 8, 9, and 10, exhausts all the possibilities for η. Therefore lemma 6 is proved and so is theorem 1. □

References

[1] R. Bellman and K. Cooke (1963): *Differential Difference Equations*, Academic Press, New York.

[2] J. K. Hale (1977): *Theory of Functional Differential Equations*, Springer-Verlag, New York.

[3] J. Mallet-Paret and R. Nussbaum (1986): "Global continuation and asymptotic behavior for periodic solutions of a diff-delay equation", *Annali di Matematica Pura ed Applicada* (4) **CXLV**, 33-128.