UNIVERSIDADE DE SÃO PAULO

INSTITUTO DE FÍSICA
CAIXA POSTAL 66318
05389-970 SÃO PAULO - SP
BRASIL

PUBLICAÇÕES

IFUSP/P-1235

QUANTUM CORRECTIONS TO THE CLASSICAL DYNAMICS: APPLICATION TO THE SU(2) LIPKIN MODEL

M. Trindade dos Santos
Instituto de Física, Universidade de São Paulo

M.C. Nemes
Universidade Federal de Minas Gerais,
Departamento de Física, ICEX,
C.P. 702, 30161-970 Belo Horizonte, MG, Brazil

Agosto/1996

Quantum Corrections to the Classical Dynamics: Application to the SU(2) Lipkin Model [1]

M.Trindade dos Santos[(1)], M.C.Nemes[(2)]

[(1)] Universidade de São Paulo,
Instituto de Física,
Departamento de Física Matemática,
C.P. 66318 05389-970 São Paulo, SP, Brazil.

[(2)] Universidade Federal de Minas Gerais,
Departamento de Física, ICEX,
C.P. 702, 30161-970 Belo Horizonte, MG, Brazil.

Abstract

We show how nonrelativistic many body techniques can be used to study quantum corrections to the classical limit, in particular of the $SU(2)$ Lipkin Model. We show that the quantum corrections are essentially of two types: unitary and nonunitary. In this work we perform a detailed study of the unitary corrections. They can be cast in Hamiltonian form and are shown to double the number of degrees of freedom. As a consequence chaotic behavior emerges. We show that this semiquantal chaos is the mechanism trough which tunneling is effected. We also show that these corrections systematically improve the classical results and propose some quantitative measure of this improvement.
PAC number (21.10)

[1] *This work was partially supported by FAPESP and CNPq.*

1 Introduction

One of the most fascinating open problems in the area of dynamical systems is the search for an adequate semiclassical description of quantum systems. Several attempts focus on obtaining $\hbar$ corrections to the dynamics of the corresponding classical system. A pioneer work along this idea is the semiclassical method proposed by Einstein-Brilloin-Keller [1] which leads to quantization rules for integrable systems. Other important contributions are approximations to the Feynman path integral formulation [2], used to derive the periodic orbit trace formula for chaotic systems: The Gutzwiller trace formula [3]. This relates the spectrum of quantum systems to a weighted sum over the unstable periodic orbits of the classical system.

Recently semiquantal dynamics has been derived via Ehrenfest's theorem and recast as an extended classical system with the fluctuation variables coupled to the average variables [4] [5]. A different approach which can be shown to yield identical results is the time dependent variational principle where the true solution is approximated by squeezed states, the so called gaussian variational approximation [6], [7], [8] and [9]. Other methods for this propose are quoted in ref. [10]. The Wigner formalism has also recently been applied to study diffusive and dissipative type of corrections [11] and [12] .

The purpose of the present contribution is twofold. The first one is of formal character. We show that a mean field expansion in the sense of nonrelativistic many body theories can be used as a consistent and systematic tool to analyse the nature of quantum corrections to the classical limit. We show that such corrections can be classified in two types: unitary quantum corrections, which amount to considering (on top of the dynamics of the average values) the dynamics of the width of gaussian wave packets. The number of degrees of freedom of this system is therefore doubled, and the semiclassical limit can be cast into the form of Hamilton's equations. This unitary dynamics reveals in particular general features of quantum kinematics - a centrifugal type barrier involving the width of the packet is the classical counterpart of the uncertainty principle - and the resulting dynamics is in general chaotic even if we start from a simple integrable system, as we will show. The same result in spirit has been obtained in the context of a variational calculation by Kerman and Pattanyak [4] in the framework of Heisemberg's equations of motion. The second type of corrections to the classical limit in the context of many body mean field calculation is non unitary in character. It arises from the inclusion of quantum correlation contributions. We show that their inclusion induce, given adequate approximations, a Langevin-type force

on top of the Hamiltonian description provided by the unitary evolution. They can thus formally account for diffusive and dissipative type behaviors, making immediate contact with the work of reference [11].

The second purpose of the present work is to explore in detail the unitary type corrections in the context of the integrable $SU(2)$ Lipkin Model [13]. The reason for choosing this particular model, aside its simplicity, resides in the fact that it possesses a well defined classical limit when the number of particles N goes to infinity [14]. In this case it can be rigorously shown that the classical limit corresponds to constructing the Hamiltonian dynamics for the parameters of coherent states, a special case of gaussian wave packets. In this case we will be safe to perform comparisons between the classical, unitary semiclassical and exact dynamics. We show that the derived corrections improve the classical results both qualitatively and quantitatively. From the qualitative point of view we show that the presence of quantum degrees of freedom (widths of wave packet) induces chaotic motion and this is the mechanism through which quantum tunneling is effected, a clear quantum correction to the classical description. Also from the qualitative point of view we analytically study the time evolution of observable for short times, investigating in this way modifications introduced by quantum corrections. From the quantitative point of view, in order to assess the time of validity of the approximation we perform a comparative numerical study of the time evolution given by the approximations and the exact result for the time evolution of observable. We show that despite of chaotic behavior the semiclassical or gaussian approximation gives a better description of the dynamics. We also set up a quantitative measure of the time of validity of the approximation according to which the gaussian approximation is better than the classical one and this quality increases with N, as it should.

This paper is divided as follows: section 2 contains the formalism based on which we define the semiclassical limit, section 3 contains an application to the $SU(2)$ Lipkin Model, section 4 contains a discussion of our numerical results and section 5 contains some conclusions.

2 The Formalism:

One of the most widely used method to construct the classical limit of a quantum system is by means of coherent states [15]. As is well known such method can be viewed as a mean field approximation where the width of the wave packet is minimal and remains unchanged during the time evolution.

The idea of defining the semiclassical limit as some kind of mean field approximation is appealing. A very general and unique definition of mean field approximation can be given once one requires that the density function which should be used to calculate traces be the one which reproduces in exact form all expectation values of one body operators. In other words a gaussian density operator [16].

The formalism stated below is a simple application of techniques developed before for the treatment of the reduced dynamics of gaussian observable of interacting many boson systems in Many Body Nuclear Physics and Quantum Field Theory [17]. We make here an option for self-containedness. In the eventual lack of technical details the reader is referred to the works in references [9], [17], and [18].

Quantum bosonic states are represented by density operators F so that mean values of a chosen operator $\mathcal{O}$ are given in terms of traces, e.g.

$$< \mathcal{O} >= Tr(\mathcal{O}F) \quad , \tag{1}$$

and the basic dynamical equation is the Liouville-von Neumann equation for F

$$\imath \dot{F} = [H, F] \quad , \tag{2}$$

where H is the Hamiltonian of the system.

Gaussian states are particular densities which are fully determined by the mean values of the field operators and their bilinear or quadratic combinations, in addition to statistics. For bosonic systems ($[a, a^\dagger] = 1$) the relevant quantities are $< a^\dagger >$, $< a^\dagger a >$, and $< a^\dagger a^\dagger >$. The first of these can be conveniently parameterized in terms of two real quantities q and p

$$< a^\dagger >= Tr(a^\dagger F_0) = \sqrt{\frac{\mu_0}{2}}(q - i\frac{p}{\mu_0}) \quad , \tag{3}$$

where μ_0 is a scale parameter (we set μ_0=1). To deal with the other two one can define new Bogoliubov quasiboson operators as

$$\begin{pmatrix} \tilde{\eta} \\ \tilde{\eta}^\dagger \end{pmatrix} = \mathbf{X}^\dagger \begin{pmatrix} \tilde{a} \\ \tilde{a}^\dagger \end{pmatrix} \quad , \tag{4}$$

where $\tilde{\eta} = \eta - <\eta>$, $\tilde{a} = a - <a>$ and $\mathbf{X}$ define the Bogoliubov transformation. It is given by

$$\mathbf{X} = \begin{pmatrix} x^* & y \\ y^* & x \end{pmatrix} \quad , \tag{5}$$

satisfying the normalization condition

$$\mathbf{X}^\dagger \mathbf{G} \mathbf{X} = \mathbf{G} \tag{6}$$

with

$$\mathbf{G} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad \mathbf{G}^2 = \mathbf{1} \quad . \tag{7}$$

The preservation of the commutation relations $[\eta, \eta^\dagger] = 1$ requires as usual that the transformation coefficients x and y be chosen so that $|x|^2 - |y|^2 = 1$. This is guaranteed by equation (6).

For the bilinear operators equation (4) implies that

$$\mathbf{N} = \mathbf{X}^\dagger \mathbf{R} \mathbf{X} \quad , \tag{8}$$

where

$$\mathbf{N} = \begin{pmatrix} <\tilde{\eta}^\dagger \tilde{\eta}> & <\tilde{\eta}\tilde{\eta}> \\ <\tilde{\eta}^\dagger \tilde{\eta}^\dagger> & <\tilde{\eta}\tilde{\eta}^\dagger> \end{pmatrix} \quad , \quad \mathbf{R} = \begin{pmatrix} <\tilde{a}^\dagger \tilde{a}> & <\tilde{a}\tilde{a}> \\ <\tilde{a}^\dagger \tilde{a}^\dagger> & <\tilde{a}\tilde{a}^\dagger> \end{pmatrix} \quad . \tag{9}$$

This Bogoliubov transformation is so chosen that from a direct calculation one has

$$Tr(\tilde{\eta}\tilde{\eta} F_0) = Tr(\tilde{\eta}^\dagger \tilde{\eta}^\dagger F_0) = 0 \quad . \tag{10}$$

This gives us also the quantity $\nu = <\tilde{\eta}^\dagger \tilde{\eta}>$ which is the occupation probability. We can thus parameterize $<a^\dagger a^\dagger>$ and $<a^\dagger a>$ in terms of x, y and ν. As a consequence of this the dispersion of the quadratures Δq and Δp must depend on x,y and ν only

$$\Delta q^2 = \frac{1}{2\mu_0}[<(a^\dagger + a)>^2 - (<a>^* + <a>)^2] =$$

$$= \frac{1}{2\mu_0}[1 + 2|y|^2 - (x^* y + y^* x)](1 + 2\nu) \quad , \tag{11}$$

$$\Delta p^2 = -2\mu_0[<(a^\dagger - a)^2> - (<a>^* - <a>)^2] =$$

$$= 2\mu_0[1 + 2|y|^2 + (x^* y + y^* x)](1 + 2\nu) \quad . \tag{12}$$

As we will show in the sequel it is possible to define a pair of canonically conjugate variables associated to Δq and Δp as follows

$$\Delta q^2 = Q^2 \quad ,$$

$$\Delta p^2 = P^2 + \frac{(2\nu + 1)}{Q^2} \quad .$$

The gaussian state so specified is given by

$$F_0 = \frac{1}{1+\nu}\left(\frac{\nu}{1+\nu}\right)^{\tilde{\eta}^\dagger \tilde{\eta}} \tag{13}$$

$$= \sum_n |n> \frac{1}{1+\nu}\left(\frac{\nu}{1+\nu}\right)^n <n| \quad ,$$

where $\tilde{\eta}^\dagger \tilde{\eta}|n> = \nu|n>$, is the state which contains all the information about the operators in question and only this. Notice that the operator $\tilde{\eta}^\dagger \tilde{\eta}$ in the expression of F_0 is a linear combination of the bosonic operators $a^\dagger, a, a^\dagger a, aa, a^\dagger a^\dagger$ and the coefficients are functions of q, p, x, y and ν. More technical details are given in the didactic work by de Cloizeaux [19]. It is a simple matter to check that F_0 need not be a pure state and that in the limit $\nu \to 0$ it goes to the pure gaussian state $|0><0|$. This state precisely corresponds to the variational state used in refs. [6] and [8]. The usual coherent state approximation corresponds to setting $\nu = 0$ and moreover requiring the fullfilment of the minimal uncertainty relation, $x = 1$ and $y = 0$ for all times.

Notice that the equation $Tr(\tilde{\eta} F_0) = Tr(\tilde{\eta}^\dagger F_0) = 0$ is a consequence of the definition of $\tilde{\eta}$ and therefore imposes no constraints on x and y. Therefore we have four undetermined parameters Re(x) ,Im(x), Re(y) and Im(y) and three conditions to fix them, namely, eq. (10) and the normalization condition. The fourth one can be obtained by imposing an overall null phase in the state F_0.

We now sketch the mean field expansion method: In general the state F is not of the form F_0, but even so can be used to evaluate $<a^\dagger>, <a^\dagger a^\dagger>$ and $<a^\dagger a>$ and hence a set of gaussian parameters. In terms of these we can set up a gaussian state F_0 and split the complete state F as

$$F = F_0 + F' \tag{14}$$

where $Tr(F') = 0$, so that F' is a pure correlation part of F. We make essential use of the fact that the decomposition (14) can be implemented in terms of a projection operator $P(t)$ such that $F_0 = P(t)F$ (the explicit form of $P(t)$ for bosons is given in refs. [9] and [17]) to rewrite eq. (14) as a set of coupled equations for F_0 and $F'(t)$. This eventually allows one to write $F'(t)$ as a function of the past history of the gaussian projection $F_0(t)$:

$$F'(t) = F'[F_0(t' < t)] \tag{15}$$

Using this result one can close the equations of motion for the gaussian parameters again by taking appropriate traces of the Liouville - von Neuman equation (2) and using $F'(t) = F_0(t) + F'[F_0(t' < t)]$. The first term will reproduce the mean field result, while the second will give rise to additional terms involving memory integrals (refs [9] and [17]). As a matter of fact, the expression for equation (15) is in general not computable even in simple model problems without approximations. However a consistent and systematic approximation scheme has been constructed based on a criterium of energy conservation [20]

$$< H >= Tr(HF_0) + Tr(HF')$$

so that to every order of approximation one makes sure that $d < H > /dt = 0$. As shown in ref. [20] and implemented in various systems [18] [21] this criterium leads to a systematic, controllable expansion around the mean field approximation.

Since we have defined $\nu = Tr(F_0\tilde{\eta}^\dagger\tilde{\eta})$ the occupation probability does not evolute on time on the mean field approximation level (unitary contributions). We have

$$\imath\dot{\nu} = Tr[\tilde{\eta}^\dagger\tilde{\eta}, H]F_0 = TrH[F_0, \tilde{\eta}^\dagger\tilde{\eta}] \quad ,$$

and $[F_0, \tilde{\eta}^\dagger\tilde{\eta}] \equiv 0$ (see eq.(13)) The nonunitary contributions will come from the time evolution of quantum correlations,

$$\imath\frac{d}{dt}\tilde{\eta}^\dagger\tilde{\eta} = \imath\dot{\nu}(t) = Tr([\tilde{\eta}^\dagger\tilde{\eta}, H]F') \quad , \tag{16}$$

in the form of an explicit time dependence on the occupation probability ν. This has been explicitly studied in the context of the anharmonic oscillator [17] and of the Maser Model [21], showing both qualitatively and quantitatively what are the effects of many body

correlations at the level of one body observables. In the present work these nonunitary contributions have been neglected, since they are of higher order than the unitary ones. It is however worthwhile noticing that the inclusion of such corrections may lead to Fokker-Planck type equations making thus immediate contact with the works of refs. [11] and [12].

Let us now make the connection between this general mean field expansion and the classical limit plus corrections. From the point of view of this formalism the classical limit corresponds to the following scheme:

Exact

Quantum Mechanics of Many Body Systems

State Space:
F=$F_0(q, p, Q, P; \nu)$ + F'(correlations)

Dynamics:
$\imath\dot{F} = [H, F]$

↓

Semiclassical

Most General One Body Mean Field Approximation

State Space:
$F = F_{0sc} = (q, p, Q, P; \nu = 0)$

Dynamics:

Mean Values Dynamics: $\dot{p} = -\frac{\partial}{\partial q}\mathcal{H}_{sc}$, $\dot{q} = \frac{\partial}{\partial p}\mathcal{H}_{sc}$

Width Dynamics: $\dot{P} = -\frac{\partial}{\partial Q}\mathcal{H}_{sc}$, $\dot{Q} = \frac{\partial}{\partial P}\mathcal{H}_{sc}$

$\mathcal{H}_{sc} = Tr(F_{0sc}H) = \mathcal{H}_{cl} + \mathcal{H}_{correc}$

↓

Classical

One Body Coherent State Approximation
Kinematics: $F = F_{0cl}(q,p;\nu=0)$ **Dynamics:** $\dot{p} = -\frac{\partial}{\partial q}\mathcal{H}_{cl}$, $\dot{q} = \frac{\partial}{\partial p}\mathcal{H}_{cl}$ $\mathcal{H}_{cl} = Tr(F_{0cl}H)$

where F_{0cl} is obtained from eq.(13) and the Bogoliubov transform (4) by setting $x = 1$, $y = 0$ and $\nu = 0$, *i.e.*, F_{0cl} corresponds to a coherent minimum uncertainty state. On the other hand F_{0sc} incorporates the dynamics of the quadratures $< a^\dagger a >$ and $< a^\dagger a^\dagger >$ related to the variances (eqs. (11) and (12)) and is a pure state ($\nu = 0$). The dynamics of $< a^\dagger a >$ and $< a^\dagger a^\dagger >$ enter as a correction ($\mathcal{H}_{correc}$) to the classical one ($\mathcal{H}_{cl}$). Of course considering $\nu \neq 0$ would enlarge the class of gaussian states so as to encompass mixtures. Since there are no rigorous classical results available which cover this generality we restrict ourselves to $\nu = 0$.

Since on the one body mean field approximation the occupation probability ν does not depend on time we have an even number of parameters which characterize the time evolution of F_0, namely q, p, x and y - in this case ν enter the dynamics as a free parameter. This enable one to cast (in a simple way) the variances Δq and Δp in the form of canonical variables (Q, P) [18] quoted in the above scheme. We define

$$x = \cosh\sigma + \imath\frac{\tau}{2} \quad , \tag{17}$$

$$y = \sinh\sigma + \imath\frac{\tau}{2} \quad , \tag{18}$$

and then

$$P = \sqrt{\frac{(1+2\nu)}{2}}\ \tau \quad , \tag{19}$$

$$Q = \sqrt{\frac{(1+2\nu)}{2}}\ e^{-\sigma} \quad . \tag{20}$$

In the semiclassical level, defined by F_{0sc}, the transformations are the same, setting $\nu = 0$. The connection of these semiclassical variables with the variables used in refs. [5] and [6] is made by a simple transformation

$$G = Q^2 \quad ; \quad \Pi = \frac{P}{2Q} \quad ; \nu = 0 \quad . \tag{21}$$

It is a simple matter to check that eqs. (11) and (12), in terms of these variables acquire a very simple interpretable form

$$\Delta q^2 = Q^2 \quad , \tag{22}$$

$$\Delta p^2 = P^2 + \frac{(1+2\nu)^2}{4Q^2} \quad . \tag{23}$$

It becomes clear how the uncertainty principle will manifest in the semiclassical approximation: The fact that the width of the wave packet cannot be zero is translated in classical terms by a centrifugal barrier

$$\frac{(1+2\nu)^2}{4Q^2} \quad . \tag{24}$$

of course this term will be introduced also into the dynamics and coupled to the other degrees of freedom. The uncertainty product is given by

$$\Delta q\Delta p = \sqrt{\frac{(1+2\nu)^2}{4} + P^2Q^2} \ \geq \ \frac{(1+2\nu)}{2} \quad , \tag{25}$$

and the minimal uncertainty situation in the limit $\nu \to 0$ is $\Delta q = \Delta p = \sqrt{\frac{1}{2}}$. We get it with $P = 0$ and $Q = \sqrt{\frac{1}{2}}$.

The presence of the centrifugal barrier in equation (25) shows that it came from the gaussian approximation in a purely kinematical way and therefore does not depend on the dynamics H.

The next step is to obtain the time evolution of the complete set of gaussian parameters. This is done by means of the quantum equation of motion

$$\imath\frac{d}{dt} < \mathcal{O} >= Tr(F[\mathcal{O}, H]) \quad , \tag{26}$$

where $\mathcal{O}$ is any operator we have chosen as relevant. Let us calculate the l.h.s. of eq. (26).

We calculate the time derivative of the matrix N (eq. (8))

$$\mathbf{N} = \mathbf{X}^\dagger \mathbf{R} \mathbf{X} = \begin{pmatrix} \nu & 0 \\ 0 & 1+\nu \end{pmatrix} \quad , \tag{27}$$

$$\frac{d}{dt}\mathbf{N} = \frac{d}{dt}\mathbf{X}^\dagger \mathbf{R} \mathbf{X} \tag{28}$$

which gives

$$\mathbf{X}^\dagger \dot{\mathbf{R}} \mathbf{X} = \dot{\mathbf{N}} - \dot{\mathbf{X}}^\dagger \mathbf{R} \mathbf{X} - \mathbf{X}^\dagger \mathbf{R} \dot{\mathbf{X}} \quad , \tag{29}$$

and with help of the normalization condition $\mathbf{X}^\dagger \mathbf{G} \mathbf{X} = \mathbf{G}$,

$$\mathbf{X}^\dagger \dot{\mathbf{R}} \mathbf{X} = \dot{\mathbf{N}} - \dot{\mathbf{X}}^\dagger \mathbf{G} \mathbf{X} \mathbf{G} \mathbf{N} - \mathbf{N} \mathbf{G} \mathbf{X}^\dagger \mathbf{G} \dot{\mathbf{X}} \quad . \tag{30}$$

Writing explicitly the above matrix equation and comparing them element by element, we obtain

$$i\frac{d}{dt}\tilde{\eta}^\dagger \tilde{\eta}^\dagger = i(\dot{x}y - x\dot{y})(1+2\nu) = Tr([\tilde{\eta}^\dagger \tilde{\eta}^\dagger, H]F_0) \quad . \tag{31}$$

For the condensate $< a >$ the equation of motion is obtained directly from the parameterization (3) and the Bogoliubov transformation as

$$i\frac{d}{dt}\{\sqrt{\frac{\mu_0}{2}}(q + \frac{i}{\mu_0}p)\} = xTr([\eta, H]F_0) - y^* Tr([\eta^\dagger, H]F_0) \quad . \tag{32}$$

The Hamiltonian dynamics in terms of canonical variables is obtained with transformations (17)-(20) which splits eqs. (30) and (31) into a set of four equations for the corresponding real quantities q, p, Q and P.

3 The Model and the Semiclassical Dynamics: 1/N corrections

The $SU(2)$ Lipkin Model [13]

$$H = \epsilon J_z + \frac{V}{2}(J_+^2 + J_-^2) \quad , \tag{33}$$

is characterized in Schwinger's representation by [22]

$$H = \frac{1}{2}\epsilon(b^\dagger b - a^\dagger a) + \frac{V}{2}[b^\dagger b^\dagger a a + a^\dagger a^\dagger b b] \quad , \tag{34}$$

where $b^\dagger$ and $a^\dagger$ are the creation operators for bosons

$$[a, a^\dagger] = 1 \qquad\qquad [b, b^\dagger] = 1 \quad . \tag{35}$$

Here we have $\hbar = 1$. The realization of SU(2) algebra in this representation is

$$J_z = \frac{1}{2}(b^\dagger b - a^\dagger a) \quad , \tag{36}$$

$$J_+ = b^\dagger a \quad , \quad J_- = a^\dagger b. \tag{37}$$

The number of particles in the system is given by $N = b^\dagger b + a^\dagger a$ and the Casimir operator J^2 have eigenvalues equal to $N/2(N/2+1)$, with $N = 2J$.

In order to obtain the mean field approximation we first define the normalized density matrix $F_0(\tilde{\alpha}^\dagger \tilde{\alpha}, \tilde{\beta}^\dagger \tilde{\beta}; \nu_\alpha, \nu_\beta)$, *i.e.*, encompassing mixtures. The corresponding semiclassical approximation may be immediately obtained by setting $\nu_i = 0$. We have

$$F_0 = F_{0\alpha} \otimes F_{0\beta} \quad , \tag{38}$$

where

$$F_{0\alpha} = \frac{1}{1+\nu_\alpha}(\frac{\nu_\alpha}{1+\nu_\alpha})^{\tilde{\alpha}^\dagger \tilde{\alpha}} \quad ; \quad F_{0\beta} = \frac{1}{1+\nu_\beta}(\frac{\nu_\beta}{1+\nu_\beta})^{\tilde{\beta}^\dagger \tilde{\beta}} \tag{39}$$

and

$$\tilde{\alpha} = x_a(a - < a >) + y_a(a^\dagger - < a^\dagger >) \quad , \tag{40}$$

$$\tilde{\beta} = x_b(b - < b >) + y_b(b - < b^\dagger >) \quad . \tag{41}$$

The next tedious but straightforward step is to rewrite the Hamiltonian (33) in terms of the operators in the equations above ($H(a^\dagger a, a^\dagger a^\dagger, aa, b^\dagger b, b^\dagger b^\dagger, bb) \rightarrow H(\tilde{\alpha}^\dagger\tilde{\alpha}, \tilde{\alpha}^\dagger\tilde{\alpha}^\dagger, \tilde{\alpha}\tilde{\alpha}, \tilde{\beta}^\dagger\tilde{\beta}, \tilde{\beta}^\dagger\tilde{\beta}^\dagger, \tilde{\beta}\tilde{\beta})$) so that the necessary traces are easily calculated. We get

$$\imath\dot{\nu}_\alpha = \imath\frac{d}{dt} < \tilde{\alpha}^\dagger\tilde{\alpha} >= TrF_0[\tilde{\alpha}^\dagger\tilde{\alpha}, H] = 0 \tag{42}$$

$$\imath\dot{\nu}_\beta = \imath\frac{d}{dt} < \tilde{\beta}^\dagger\tilde{\beta} >= TrF_0[\tilde{\beta}^\dagger\tilde{\beta}, H] = 0 \tag{43}$$

$$i\frac{d}{dt} < \tilde{\alpha}^\dagger\tilde{\alpha}^\dagger >= TrF_0[\tilde{\alpha}^\dagger\tilde{\alpha}^\dagger, H] \Rightarrow$$

$$\imath(\dot{x}_a y_a - x_a\dot{y}_a)(1+2\nu_\alpha) = -\epsilon x_a y_a(1+2\nu_\alpha) + \tag{44}$$

$$-Vx_a^2(1+2\nu_\alpha)[x_b^{*2} < \beta^\dagger >^2 + y_b^2 < \beta >^2 - x_b^* y_b(2\nu_\beta + 2 < \beta^\dagger >< \beta > +1)] +$$

$$-Vy_a^2(1+2\nu_\alpha)[x_b^2 < \beta >^2 + y_b^{*2} < \beta^\dagger >^2 - x_b y_b^*(2\nu_\beta + 2 < \beta^\dagger >< \beta > +1)] \quad ,$$

$$i\frac{d}{dt} < \tilde{\beta}^\dagger\tilde{\beta}^\dagger >= TrF_0[\tilde{\beta}^\dagger\tilde{\beta}^\dagger, H] \Rightarrow$$

$$\imath(\dot{x}_b y_b - x_b\dot{y}_b)(1+2\nu_\beta) = -\epsilon x_b y_b(1+2\nu_\beta) + \tag{45}$$

$$-Vx_b^2(1+2\nu_\beta)[x_a^{*2} < \alpha^\dagger >^2 + y_a^2 < \alpha >^2 - x_a^* y_a(2\nu_\alpha + 2 < \alpha^\dagger >< \alpha > +1)] +$$

$$-Vy_b^2(1+2\nu_\beta)[x_a^2 < \alpha >^2 + y_a^{*2} < \alpha^\dagger >^2 - x_a y_a^*(2\nu_\alpha + 2 < \alpha^\dagger >< \alpha > +1)] \quad .$$

From equations (42) and (43) we see that $\dot{\nu}_a = \dot{\nu}_b = 0$. This is directly a consequence of the mean field approximation. Thus in the semiclassical level, ν_i enters the dynamics as a free parameter. For the bosonic condensate the equations of motion read

$$i\frac{d}{dt} < a >= x_a Tr[\alpha, H]F - y_a^* tr[\alpha^\dagger, H]F_0 \quad , \tag{46}$$

$$i\frac{d}{dt} < b >= x_b Tr[\beta, H]F - y_b^* tr[\beta^\dagger, H]F_0 \quad , \tag{47}$$

where

$$TrF_0[\alpha^\dagger, H] = -\frac{1}{2}\epsilon[2x_a y_a < \alpha > -(x_a^* x_a + y_a^* y_a) < \alpha^\dagger >] + \tag{48}$$

$$-V(y_a^2 < \alpha > -y_a x_a^* < \alpha^\dagger >)*$$

$$*[x_b^{*2} < \beta >^2 + y_b^{*2} < \beta^\dagger >^2 - x_b y_b^*(2\nu_\beta + 2 < \beta^\dagger >< \beta > +1]$$

$$-V(x_a^2 < \alpha > -x_a y_a^* < \alpha^\dagger >)*$$

$$*[x_b^{*2} < \beta^\dagger >^2 + y_b^2 < \beta >^2 - x_b^* y_b(2\nu_\beta + 2 < \beta^\dagger >< \beta > +1)]$$

$$TrF_0[\beta^\dagger, H] = -\frac{1}{2}\epsilon[2x_b y_b < \beta > -(x_b^* x_b + y_b^* y_b) < \alpha^\dagger >] + \tag{49}$$

$$-V(x_b^2 < \beta > -x_b y_b^* < \beta^\dagger >)*$$

$$*[x_a^{*2} < \alpha^\dagger >^2 + y_a^2 < \alpha >^2 - x_a^* y_a(2\nu_\alpha + 2 < \alpha^\dagger >< \alpha > +1)]$$

$$-V(y_b^2 < \beta > -x_b^* y_b < \beta^\dagger >)*$$

$$*[x_a^2 < \alpha >^2 + y_a^{*2} < \alpha^\dagger >^2 - x_a y_a^*(2\nu_\alpha + 2 < \alpha^\dagger >< \alpha > +1)]$$

Now the formal definition of the classical limit for $1/N$ type models is in order.

From the $SU(2)$ quasispin operators which characterize the algebra of the model one may write [23]

$$[\mathcal{J}_i, \mathcal{J}_j] = \frac{i\hbar}{J}\epsilon_{ijk}\mathcal{J}_k \tag{50}$$

where $\mathcal{J}_i = J_i/J$ and $J = N/2$. The classical limit is mathematically defined as

$$\lim_{J\to\infty}\frac{J}{i\hbar}[\mathcal{J}_i, \mathcal{J}_j] = \epsilon_{ijk}\mathcal{J}_k \quad . \tag{51}$$

For finite J the eigenvalues of the $\mathcal{J}_i$'s are mapped into the interval $[-1, 1]$ and the spectra get denser as J increases while $\hbar/J$ decreases. In Schwinger's representation this procedure implies that the bosonic operators a and b be scaled as [24]

$$a^{(\dagger)} \rightarrow \frac{a^{(\dagger)}}{\sqrt{J}} \tag{52}$$

$$b^{(\dagger)} \rightarrow \frac{b^{(\dagger)}}{\sqrt{J}} \quad . \tag{53}$$

Since we have the Hamiltonian as a function of the Bogoliubov operators $H(\tilde{\alpha}^\dagger\tilde{\alpha}, \tilde{\alpha}^\dagger\tilde{\alpha}^\dagger, \tilde{\alpha}\tilde{\alpha}, \tilde{\beta}^\dagger\tilde{\beta}, \tilde{\beta}^\dagger\tilde{\beta}^\dagger, \tilde{\beta}\tilde{\beta})$ writing down the corresponding equations of motion in the canonical variable phase space is straightforward. Firstly we invert the Bogoliubov transformation to obtain from parameterization (3) the mean values $< \alpha >$ and $< \beta >$ as a function of the scaled quantities $q_i' = q_i/\sqrt{J}$ and $p_i' = p_i/\sqrt{J}$ $(i = a, b)$.

$$< \alpha^\dagger >= \sqrt{\frac{1}{2}}[q_a'(x_a + y_a) - ip_a'(x_a - y_a)] \quad , \tag{54}$$

$$< \beta^\dagger >= \sqrt{\frac{1}{2}}[q'_b(x_b + y_b) - ip'_b(x_b - y_b)] \quad . \tag{55}$$

Then we use equations (17)-(20) writing the Bogoliubov parameters x_i and y_i in terms of Q_i, P_i and ν_i. In this way we get

$$\mathcal{H}_{sc}(q'_i, p'_i, Q_i, P_i) = \mathcal{H}_{cl}(q'_i, p'_i) +$$
$$+\frac{1}{J}\mathcal{H}^1_{correc}(q'_i, p'_i, Q_i, P_i; \nu_i = 0) + \frac{1}{J^2}\mathcal{H}^2_{correc}(Q_i, P_i; \nu_i = 0) \quad , \tag{56}$$

where

$$\mathcal{H}_{cl}(q'_i, p'_i) = \frac{1}{2}\epsilon\{\frac{1}{2}(q'^2_b + p'^2_b) - \frac{1}{2}(q'^2_a + p'^2_a)\} + \tag{57}$$
$$\chi\{\frac{1}{2}(q'^2_a - p'^2_a)\frac{1}{2}(q'^2_b - p'^2_b) + q'_b p'_b q'_a p'_a\} \quad ,$$

$$\mathcal{H}^1_{correc}(q'_i, p'_i, Q_i, P_i) = \frac{1}{2}\epsilon\{\frac{1}{2}(Q^2_b + P^2_b) - \frac{1}{2}(Q^2_a + P^2_a) + \frac{(1+2\nu_\beta)^2}{8Q^2_b} - \frac{(1+2\nu_\alpha)^2}{8Q^2_a}\} + \tag{58}$$
$$-\chi\{\frac{1}{2}(q'^2_b - p'^2_b)[\frac{(1+2\nu_\alpha)^2}{8Q^2_a} - \frac{1}{2}(Q^2_a - P^2_a)] + q'_b p'_b Q_a P_a\} +$$
$$-\chi\{\frac{1}{2}(q'^2_a - p'^2_a)[\frac{(1+2\nu_\beta)^2}{8Q^2_b} - \frac{1}{2}(Q^2_b - P^2_b)] + q'_a p'_a Q_b P_b\} \quad ,$$

$$\mathcal{H}^2_{correc}(Q_i, P_i) = \chi\{[\frac{(1+2\nu_\beta)^2}{8Q^2_b} - \frac{1}{2}(Q^2_b - P^2_b)][\frac{(1+2\nu_\alpha)^2}{8Q^2_a} - \frac{1}{2}(Q^2_a - P^2_a)] + Q_b P_b Q_a P_a\} \quad . \tag{59}$$

In the above equations $\chi = V(2J)$ is the scaled interaction parameter.

In the limit $J \to \infty$ the scaled number of particles

$$\mathcal{N} = Tr[(a^\dagger a + b^\dagger b)F]/J \quad ,$$

$$\mathcal{N} = \frac{1}{2}(q'^2_b + p'^2_b) + \frac{1}{2}(q'^2_a + p'^2_a) + \frac{1}{J}[\frac{(1+2\nu_\beta)^2}{8Q^2_b} + \frac{1}{2}(Q^2_b + P^2_b)] + \frac{1}{J}[\frac{(1+2\nu_\alpha)^2}{8Q^2_a} + \frac{1}{2}(Q^2_a + P^2_a)] \quad , \tag{60}$$

is a constant of motion. We have

$$\frac{d}{dt}\mathcal{N} = \{\mathcal{H}, \mathcal{N}\} = 0 \quad , \tag{61}$$

with the Poisson brackets defined as

$$\{f(q_i, p_i, Q_i, P_i), g(q_i, p_i, Q_i, P_i)\} = \sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} + \frac{\partial f}{\partial Q_i}\frac{\partial g}{\partial P_i} - \frac{\partial g}{\partial Q_i}\frac{\partial f}{\partial P_i} \quad . \tag{62}$$

Although if J is finite, the Poisson bracket $\{\mathcal{H}_{sc}, \mathcal{N}\}$ do not vanishes and we have $\frac{d}{dt}\mathcal{N} \neq 0$.

As is well known the number of particles $N = 2J$ plays the role of a semiclassical expansion parameter in the $SU(2)$ Lipkin Model [25]. Therefore in the limit $N \to \infty$ the Hamiltonian $\mathcal{H}_{cl}(q_i, p_i)$ precisely corresponds to the classical limit of the model usually taken by means of coherent states [25].

The term $\mathcal{H}^1_{correc}(q'_i, p'_i, Q_i, P_i)$ in eq.(56) which is the first order quantum correction contains the dynamics of the quadratures coupled to the mean values. Notice that for the minimal uncertainty initial condition $\{q'_i(0), p'_i(0), Q_i(0) = \sqrt{\frac{1}{2}}, P_i(0) = 0\}$, we have the total energy equal to the classical one $\mathcal{H}_{sc} = \mathcal{H}_{cl}|_{t=0}$ $(\mathcal{H}^1_{correc}|_{t=0} = 0)$. The term $\mathcal{H}^2_{correc}(Q_i, P_i)$ contains only the dynamics of the quadratures, and if one chose an initial condition with the minimal uncertainty situation it does not contribute to the dynamics $\mathcal{H}^2_{correc}(t) = 0$ (for the sake of clarity we now reset the variables $q'_i, p'_i \to q_i, p_i$).

The equations of motion on the $\{q_i, p_i, Q_i, P_i\}$ phase space are obtained as follows: we use (17)-(20) to rewrite the l.h.s. of eqs. (44) and (45) in terms of $\dot{Q}_i$ and $\dot{P}_i$ and parameterization (3) to rewrite the l.h.s. of eqs. (46) and (47) in terms of $\dot{q}_i$ and $\dot{p}_i$. Doing the same transformations on the r.h.s. of (44)-(47) and comparing their real and imaginary parts the two equations for each level a and b split into a set of four real equations for the corresponding real quantities $\dot{q}_i, \dot{p}_i, \dot{Q}_i$ and $\dot{P}_i$ $(i = a, b)$.

$$\dot{q}_a = -\frac{\epsilon}{2}p_a + \chi[q_a(\frac{Q_b P_b}{J} + q_b p_b) - p_a\frac{1}{2}(q^2_b - p^2_b)] + \tag{63}$$
$$+\chi p_a[\frac{(1+2\nu_\beta)^2}{8JQ^2_b} - \frac{1}{2J}(Q^2_b - P^2_b)] \quad ,$$

$$\dot{p}_a = \frac{\epsilon}{2}q_a - \chi[p_a(\frac{Q_b P_b}{J} + q_b p_b) + q_a\frac{1}{2}(q^2_b - p^2_b)] + \tag{64}$$
$$+\chi q_a[\frac{(1+2\nu_\beta)^2}{8JQ^2_b} - \frac{1}{2J}(Q^2_b - P^2_b)] \quad ,$$

$$\dot{q}_b = \frac{\epsilon}{2}p_b + \chi[q_b(\frac{Q_a P_a}{J} + q_a p_a) - p_b\frac{1}{2}(q^2_a - p^2_a)] + \tag{65}$$
$$+\chi p_b[\frac{(1+2\nu_\alpha)^2}{8JQ^2_a} - \frac{1}{2J}(Q^2_a - P^2_a)] \quad ,$$

$$\dot{p}_b = -\frac{\epsilon}{2}q_b - \chi[p_b(\frac{Q_a P_a}{J} + q_a p_a) + q_b\frac{1}{2}(q^2_a - p^2_a)] + \tag{66}$$
$$+\chi q_b[\frac{(1+2\nu_\alpha)^2}{8JQ^2_a} - \frac{1}{2J}(Q^2_a - P^2_a)] \quad ,$$

$$\dot{Q}_a = -\frac{\epsilon}{2J}P_a + \chi\frac{P_a}{J^2}[\frac{(1+2\nu_\beta)^2}{8Q_b^2} - \frac{1}{2}(Q_b^2 - P_b^2)] + \tag{67}$$

$$+\chi[\frac{Q_a}{J}(\frac{Q_bP_b}{J} + q_bp_b) - \frac{P_a}{J}\frac{1}{2}(q_b^2 - p_b^2)] \ ,$$

$$\dot{P}_a = \frac{\epsilon}{2J}(Q_a - \frac{(1+2\nu_\alpha)^2}{4Q_a^3}) + \tag{68}$$

$$+\chi\{\frac{1}{J^2}[\frac{(1+2\nu_\alpha)^2}{4Q_a^3} + Q_a][\frac{(1+2\nu_\beta)^2}{8Q_b^2} - \frac{1}{2}(Q_b^2 - P_b^2)] +$$

$$-\frac{1}{2J}(q_b^2 - p_b^2)[\frac{(1+2\nu_\alpha)^2}{4Q_a^3} + Q_a] - \frac{P_a}{J}(\frac{Q_bP_b}{J} + q_bp_b)\} \ ,$$

$$\dot{Q}_b = +\frac{\epsilon}{2J}P_b + \chi\frac{P_b}{J^2}[\frac{(1+2\nu_\alpha)^2}{8Q_a^2} - \frac{1}{2}(Q_a^2 - P_a^2)] + \tag{69}$$

$$+\chi[\frac{Q_b}{J}(\frac{Q_aP_a}{J} + q_ap_a) - \frac{P_b}{J}\frac{1}{2}(q_a^2 - p_a^2)] \ ,$$

$$\dot{P}_b = \frac{\epsilon}{2J}(\frac{(1+2\nu_\beta)^2}{4Q_b^3} - Q_b) + \tag{70}$$

$$+\chi\{\frac{1}{J^2}[\frac{(1+2\nu_\beta)^2}{4Q_b^3} + Q_b][\frac{(1+2\nu_\alpha)^2}{8Q_a^2} - \frac{1}{2}(Q_a^2 - P_a^2)] +$$

$$-\frac{1}{2J}(q_a^2 - p_a^2)[\frac{(1+2\nu_\beta)^2}{4Q_b^3} + Q_b] - \frac{P_b}{J}(\frac{Q_aP_a}{J} + q_ap_a)\} \ .$$

We next analyze the consequences of these $1/N$ corrections.

4 Results:

What are then the effects of the dynamics of the width (and therefore the uncertainty principle) on the classical dynamics of the Lipkin Model? We will discus our results both qualitatively and quantitatively.

Firstly the semiclassical manifestation of the uncertainty principle is the appearance of new degrees of freedom whose position Q_i cannot be zero. This is formally achieved by a repulsive centrifugal-type potential. The inclusion of these new degrees of freedom destroys the integrability of the classical dynamics and chaotic behavior emerges. Since the Lipkin Model has a regular motion both in the quantum regime and classical limit the semiclassical chaos arises as an artifact of the approximation. The main result we want to stress here is that despite of inducing chaotic behavior quantum $(1/N)$ correction gives a better description of the time evolution of observables. Other interesting feature is: Chaos is the mechanism through which quantum properties are effected on the semiclassical phase space (such as quantum tunneling effect).

Let us now describe the classical dynamics in the four dimensional phase space. Beside the conservation of energy $\mathcal{H}_{cl}$ we have also the constraint in $\mathcal{N}$. Thus, once it is fixed, there exists one, and only one trajectory satisfying $\mathcal{N} =< N > /(J) = 2$ with a given value of $\mathcal{H}_{cl}$. The existence of these two constants of motion enables one to show in the same Poincaré section the trajectories for all available energies for a given value of the interaction parameter χ. In figure 1 we show the well known second order phase transition exhibited by the model in its classical limit. For any value of χ below the critical one $|\chi| < |\chi_{crit}| = 1.0$, the invariant tori are all of the same kind and represent the *rotational* aspect of the dynamics (see figure 1(a)). In this case the possible range of energies is $|\mathcal{H}_{cl}| < 1.0$. In figure 1(a) the energies increase from the boundary (where $\mathcal{H}_{cl} = -1.0$) to the origin (where $\mathcal{H}_{cl} = 1.0$). For $|\chi| > |\chi_{crit}| = 1.0$ the possible range of energies is enlarged $|\mathcal{H}_{cl}| < |(1+\chi^2)/(2\chi)|$. We still have the *rotating* trajectories $|\mathcal{H}_{cl}| < 1.0$ (which we label by E_{rot} in figure 1(b)) and we also have the *deformed* ones $-1.0 < \mathcal{H}_{cl} < (1+\chi^2)/(2\chi)$ and $1.0 < \mathcal{H}_{cl} < -(1+\chi^2)/(2\chi)$ which we label by E_{min} and E_{max} respectively in figure 1(b) ($\chi = -6.0$). The fixed points of the Poincaré map associated to the extreme energies are

$$q_a = 0 \quad ; \quad p_a = \pm\sqrt{(1-\frac{1}{\chi})}$$

$$q_b = 0 \quad ; \quad p_b = \pm\sqrt{(1+\frac{1}{\chi})}\,, \tag{71}$$

which give us

$$\mathcal{H}_{cl} = \frac{1}{2}\frac{1+\chi^2}{\chi} \quad ,$$

and

$$\begin{aligned} q_a &= \pm\sqrt{(1+\frac{1}{\chi})} \quad ; \quad p_a = 0 \\ q_b &= 0 \qquad\qquad\quad ; \quad p_b = \pm\sqrt{(1-\frac{1}{\chi})} \end{aligned} \tag{72}$$

which gives

$$\mathcal{H}_{cl} = -\frac{1}{2}\frac{1+\chi^2}{\chi} \quad .$$

The *rotational* trajectories are isolated from the *deformed* ones by two separatrices S_- ($\mathcal{H}_{cl}(S_-)$=-1.0) and S_+ ($\mathcal{H}_{cl}(S_+)$=1.0). See figure 1(b).

Introducing the correction terms in $\mathcal{H}_{sc}$, and therefore the new degrees of freedom related to widths (Q_i, P_i) coupled to mean values (q_i, p_i), the geometrical structure of the integrable system is destroyed and chaotic behavior emerges. The quantity $\mathcal{N}$ is not a constant of motion any more. For increasing J the integrability of the classical limit is gradually recovered (see figure 2(a)-(c)) and takes place again only in the limit $J \to \infty$.

Of course in the classical domain tunneling effects are completely forbidden. However this it is not the case when the quantum corrections we are dealing with are taken into account. Let us now look at the low energy orbits $(1+\chi^2)/(2\chi) < \mathcal{H}_{cl} < -1.0$ ($\chi = -6.0$) . The classically corresponding invariant tori are localized in the two symmetric regions E_{min} with $p_a < 0$ or $p_a > 0$ exclusively and because of classical integrability the time evolution of any chosen initial condition will be confided on its respective region. Therefore, destroying integrability is the way the quantum correction works to effect quantum tunneling. Choosing an initial condition with energy $\mathcal{H}_{cl} < -1.0$ in the $p_a < 0$ semiplane as an example we show in figure 3(a) its classical Poincaré section. For finite values of J this same initial condition (evolved semiclassically) is able to access the symmetrical region $p_a > 0$ (figure 3(b)). The quantum observable associated to the transition between the semiplanes $p_a < 0$ and $p_a > 0$ is $J_x = (J_+ + J_-)/2$. Its mean value sign on the Poincaré section ($q_b = 0$, $p_b > 0$) is determined by the p_a sign.

$$Sign[< J_x >] = Sign[\frac{1}{2}(q_b q_a + p_b p_a)] = Sign[p_a] \quad .$$

The frequency of such transitions also depends on energy, increasing for energies near the separatrix and decreasing as J increase. A quantitative measure of this process may be achieved by studying its time scale. We define the confinement time T_c as the time interval between two transitions $p_a < 0 \leftrightarrow p_a > 0$. We then divide it by the Poincaré time T_p ,*i.e.*, the amount of time required by starting with an initial condition on the Poincaré plane and evolving until it reaches the plane again. We take the average over the first thousand values of $\bar{T}_c$ and evaluate the dimensionless quantity $\bar{T}_c//\bar{T}_p$ as a function of J for different energies (see figure 4). This quantity can also be interpreted as the average number of iterations of the Poincaré map necessary for a transition to occur. From the figure we note that the transitions become scarce as J increase. For any finite value of J there must occur a transition, although, for large values of J (or energies close to the minimum value) this may require numerically integrating the equations of motion for an *enormous* amount of time. Another interesting feature in figure 4 is the dependence on energy. For an energy close to the separatrix ($\mathcal{H}_{cl} = -1.01$) $\bar{T}_c/\bar{T}_p$ increases slower than for lower energies ($\mathcal{H}_{cl} = -1.1$ and $\mathcal{H}_{cl} = -1.2$). Since tunneling effect manifests itself through chaotic motion it must be more conspicuous where chaos (roughly speaking) persists, *i.e*,, near the separatrix.

Notice that here we do not intend to rigorously define a tunneling rate (in terms of energy splittings), which is an interesting problem in itself in particular for spin systems, we are just characterizing the phenomenon in the gaussian representation.

An important question that naturally arises in this work concerns the time evolution of observables. Since the nonintegrability of the semiclassical description is alien both to the quantum and classical dynamics of the system, it is natural to ask whether the approximation makes sense quantitatively. We show that despite of introducing chaos the inclusion of the quantum degrees of freedom (Q_i, P_i) gives a better approximation to the time evolution of the observable $< J_z > (t)$ which we analyse as an example.

In figure 5 we display $< J_z > (t)$ ($\chi = -6.0$, $J = 4$) for the three cases: The exact calculation, the semiclassical approximation and the classical result. As can be seen from the figure the semiclassical approximation represents an improvement over the classical result. We have checked that this is always true for short enough times. The validity of the approximation is of course sensitive to the value of χ. We next arbitrarily define one possible quantitative measure of the accuracy of the approximations. Consider the expression

$$\Delta_{approx} = \frac{\int | < J_z >_{exact} (t) - < J_z >_{approx} (t)| dt}{\int | < J_z >_{exact} (t)| dt} \qquad (73)$$

In figure 6 we display for three different values of time, the value of Δ_{approx} as a function of $1/J$ for both the classical as well as for the semiclassical approximation. Notice that the error so defined depends linearly on $1/J$ and the classical calculation lies always above the semiclassical one. Figure 6 also shows explicitly that in both cases Δ_{approx} goes to zero as $J \to \infty$. We have also defined a *breakdown* time in the following way: we fix a maximum value for the error $\Delta^{max}_{approx} = 0.12$ and plot the time T_b when this occurs for several values of J in both cases, the classical and the semiclassical (gaussian) approximations. Again, according to this measure we see that the gaussian approximation is systematically better, *i.e.*, it is valid for longer times (see figure 7). It is interesting to notice that the form of the curve is the same for both approximations. The problem of the form of the curve for the breakdown time has been the subject of recent investigation [26].

5 Conclusions:

In the present contribution we have shown how a mean field expansion in the sense of nonrelativistic many body theories can be used to obtain quantum corrections to the classical limit. The unitary time evolution of a gaussian state is shown to contain the classical limit plus corrections coming from allowing the width of the wave packet to become an independent variable. We have discussed the connection between this approach and other approaches in the literature and applied it to the $SU(2)$ Lipkin model. We have performed a detailed analysis of the unitary quantum corrections showing that they give rise to chaotic behavior, which is essentially the mechanism through which the tunneling phenomenon can happen in this context. We have also shown that the quantum corrections systematically improve the results obtained in the classical limit. The question left unexplored in the present work is the effect of nonunitary contributions. We believe this is an important next step, *i.e.*, including the time evolution of occupation probabilities in the dynamics which is rather natural in the present formalism. It would be interesting to cast these contributions in the form of diffusive and dissipative processes. In particular, as can be seen from the equations of motion, when $\nu_i \neq 0$, $\nu = \nu(t)$ the centrifugal barrier will be time dependent, affecting thus in particular the tunneling rates. Work along these lines is in progress.

Acknowledgment: We are indebted to Prof. A. F. R. de Toledo Piza for many fruitful discussions.

References

[1] I. C. Percival, Adv. Chem. Phys. 36, (1977), 1; for a review of this work

[2] L.S. Schulman , Technics and Applications of Path Integration(Wiley - Interscience, New York, 1981)

[3] M.C. Gutzwiller, Journal of Mathematical Physics Vol. 12 N 3 (1971) 343

[4] A. K. Pattanyak and W. C. Schieve, Phys. Rev. Letters 72 (1994), 2855

[5] A. K. Pattanyak and W. C. Schieve, Phys. Rev. E 50, 3601 (1994)

[6] R. Jackiw and Kerman Phys. Lett. 71A (1979), 158

[7] F. Cooper, S. Y. Pi and P. N. Stancioff, Phys Rev. D 34, (1986) 3831

[8] A. Kovner and B Rosenstein, Phys. Rev D 39 (1989) 2332

[9] Chi Yong , A. F. R. de Toledo Piza - Modern Physics Letters A, Vol.5 , No.20, (1990) 1605-1614

[10] B. Eckhardt, Phys. Rep. 163 (1988) 205-297

[11] Luca Bonci, Roberto Roncaglia, Bruce J. West and Paolo Grigolini, Phys. Rev. Lett. Vol. 67 No. 19 (1991) 2593-2596

[12] Luca Bonci, Roberto Roncaglia, Bruce J. West and Paolo Grigolini, Phys. Rev. A Vol. 45, No. 12 (1992) 8490-8500

[13] H.Lipkin,N. Meshkov e A. Glick, Nucl. Phys **62** (1965) 188

[14] L. Yaffe, Rev. Mod. Phys. 54 (1982) 407-435

[15] A. M. Perelomov, Sov. Phys. Usp. 20(9), (sept. 1997) 703-720

[16] C.W.Gardiner, Quantum Noise, Springer-Verlag, Berlin Heidelberg (1991) 128

[17] Lin Chi Yong , A. F. R. de Toledo Piza - Physical Review D, Vol.46, No.2, (1992) 742-756

[18] A.F.R. de Toledo Piza - Physical Review A **51** N 2 (1995) 1612-1616.

[19] J. de Cloizeaux, in *Many Body Physics*, C. de Witt and R. Balian, eds., Gordon and Breach, N. Y. (1968).

[20] P. Buck, H Feld Meier and M. C. Nemes, Annals of Physics Vol. 185, No. 1, (July 1988) 170-190

[21] E. R. Takano Natti and A.F.R. de Toledo Piza, Submitted to Phys. Rev. A.

[22] P. Ring , P, Schuck - The Nuclear Many-Body Problem - Springer-Verlag New York inc. 1980

[23] D Galleti, A.F.R. de Toledo Piza - Physica A **214** (1995) 207-228.

[24] J.L.Van Hemmen, W.Von Waldenfels Physica **100 A** (1980) 83.

[25] M. C. Cambiaggio, G.C. Dussel e M. Saraceno-Condensed Matter Theories, Vol.1 ed. F.B. Malik (Plenum, N. Y., 1986) 57.

[26] F. M. Dittes, E. Doron and U. Smilansky, Phys. Rev. E Vol. 49, No. 2 (1994) R963

Figure 1.a

Captions:

ection on the plane (q_a, p_a) (with $q_b = 0$ and $pb > 0$) for the classical (q_i, p_i) :(a) for $\chi = -0.5 < \chi_{crit}$. (b)for $\chi = -6.0 > \chi_{crit}$. See text for

ection on the plane (q_a, p_a) (with $q_b = 0$ and $pb > 0$) for the semiclassical (q_i, p_i, Q_i, P_i) in the case $\chi = -6.0 > \chi_{crit}$. Initial conditions for the widths the minimal uncertainty situation $(Q_i(0) = \sqrt{1/2}, P_i = 0.0)$. For each initial energies are the same as in figure 1(b) $\mathcal{H}_{sc} = \mathcal{H}_{cl}(t = 0)$. The values of J (b) $J = 8$ and (c) $J = 12$. See text for more details.

ection on the plane (q_a, p_a) (with $q_b = 0$ and $pb > 0$) with $\chi = -6.0 > \chi_{crit}$ itial condition near the separatrix energy $\mathcal{H}_{cl}(\mathrm{S}_-) = -1.0$ in region E_{min}, classical evolution with $\mathcal{H}_{cl} = -1.1$ (in arbitrary ϵ units, $\epsilon = 1.0$). (b) volution with $\mathcal{H}_{sc} = -1.1$ and $J = 9$. The initial conditions for the widths minimal uncertainty

uated over the first thousand values of T_c as a function of J for various below the classical separatrix energy $\mathcal{H}_{cl}(\mathrm{S}_-) = -1.0$.

Time evolution of $< J_z > (t)$ $(\chi = -6.0)$ for the three cases: The exact $= 4)$, semiclassical approximation $(J = 4)$ and classical result. Time t is trary units.

$_{rox}$ evaluated at three different times t (in arbitrary units) and plotted as a J for both semiclassical and classical approximations. $\chi = -0.5$.

Breakdown time T_b (in arbitrary units) for $\Delta^{max}_{approx} = 0.1$ plotted as a J for both semiclassical and classical calculations.$\chi = -0.5$.