

UNIVERSIDADE DE SÃO PAULO

PUBLICAÇÕES

INSTITUTO DE FÍSICA  
CAIXA POSTAL 66318  
05389-970 SÃO PAULO - SP  
BRASIL

IFUSP/P-1257

ON DUALITY SYMMETRY IN THE SCHWARZ-SEN  
MODEL

**H.O. Girotti**

Instituto de Física, Universidade Federal do Rio Grande do Sul,  
CP 15051, 91501-970, Porto Alegre, RS

**V.O. Rivelles and A.J. da Silva**

Instituto de Física, Universidade de São Paulo

Fevereiro/97

Pág. 1 - 10

## On Duality Symmetry in the Schwarz-Sen Model\*

H. O. Girotti<sup>a</sup>, M. Gomes<sup>b</sup>, V. O. Rivelles<sup>b</sup> and A. J. da Silva<sup>b</sup>

<sup>a</sup> *Instituto de Física, Universidade Federal do Rio Grande do Sul, CP15051, 91501-970 Porto*

*Alegre, RS, Brazil.*

<sup>b</sup> *Instituto de Física, Universidade de São Paulo, CP66318, 05315-970, São Paulo, SP, Brazil.*

### Abstract

The continuous extension of the discrete duality symmetry of the Schwarz-Sen model is studied. The corresponding infinitesimal generator  $Q$  turns out to be local, gauge invariant and metric independent. Furthermore,  $Q$  commutes with all the conformal group generators. We also show that  $Q$  is equivalent to the non-local duality transformation generator found in the Hamiltonian formulation of Maxwell theory. We next consider the Batalin-Fradkin-Vilkovisky formalism for the Maxwell theory and demonstrate that requiring a local duality transformation lead us to the Schwarz-Sen formulation. The partition functions are shown to be the same which implies the quantum equivalence of the two approaches.

It is nowadays accepted that the known string theories are different perturbative versions of an underlying M-theory [1]. This idea was originated by the several duality symmetries present in the string theories. It is therefore important to study in detail duality symmetries in field theory and understand its implications. In this respect the old electric-magnetic duality present in Maxwell's equations has again been the subject of intensive study recently [2, 3, 4, 5, 6, 7]. These studies show that there is a conflict between electric-magnetic duality symmetry and manifest Lorentz covariance when we attempt to implement duality at the action level. If manifest Lorentz covariance is maintained then the action is either non-polynomial [2] or requires an infinite set of fields [3]. If we give up manifest Lorentz covariance then duality symmetry can be implemented in the Hamiltonian formulation of Maxwell theory in a non-local way [4]. However a recent proposal made by Schwarz and Sen implements duality in a local way at the expenses of introducing one more potential [5]. Although Schwarz-Sen's formulation is not manifestly covariant it is Poincaré covariant both at the classical [5] and quantum [7] levels. In this paper we shall investigate some consequences of such proposal. In particular, we will construct the generator of duality transformations and discuss its meaning and relation with the corresponding non-local generator found in the first order Hamiltonian formulation described in [4].

The Schwarz-Sen action [5] involves two gauge potentials  $A^{\mu a}$  ( $1 \leq a \leq 2$  and  $0 \leq \mu \leq 3$ ) and is given by

$$S = -\frac{1}{2} \int d^4x (B^{a,i} \epsilon^{ab} E^{b,i} + B^{a,i} B^{a,i}), \quad (1)$$

where

$$E^{a,i} = -F^{a,0i} = -(\partial^0 A^{a,i} - \partial^i A^{a,0}), \quad (2)$$

$$B^{a,i} = -\frac{1}{2} \epsilon^{ijk} F_{jk}^a = -\epsilon^{ijk} \partial_j A_k^a \quad (3)$$

and  $\epsilon$  is the Levi-Civita symbol ( $\epsilon^{12} = 1, \epsilon^{123} = 1$ ) and  $1 \leq i, j, k \leq 3$ . This action is separately invariant under the local gauge transformations,

$$A^{a,0} \rightarrow A^{a,0} + \psi^a, \quad (4)$$

$$A^{a,i} \rightarrow A^{a,i} - \partial^i \Lambda^a \quad (5)$$

and the global SO(2) rotations,

$$A'^{\mu a} = A^{\mu a} \cos \theta + \epsilon^{ab} A^{\mu b} \sin \theta, \quad (6)$$

which reduces to the usual discrete duality transformation for  $\theta = \pi/2$ .

The Noether's charge associated with this SO(2) symmetry is

$$Q = -\frac{1}{2} \int d^3x \epsilon^{ijk} (\partial_j A_i^a) A_k^a = \frac{1}{2} \int d^3x B^{ak} A_k^a. \quad (7)$$

Notice that  $Q$  is a SO(2) invariant Chern-Simons term. Hence, up to surface terms, it is gauge invariant. It is also metric independent and so its algebraic form also holds for curved spaces.

Using the equal time commutators [7],

$$[A^{a,i}(\vec{x}), A^{b,j}(\vec{y})] = -i \epsilon_{ab} \epsilon^{ijk} \frac{\partial_k^x}{\nabla^2} \delta(\vec{x} - \vec{y}), \quad (8)$$

it is straightforward to verify that  $Q$  indeed generates infinitesimal SO(2) rotations,

$$[Q, A_j^b(\vec{y})] = -i \epsilon^{ba} A_j^a(\vec{y}). \quad (9)$$

The Fock space of states is constructed through the action of creation and annihilation operator  $a_\lambda^\dagger$  and  $a_\lambda$  introduced via the Fourier decomposition of  $A^{a,i}$ ,

$$A^{a,i}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2|\vec{p}|}} \sum_{\lambda=1}^2 (e^{-ipx} \epsilon_\lambda^{ai} a_\lambda(p) + e^{ipx} \epsilon_\lambda^{ai} a_\lambda^\dagger(p)), \quad (10)$$

where  $px \equiv |\vec{p}|x^0 - \vec{p} \cdot \vec{x}$  and  $\epsilon_\lambda^{a,i}(p)$ ,  $\lambda = 1, 2$  are unit norm polarization vectors, orthogonal to  $\vec{p}$  and satisfying

$$(g_{ij} \delta_{ab} p_0 - \epsilon_{ab} \epsilon_{ijl} p^l) \epsilon_\lambda^{bj}(p) = 0. \quad (11)$$

This means that  $(\hat{p}, e_1^a(\vec{p}), e_2^a(\vec{p}))$ ,  $a = 1, 2$ , are two orthonormal basis rotated by  $\pi/2$  in the direction of  $\hat{p}$ . The operators  $a_\lambda$  and  $a_\lambda^\dagger$  satisfy the usual algebra

$$[a_\lambda(\vec{p}), a_{\lambda'}^\dagger(\vec{p}')] = \delta_{\lambda\lambda'}\delta(\vec{p} - \vec{p}') \quad (12)$$

In terms of these operators the charge  $Q$  can be rewritten as

$$Q = i \int d^3k (a_1^\dagger a_2 - a_2^\dagger a_1) \quad (13)$$

and becomes diagonal,

$$Q = \int d^3k (a_L^\dagger a_L - a_R^\dagger a_R) \quad (14)$$

in the base of circularly polarized operators, defined by

$$a_R^\dagger = \frac{a_1^\dagger + i a_2^\dagger}{\sqrt{2}} \quad (15)$$

$$a_L^\dagger = \frac{a_1^\dagger - i a_2^\dagger}{\sqrt{2}}. \quad (16)$$

From (14) one sees that, in a generic state,  $Q$  counts the number of left minus right polarized photons.

It is easily checked that  $Q$  commutes with the components  $\theta^{0\mu}$  of the energy momentum tensor. Hence, it commutes with all the generators of the conformal group as should be expected from an internal symmetry.

Identifying the operators  $a_\lambda$  with the corresponding ones in Maxwell's theory one could work backward and find that the charge  $Q$  has the non-local expression,

$$Q = \frac{1}{2} \int d^3x (-\vec{A} \cdot \nabla \times \vec{A} + \vec{E} \cdot \nabla^{-2} \nabla \times \vec{E}), \quad (17)$$

where  $\vec{E}$  is the electric field and  $\vec{A}$  the vector potential in the Coulomb gauge. As described in [4], (17) arises from a first order Hamiltonian formulation of Maxwell action.

The expression (17) for the charge  $Q$  can also be arrived through formal manipulations using the equations of motion. In fact, using the gauge freedom (4) Schwarz and Sen have shown that

$$\vec{B}^2 = \vec{E}^2. \quad (18)$$

Thus, taking the curl of this equation and using the Coulomb gauge condition one has formally

$$\vec{A}^2 = -\nabla^{-2}\nabla \times \vec{E}^1. \quad (19)$$

Equation (17) follows from the replacement of (18) and (19) into (7).

Our discussion indicates that Deser and Teitelboim and Schwarz and Sen implementations of duality are equivalent and both formulations have the same physical content. We now show how this equivalence can be understood in the path integral framework. To see that let us use the Batalin–Fradkin–Vilkovisky formalism [8] for constrained systems. The generating functional for the Maxwell theory is

$$Z = \int \mathcal{D}A_\mu \mathcal{D}\pi_\mu \mathcal{D}c \mathcal{D}\bar{c} \mathcal{D}\mathcal{P} \mathcal{D}\bar{\mathcal{P}} e^{iS_{eff}} \quad (20)$$

with the effective action given by

$$S_{eff} = \int d^4x (\pi^i \dot{A}_i + \pi_0 \dot{A}_0 + \dot{\mathcal{P}}\bar{c} + \dot{\bar{\mathcal{P}}}c - \mathcal{H}_0 - \{\Psi, Q_B\}). \quad (21)$$

As usual,  $\pi_\mu$  is the conjugate momentum of  $A_\mu$ ,  $c$  and  $\bar{c}$  are ghosts and  $\mathcal{P}$  and  $\bar{\mathcal{P}}$  their conjugate momenta. At equal times they satisfy

$$\{\bar{\mathcal{P}}(x), c(y)\} = -\delta(\vec{x} - \vec{y}) \quad \{\mathcal{P}(x), \bar{c}(y)\} = -\delta(\vec{x} - \vec{y}). \quad (22)$$

The BRS charge and Hamiltonian densities are given by

$$Q_B = \partial_i \pi^i c - i\mathcal{P}\pi_0 \quad (23)$$

$$\mathcal{H}_0 = -\frac{1}{2}(\pi^i \pi_i + B^i B_i). \quad (24)$$

It is convenient to choose the gauge fixing function  $\Psi$  as

$$\Psi = \int d^3x (i\bar{c}\chi + \mathcal{P}A_0) \quad (25)$$

and fix  $\chi$  in such way that the Coulomb condition holds. This can be achieved if  $\chi = \frac{1}{\epsilon}\partial_i A^i$ , we redefine the fields  $\pi_0 \rightarrow \epsilon\pi_0$ ,  $\bar{c} \rightarrow \epsilon\bar{c}$  and let  $\epsilon$  go to zero. Notice that this scaling transformation produces a trivial Jacobian and is compatible with the BRS transformation,

$$\delta_{BRS}A_i = \partial_i c, \quad \delta_{BRS}\pi_i = 0, \quad \delta_{BRS}A_0 = i\mathcal{P}, \quad \delta_{BRS}\pi_0 = 0, \quad (26)$$

$$\delta_{BRS}c = 0, \quad \delta_{BRS}\bar{c} = i\pi_0, \quad \delta_{BRS}\bar{\mathcal{P}} = -\partial_i\pi^i, \quad \delta_{BRS}\mathcal{P} = 0 \quad (27)$$

which leaves invariant the generating functional  $Z$ . After taking the limit,  $\epsilon \rightarrow 0$  we get,

$$S_{eff} = \int d^4x (\pi^i \dot{A}_i + \dot{c}\bar{\mathcal{P}} - \mathcal{H}_0 + A_0 \partial_i \pi^i + \pi_0 \partial_i A^i + i\bar{\mathcal{P}}\mathcal{P} - i\bar{c}\nabla^2 c) \quad (28)$$

It is interesting to observe that already at this level there is a duality transformation of the fields which leaves  $Z$  invariant. The infinitesimal form of the transformation is given by

$$\delta A_i = \theta \nabla^{-2} \epsilon_{ijk} \partial^j \pi^k \quad \delta \pi_i = \theta \epsilon_{ijk} \partial^j A^k \quad (29)$$

the variations of the remaining fields being zero. Notice that the variation of  $A_i$  and  $\pi_i$  is the non-local duality transformation introduced by Deser and Teitelboim.  $\delta$  commutes with the BRS transformation, as can be checked.

Integrating (20) in  $\pi_0$  and  $A_0$  produces the usual delta functions which characterize the Coulomb gauge. Performing also the trivial integrations in the ghosts  $\bar{\mathcal{P}}$  and  $\mathcal{P}$  we finally have

$$Z = \int \mathcal{D}A_i \mathcal{D}\pi_i \mathcal{D}c \mathcal{D}\bar{c} e^{iS_{eff}} \delta(\partial_i A^i) \delta(\partial_i \pi^i) \quad (30)$$

where now  $S_{eff}$  is the action considered by Deser and Teitelboim,

$$S_{eff} = \int d^4x (\pi^i \dot{A}_i - \mathcal{H}_0 - i\bar{c}\nabla^2 c). \quad (31)$$

This action is of course invariant under the Deser-Teitelboim duality transformation, given by (29) and  $\delta c = \delta \bar{c} = 0$ .

We will now show that the Schwarz-Sen formulation can be obtained by requiring that the duality transformation be local. To this end we introduce two auxiliary fields  $C_i$  and  $P_i$  so that

$$\delta A_i = \theta C_{iT} \quad \delta \pi_i = \theta P_{iT} \quad (32)$$

where the index  $T$  indicates the transversal part of the field, as required by the Coulomb gauge condition on  $A_i$  and  $\pi_i$ . These fields are fixed through the equations

$$\nabla^2 C_{iT} = -\epsilon_{ijk} \partial^j \pi^k \quad P_{iT} = \epsilon_{ijk} \partial^j A^k \quad (33)$$

and the transversality conditions  $\partial^i C_i = -\partial^i P_i = 0$ .

From (33) and (29) we have,

$$\nabla^2 \delta C_{iT} = -\theta \epsilon_{ijk} \epsilon^{klm} \partial^j \partial_l A_m = -\theta \nabla^2 A_i. \quad (34)$$

Hence, the transversality condition implies that  $\delta C_i = -\theta A_i$ . Furthermore, proceeding in analogous way one can prove that  $\delta P_i = -\theta \pi_i$ .

Comparing these equations with the infinitesimal form of (6) for the Schwarz-Sen approach allow us to identify their fields as follows,

$$A_i^{(1)} = A_i \quad A_i^{(2)} = C_i. \quad (35)$$

The conjugate momenta to these fields are,

$$\pi_i^{(1)} = \frac{1}{2} \pi_i \quad \pi_i^{(2)} = \frac{1}{2} P_i \quad (36)$$

To construct the Schwarz-Sen action first notice that

$$P^i \dot{C}_i = -\epsilon^{ijk} \partial_j A_k \epsilon_{ilm} \nabla^{-2} \partial^l \dot{\pi}^m = -A^i \dot{\pi}_i \quad (37)$$

and

$$\epsilon^{ijk} \nabla^2 \partial_j C_{kT} = -\nabla^2 \pi^i \quad (38)$$

from which  $\pi_i = -\epsilon_{ijk} \partial^j C_T^k$ . We can therefore write (31) as

$$S_{eff} = \int d^4x \left( \frac{1}{2} \pi^i \dot{A}_i + \frac{1}{2} P^i \dot{C}_i - \mathcal{H}_0 - i\bar{c} \nabla^2 c \right), \quad (39)$$

where now  $\mathcal{H}_0 = 1/2((\nabla \times \vec{C})^2 + (\nabla \times \vec{A})^2)$ .

Up to the ghost term, equation(39) is Schwarz-Sen's action in Hamiltonian form. Although our discussion already demonstrates the equivalence of the two approaches, we will prove now that the generating functional  $Z$  given by (30) is equal to the corresponding



functional obtained using the Batalin–Fradkin–Vilkovisky formalism for the Schwarz–Sen action. After the introduction of the auxiliary fields  $C_i$  and  $P_i$ ,  $Z$  is obviously equal to

$$Z = \int \mathcal{D}A_i \mathcal{D}\pi_i \mathcal{D}C_i \mathcal{D}P_i \mathcal{D}c \mathcal{D}\bar{c} \delta(C_{iT} + \nabla^{-2} \epsilon_{ijk} \partial^j \pi^k) \delta(P_{iT} - \epsilon_{ijk} \partial^j A^k) \times \delta(\partial_i A^i) \delta(\partial_i \pi^i) \delta(\partial_i C^i) \delta(\partial_i P^i) e^{iS_{eff}}. \quad (40)$$

Now, we can write

$$\delta(C_{iT} + \nabla^{-2} \epsilon_{ijk} \partial^j \pi^k) = \det^{-1}(\nabla^{-2} \epsilon_{ijk} \partial^j) \delta(\pi_i + \epsilon_{ijk} \partial^j C^k). \quad (41)$$

But, because of the transversality property guaranteed by the delta functions in (40),  $\det^{-1}(\nabla^{-2} \epsilon_{ijk} \partial^j) = \det(\epsilon_{ijk} \partial^j)$ . So,

$$Z = \int \mathcal{D}A_i \mathcal{D}\pi_i \mathcal{D}C_i \mathcal{D}P_i \mathcal{D}c \mathcal{D}\bar{c} \delta(\partial_i A^i) \delta(\partial_i \pi^i) \times \quad (42)$$

$$\det(\epsilon_{ijk} \partial^j) \delta(\pi_i + \epsilon_{ijk} \partial^j C^k) \delta(P_{iT} - \epsilon_{ijk} \partial^j A^k) \delta(\partial_i C^i) \delta(\partial_i P^i) e^{iS_{eff}} \quad (43)$$

As shown in [7] the Schwarz–Sen action exhibits both first and second class constraints.

They are, respectively,

$$\Omega_0^a \equiv \pi_0^a \approx 0, \quad \Omega^a \equiv \partial^i \pi_i^a \approx 0 \quad (44)$$

and

$$\Omega_{iT}^a \equiv \pi_{iT}^a + \frac{1}{2} \epsilon_{ab} \epsilon_{ijk} \partial^j A_T^{b,k} \approx 0 \quad (45)$$

At equal times, the first class constraints satisfy an Abelian algebra whereas for the second class constraints we have

$$\{\Omega_{iT}^a(\vec{x}), \Omega_{jT}^b(\vec{y})\} = -\epsilon_{ab} \epsilon_{ijk} \partial_x^k \delta(\vec{x} - \vec{y}) \quad (46)$$

Therefore, in the gauge  $A_0^a = \partial^i A_i^a = 0$ , the generating functional for the Schwarz–Sen approach is given by

$$\tilde{Z} = \int \mathcal{D}A_i^a \mathcal{D}\pi_i^a \delta(\partial^i A_i^a) \delta(\partial^i \pi_i^a) \det(\nabla^2) \delta(\pi_T^{ia} + \frac{1}{2} \epsilon^{ab} \epsilon^{ijk} \partial_j A_{kT}^b) \det^{1/2}(\epsilon^{ab} \epsilon^{ijk} \partial_k) e^{i\tilde{S}_{eff}} \quad (47)$$

where

$$\check{S}_{eff} = \int d^4x (\pi^{ia} A_{ia} - \frac{1}{2} (\nabla \times \vec{A}^a)^2) \quad (48)$$

and the trivial sector  $A_0^a$ ,  $\pi_0^a$  has already been integrated out. We see that the  $\det(\nabla^2)$  factor in (47) arises also in (42) due to the ghost contribution. Moreover as

$$\det(\epsilon_{ab}\epsilon_{ijk}\partial^k) = \det^2(\epsilon_{ijk}\partial^k), \quad (49)$$

we see, from (35) that the two generating functionals  $Z$  and  $\check{Z}$  are actually identical.

We have thus shown the quantum equivalence of Schwarz–Sen and Maxwell theories. The Schwarz–Sen approach can be understood as a way to implement duality as a local symmetry in Maxwell theory. This is possible thanks to the introduction of the auxiliary fields  $C_i$  and  $P_i$ . There are, of course, other possibilities to turn the duality transformation local and presumably this may be related to the already known formulations where duality is realized locally.

## REFERENCES

- \* Supported in part by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq).
- [1] For a review see J. H. Schwarz, Lectures on Superstring and M Theory Dualities, hep-th/9607201.
- [2] A. Khoudeir and N. Pantoja, Phys. Rev. **D53** 5974 (1996); P. Pasti, D. Sorokin and M. Tonin, Phys. Lett. **B352**, 59 (1995); Phys. Rev **D52**, R4277 (1995).
- [3] N. Berkovits, Manifest Electromagnetic Duality in Closed Superstring Field Theory, hep-th/9607070; Local Actions with Electric and Magnetic Sources, hep-th/9610134.
- [4] S. Deser and C. Teitelboim, Phys. Rev **D13** 1592 (1976).
- [5] J. H. Schwarz and A. Sen, Nucl. Phys. **B411** 35 (1994).
- [6] I. Martin and A. Restuccia, Phys. Lett. **323** 311 (1994).
- [7] H. O. Girotti, "Poincaré Invariance of a Quantized Duality Symmetric Theory", to be published in Phys Rev. D.
- [8] M. Henneaux, Phys. Rep. **126** 1 (1985).