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On a q-covariant form of the BCS approximation

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Abstract

A quantum group formulation of the many-body BCS approximation for a pure pairing force in terms of $SU_q(N)$-covariant fermion operators is presented. A set of quantum BCS equations is derived, as well as a q-analog to the gap equation. The quantum occupation probabilities and gap are shown to depend explicitly on the quantum parameter.

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In the last few years, q-deformed algebraic methods have been of much interest in many-body physics [1–9]. In the framework of the q-deformed quasi-spin algebra, the phenomenology of nuclear rotational states [1–4], the pairing problem for a single j-shell [5] and the Lipkin-Meshkov-Glick model [8,9], to quote some, were studied (for a brief review on these topics, see [10]); a recent application of this formalism to boson expansion methods can be found in ref. [11]. Nonetheless, the q-fermionic theory used in previous works, following [12–15], is a generalization - or *deformation* - of the usual one, compatible with the standard Drinfel'd-Jimbo quantization of $\mathcal{U}(su(2))$ rather than strictly covariant under some linear quantum group transformations [16–18]. This fact originated some confusion, mainly with respect to the language adopted in the literature, but we hope that concepts here will be clearly defined.

Recently, Ubriaco [19] has studied thermodynamical properties of a free quantum group fermionic system with two "flavors". In particular, it was given there a $SU_q(N)$-covariant representation of the fermionic algebra for arbitrary N in terms of ordinary creation and annihilation operators. This enables one to attempt the construction of a quantum group invariant second quantized Hamiltonian for an arbitrary fermionic system. In this paper we propose a construction of the Bardeen-Cooper-Schrieffer (BCS) many-body formalism [20,21] for a pure pairing force in which the usual fermions are replaced by quantum group covariant ones satisfying appropriate anticommutation relations for a $SU_q(N)$-fermionic algebra. Our main purposes are: 1) to study, in a simple case, the effects of introducing q-covariance in a many fermion system, 2) to introduce a many-body model based on quantum group covariance rather than in q -deformed fermionic symmetry and 3) to obtain the first results concerning with the application of the BCS framework in the $q \neq 1$ realm. In what follows, we write the quantum invariant pairing Hamiltonian and BCS vacuum wave function and apply the standard variational process to the this wave function obtaining the q-analog to the BCS and gap equations.

We will work in the usual spherical basis $\{j, -j \leq m \leq j\}$ and use the BCS phases for convenience (we are allowed to use BCS phases irrespectively of quantum group angular momentum coupling coefficients because the coupling between $|j_1 j_2 m_1 m_2\rangle$ and $|j_1 j_2 I M\rangle$ states is unique and independent of q, see ref. [22]). In this basis, the usual BCS vacuum wave function is written in terms of particle operators as [21]:

$$|BCS\rangle = \prod_{jm>0} [u_j + v_j c^\dagger_{jm} c^\dagger_{j-m}] \, |0\rangle \,, \tag{1}$$

where the u_j and v_j are variational coefficients, c_{jm} and $c^\dagger_{jm}$ are the usual particle fermion operators satisfying $\{c_{jm}, c^\dagger_{j'm'}\} = \delta_{jj'}\delta_{mm'}$ with all other anticommutators vanishing and $|0\rangle$ is the bare vacuum state. We assume that we can rewrite the wave function (1) as a quantum group invariant one in the following fashion:

$$|BCS\rangle_q \doteq \prod_{jm>0} [u^q_j + v^q_j C^\dagger_{jm} C^\dagger_{j-m}] \, |0\rangle_q \,, \tag{2}$$

where the operators C_{jm} and $C^\dagger_{jm}$ play the role of creation and annihilation operators for $SU_q(2j+1)$-fermions within the j-shell with angular momentum projection m. The q-bare vacuum ket is a vector in the product Fock space defined through $C_{jm}\,|0\rangle_q = 0$. The superscripts on the occupation probabilities mean that these quantities may now depend upon the quantum parameter q. Now, the q-fermion operators C_{jm} and $C^\dagger_{jm}$ are required to satisfy an algebra covariant under quantum group transformations; we clearly want this algebra to act on *physical* vectors, that is, we want it to have a representation in the direct product Fock space generated by the eigenstates $\left|n_{j=1/2}, n_{j=3/2}, \ldots\right\rangle = \prod_{jm} \otimes |n_j\rangle$ of the operator $N = \sum_{jm} c^\dagger_{jm} c_{jm}$ (we use n_j as a shorthand for n_{jm}, $n_j = \{n_{jm}\}$). If we put the quantum group operators in one-to-one correspondence with differentials in the quantum plane, then a q-fermionic algebra explicitly invariant under linear $SU_q(2j+1)$ transformations can be cast in the form (we assume real q and consider $q > 0$) [18,23]:

$$C_{jk} C_{jl} + q \mathcal{R}_{lkmn} C_{jm} C_{jn} = 0; \tag{3}$$

$$C_{jk} C^\dagger_{jl} + q^{-1} \mathcal{R}_{kmnl} C^\dagger_{jm} C_{jn} = \delta_{kl} \tag{4}$$

(sum over repeated indices), where $-j \leq \mu \leq j$, $\mu = k, l, m, n$ with the matrix $\mathcal{R}_{klmn} = \delta_{lm}\delta_{kn}[1 + (q-1)\delta_{kl})] + \delta_{km}\delta_{ln}\theta(m-k)(q - q^{-1})$, with the usual theta function $\theta(x-y)$

($\mathcal{R} = PR$, where P is the permutation matrix and R is the R-matrix of $GL_q(2j+1)$). It is easy to check that in the classical limit $q = 1$ these expressions become the usual $SU(2j+1)$-invariant anticommutation relations for fermions. For a given j, a representation of this algebra can be given by [19]:

$$C_{jm} = c_{jm} \prod_{i=m+1}^{j} (1 + (q^{-1} - 1) c^{\dagger}_{ji} c_{ji}); \tag{5}$$

$$C^{\dagger}_{jm} = c^{\dagger}_{jm} \prod_{i=m+1}^{j} (1 + (q^{-1} - 1) c^{\dagger}_{ji} c_{ji}). \tag{6}$$

The q-fermions for various j orbits are given by $\mathcal{C} = \prod_j \otimes \mathcal{A}_j$, where $\mathcal{A}_j$ is the algebra (3, 4). The products in (5) and (6) can be written as:

$$M_{jm} \doteq \prod_{i=m+1}^{j} (1 + (q^{-1} - 1) c^{\dagger}_{ji} c_{ji}) = 1 + \sum_{i_1=m+1}^{j} (q^{-1} - 1) c^{\dagger}_{ji_1} c_{ji_1} +$$

$$+ \sum_{i_2 > i_1 = m+1}^{j} (q^{-1} - 1)^2 c^{\dagger}_{ji_1} c_{ji_1} c^{\dagger}_{ji_2} c_{ji_2} + \ldots + (q^{-1} - 1)^{j-m} c^{\dagger}_{jm+1} c_{jm+1} \ldots c^{\dagger}_{jj} c_{jj}. \tag{7}$$

It is easy to see that the c_{jm} and $c^{\dagger}_{jm}$ commute with M_{jm}. We may interpret the action of this operator on a given state as taking into account, in some effective way, not only the mean-field strength but also two-body and higher order contributions (a similar interpretation has already appeared in the literature when the consequences of q-deformation were concerned). Let us now assume that we can expand in a convergent manner the q-bare vacuum as:

$$|0\rangle_q \equiv \sum \cdots \sum_{n_j=0}^{2j+1} \cdots \sum \cdots \xi(q, n_{j=1/2}, n_{j=3/2}, \ldots) \left| n_{j=1/2}, n_{j=3/2}, \ldots \right\rangle, \tag{8}$$

where the coefficients should satisfy $\xi(q = 1, 0, 0, \ldots) = 1$ and $\xi(q = 1, \ldots, 0, 0, n_j \neq 0, 0, 0, \ldots) = 0$. Acting on the vacuum state with the operator C_{jm}, we obtain:

$$C_{jm} |0\rangle_q = \sum \ldots \sum_{n_{j'}=0}^{2j'+1} \cdots \sum \cdots \xi(q, n_{j'=1/2}, n_{j'=3/2}, \ldots) \times$$

$$\times m_{jm}(n_{j'=1/2}, n_{j'=3/2}, \ldots) c_{jm} \left| n_{j'=1/2}, n_{j'=3/2}, \ldots \right\rangle = 0, \tag{9}$$

with $m_{jm} \geq 1$ the eigenvalue of M_{jm} (which can be immediately inferred from (7)). We should note that all states with occupation number n_{jm} originally equal to zero are automatically excluded from the sum; therefore, since all the state kets are orthogonal, all the coefficients corresponding to states with n_{jm} originally equal to one must vanish. But j and m are arbitrary, which implies that:

$$|0\rangle_q = \alpha e^{i\theta} |0\rangle . \tag{10}$$

Here the factor $\alpha e^{i\theta}$ is undefined, and we choose $\alpha^2 = 1$ independently of q. (Note that, due to $M_{jm} |0\rangle = |0\rangle$, $C_{jm} |0\rangle = 0$; with this it is shown the uniqueness of a ray in the product Fock space which is annihilated by the operator C_{jm}.) Thence, the BCS q-covariant vacuum ket reads:

$$|BCS\rangle_q = \prod_{jm}[u_j^q + v_j^q C_{jm}^\dagger C_{j-m}^\dagger] |0\rangle = \prod_{jm}[u_j^q + v_j^q c_{jm}^\dagger c_{j-m}^\dagger M_{jm} M_{j-m}] |0\rangle =$$

$$= \prod_{jm}[u_j^q + v_j^q c_{jm}^\dagger c_{j-m}^\dagger] |0\rangle . \tag{11}$$

(The superscripts will hereafter be omitted in v_j^q and u_j^q). We now turn to the expression of the q-Hamiltonian. We are interested in a pure pairing Hamiltonian, whose $q = 1$ version we write as [21]:

$$H = \sum_{jm} \epsilon_j c_{jm}^\dagger c_{jm} - G \sum_{jj'm_1m_2} c_{jm_1}^\dagger c_{j-m_1}^\dagger c_{j'-m_2} c_{j'm_2}, \tag{12}$$

where G is the pairing strength and the ϵ_j are the single-particle energies; here we understand that the indices m_1 and m_2 are greater than zero. We write a q-Hamiltonian following expression (12) in the form:

$$H_q \doteq \sum_{jm} \epsilon_j c_{jm}^\dagger c_{jm} (M_{jm})^2 - G \sum_{jj'm_1m_2} c_{jm_1}^\dagger c_{j-m_1}^\dagger M_{jm_1} M_{j-m_1} c_{j'-m_2} M_{j'-m_2} c_{j'm_2} M_{j'm_2}. \tag{13}$$

One can observe that the "mean-field" term in (13) already contains explicit interaction among different levels (see also eq. (16) of ref. [19]). Using expression (7), and anticommutation properties of ordinary fermion operators, one can perform straightforwardly the calculation of the mean-value of (13) between $|BCS>_q$ states. The result is:

$$_q\!<\!BCS|H_q|BCS\!>_q = \sum_j \epsilon_j v_j^2 \times \frac{(\frac{1}{q})^{4\Omega_j}-1}{(\frac{1}{q})^2-1} - G\sum_{j\neq j'} v_j v_{j'} u_j u_{j'} \times (\frac{1}{q})^{2j+2j'} \times \Omega_j\Omega_{j'} -$$

$$-\frac{G}{2}\sum_{jm>0} v_j^2 u_j^2 [(\frac{1}{q})^{3j-m}\zeta_{2j-m} + (\frac{1}{q})^{2j}\zeta_{1jm}\zeta_{1j-m}] \times \Omega_j, \tag{14}$$

where

$$\zeta_{njm} \doteq 1 + (q^{-1}-1)(j-m-n) + (q^{-1}-1)^2 \times$$

$$\times \left[\frac{(j-m)(j-m-1)}{2!} - nj - nm - n\right] + \ldots + (q^{-1}-1)^{j-m-1} \times$$

$$\times[(j-m) - n(m+3)(m+2)\ldots(j-m-3) - n(-m-1)\times(m+2)(m+1)\ldots(j-m-2) - \ldots] \tag{15}$$

and Ω_j is the pair degeneracy of the j-shell. The coefficients of each power of $(q^{-1}-1)$ in ζ obviously have to be either positive or zero. The variational q-Hamiltonian is $\tilde{H}' = \tilde{H} - \lambda_q \sum_j 2v_j^2$. Performing a naive variation with respect to the Lagrange multiplier λ_q one obtains:

$$\lambda_q = \frac{d\, {}_q\langle BCS|\, H_q\, |BCS\rangle_q}{d\sum_j 2v_j^2}, \tag{16}$$

which means that λ_q works as the chemical potential in the frame of the q-energy $E_q = {}_q\langle BCS|\, H_q\, |BCS\,\rangle_q$. One may now calculate the variation with respect to the occupation probabilities:

$$\delta\, {}_q\langle BCS|\, H_q - \lambda_q \sum_j 2v_j^2\, |BCS\,\rangle_q = \left(\frac{\partial}{\partial v_j} + \frac{u_j}{v_j}\frac{\partial}{\partial u_j}\right) {}_q\langle BCS|\, H_q - \lambda_q\sum_j 2v_j^2\, |BCS\rangle_q\,, \tag{17}$$

which one then imposes to vanish for some fixed q. The resulting q-BCS equations are:

$$u_j v_j (\epsilon_j \times \left(\frac{(\frac{1}{q})^{4\Omega_j}-1}{[(\frac{1}{q})^2-1]\Omega_j}\right) - 2\lambda_q) + (u_j^2 - v_j^2)\Delta_j^q = 0, \tag{18}$$

where

$$\Delta_j^q \doteq G\left[\sum_{j'\neq j} u_{j'}v_{j'}(\frac{1}{q})^{2j+2j'}\Omega_{j'} + \frac{u_jv_j}{2}\sum_{m>0}(\frac{1}{q})^{3j-m}\zeta_{2j-m} + (\frac{1}{q})^{2j}\zeta_{1jm}\zeta_{1j-m}\right] \tag{19}$$

is the quantum gap parameter (which, in opposition to the standard pure pairing case, depends upon the shell label j). It is easy to verify that when $q = 1$, the quantum equations (18) are the BCS equations:

$$2u_jv_j(\epsilon_j - \lambda) + (u_j^2 - v_j^2)\Delta = 0 \tag{20}$$

for the nuclear pairing problem, with the non-quantum gap parameter $\Delta = G\sum_j u_jv_j\Omega_j$. The solution of equations (18) for the variational parameters u_j and v_j is:

$$\left.\begin{matrix} u_j^2 \\ v_j^2 \end{matrix}\right\} = \frac{1}{2}\left[1 \pm \frac{\left(\epsilon_j \times \left(\frac{(\frac{1}{q})^{4\Omega_j}-1}{2[(\frac{1}{q})^2-1]\Omega_j}\right) - \lambda_q\right)}{\sqrt{\left(\epsilon_j \times \left(\frac{(\frac{1}{q})^{4\Omega_j}-1}{2[(\frac{1}{q})^2-1]\Omega_j}\right) - \lambda_q\right)^2 + (\Delta_j^q)^2}}\right]. \tag{21}$$

The quantum gap equation is obtained in an analog way as for the standard case by substitution of (21) into (19):

$$\Delta_j^q = \frac{G}{2}\left[\sum_{j'\neq j}(\frac{1}{q})^{2j+2j'}\Omega_{j'}\frac{\Delta_{j'}^q}{\sqrt{\left(\epsilon_{j'} \times \left(\frac{(\frac{1}{q})^{4\Omega_{j'}}-1}{2[(\frac{1}{q})^2-1]\Omega_{j'}}\right) - \lambda_q\right)^2 + (\Delta_{j'}^q)^2}} + \right.$$

$$\left. + \frac{\Delta_j^q}{\sqrt{\left(\epsilon_j \times \left(\frac{(\frac{1}{q})^{4\Omega_j}-1}{2[(\frac{1}{q})^2-1]\Omega_j}\right) - \lambda_q\right)^2 + (\Delta_j^q)^2}}\sum_{m>0}(\frac{1}{q})^{3j-m}\zeta_{2j-m} + (\frac{1}{q})^{2j}\zeta_{1jm}\zeta_{1j-m}\right]. \tag{22}$$

For the case of a single j-shell, the quantum gap parameter assumes the form:

$$\Delta_j^q = \left[\frac{G^2}{4}\left(\sum_{m>0}(\frac{1}{q})^{3j-m}\zeta_{2j-m} + (\frac{1}{q})^{2j}\zeta_{1jm}\zeta_{1j-m}\right)^2 - \left(\epsilon_j \times \left(\frac{(\frac{1}{q})^{4\Omega_j}-1}{2[(\frac{1}{q})^2-1]\Omega_j}\right) - \lambda_q\right)^2\right]^{1/2}. \tag{23}$$

The qualitative behavior of Δ_j^q is, as one can see, independent of the shell label. A 3D plot shows the dependence of the curve $v_j^2 \times \epsilon_j$ upon the parameter q, for a $j = \frac{3}{2}$ shell (Fig. 1a). Figure 1b shows the behavior of $v_j^2 \times \epsilon_j$ for three different values of q.

In summary, we presented a quantum group form of the BCS method for the case of a pure pairing force, following the $SU_q(N)$-covariant representation of the fermionic algebra given by Ubriaco in ref. [19]. The quantum bare vacuum was shown to be identical (apart from a multiplicative constant) to the product Fock space vacuum. The q-analogues to the BCS equations (18) were derived along with the quantum gap equation (22). The quantum gap (19) was shown to depend explicitly on the deformation parameter; we found that the quantum gap is reduced as the deformation increases, as if the system collapsed into its ground-state and, conversely, that it goes to infinity as q tends to zero making the system unexcitable. A 3D plot was made to illustrate the dependence of the occupation probabilities v_j^2 versus the single-particle energies on the quantum parameter. One can check this dependence is qualitatively in agreement with the remark in the first paragraph below eq. (19) in ref. [19]. The study of introduction of q-covariance may be interesting in other many-body systems, in special in toy models such as the Moszkowski and the Lipkin-Meshkov-Glick ones, studied previously (in the deformed algebraic approach) in [6,7] and [8,9]. A q-analog of two-level pairing is under study and we hope to address it in a future publication.

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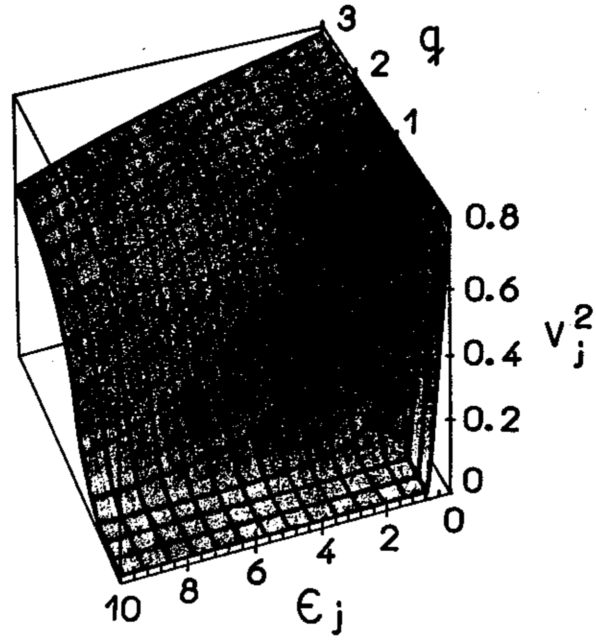
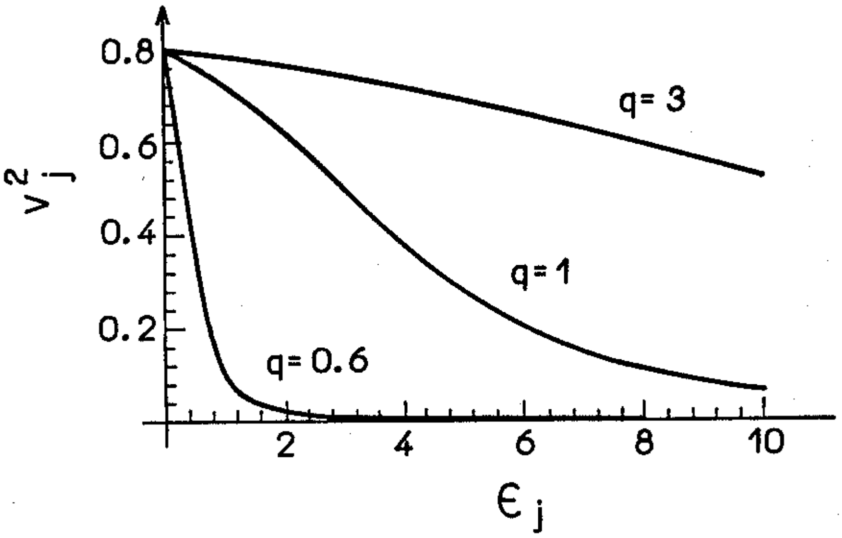
Figure Captions

Figure 1. Quantum occupation probabilities v_j^2 as a function of the single particle energy ϵ_j and the quantum parameter q for $j = 3/2$. Figure 1a is a 3D view whereas Figure 1b presents the behavior for three different values of q.



3
2
1
q
0.8
0.6
0.4
0.2
0
v_j^2
0
2
4
6
8
10
ϵ_j

(a)



0.8
0.6
0.4
0.2
v_j^2
q= 3
q= 1
q= 0.6
2
4
6
8
10
ϵ_j

(b)