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Reparametrization Invariance as Gauge Symmetry

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Abstract

Reparametrization invariance being treated as a gauge symmetry shows some specific peculiarities. We study these peculiarities both from a general point of view and on concrete examples. We consider the canonical treatment of reparametrization invariant systems in which one fixes the gauge on the classical level by means of time-dependent gauge conditions. In such an approach one can interpret different gauges as different reference frames. We discuss the relations between different gauges and the problem of gauge invariance in this case. Finally, we establish a general structure of reparametrizations and its connection with the zero-Hamiltonian phenomenon.

1 Introduction

Many actual physical theories are formulated in the so called reparametrization invariant (RI) form, for instance, models of point-like relativistic particles, gravity and string theory. Formally, reparametrization invariance can be treated as a gauge symmetry. However, this gauge symmetry shows some peculiarities, so that it is natural to separate it in a special class of gauge symmetries. Due to the same reason one has to be careful when formally applying recipes extracted from the consideration of gauge symmetries of different nature. In all known examples of RI theories the Hamiltonian vanishes on the constraint surface, in spite of the fact that explicit forms of the reparametrization transformations in these examples may look different. This issue raises a question: what is the general structure of such transformations and is there a definite relation between such a structure and zero-Hamiltonian phenomenon? The zero-Hamiltonian phenomenon in RI theories raises another well-known problem: what is time-evolution in this case? This question has a principal character for the construction of an adequate quantum theory of gravity. In the canonical schemes of consideration there

exists a possibility to introduce the evolution by means of a time-dependent gauge fixing. In turn, this demands a modification of the standard scheme of canonical quantization [1], which is adopted to stationary second-class constraints (such a modification was first proposed in [2]). Fixing the gauge in such a manner we get different evolutions depending on the selected gauge. And here we meet the question well-known in gauge theories: to what extent does the physical content of a theory depend on the gauge fixing and what is gauge invariance here? There exist, in fact, two essentially different points of view on this problem. According to the first one, which is called "local" point of view, the gauge fixing of the reparametrization gauge freedom corresponds to a certain choice of the reference frame (RF). At the same time space-time variables in the Lagrangian have to be identified namely with the coordinates of the above RF. The reparametrizations relate the description of the system in different RF. Thus, one has to admit that local physical quantities may depend on the choice of the gauge. Another, "non-local" point of view, assumes that there exists a reparametrization invariant description. Supporters of this position believe that such a description may be realized if one includes an observer in the frame of the theory. Then the physical quantities do not depend on the choice of the gauge, which fixes the reparametrization freedom, and must commute with the corresponding first-class constraints. Unfortunately, the "non-local" point of view remains, in the main, declarative. It seems that its clear and convincing realization is absent until now. An exelent and detailed survey on the subject (and relevant references) one can find in [3].

In the present paper we discuss the above and some other questions related to RI theories both from a general point of view and on specific examples. We advocate the "local" point of view considering several examples, where one can compare RI and non-RI versions of the same theory. Namely, we study a finite-dimensional theory, a field theory in a flat space-time, and a theory of the relativistic particle, all of them both in non-RI and RI form. We remind briefly on the treatment of systems with non-stationary second-class constraints and apply then this formalism to the above mentioned theories to impose time dependent (space-time dependent) gauges. We analyse the relation between different gauges both on the classical and the quantum level. Based on the considered examples we formulate an interpretation which, in fact, supports the "local" point of view. We argue that the reparametrization symmetry has to be treated specially from the gauge symmetries of different nature. In the final part of the paper, which looks more formal and independent from the previous conceptual part, we study the general structure of the reparametrizations and its relation with zero Hamiltonian phenomenon. On the example of the time reparametrization we propose a general definition of reparametrization symmetry transformations.

2 Introducing Reparametrization Invariance

The action of a point-like relativistic particle

$$S=\int_0^1 L d\tau\ ,\ L=-m\sqrt{\dot{x}^2}\ ,\ x=(x^\mu)\ ,\ \dot{x}^\mu=\frac{dx^\mu}{d\tau}\ ,\ \mu=0,...,D\ , \tag{2.1}$$

gives us a simple example of RI theory. It is invariant under reparametrizations $x^\mu(\tau)\to x'^\mu(\tau)=x^\mu(f(\tau))$, where f is an arbitrary function obeying only the following demands:

$\dot{f}(\tau) > 0, f(0) = 0, f(1) = 1$. The reparametrizations can be interpreted as gauge transformations (GT) whose infinitesimal form is

$$\delta x^\mu(\tau) = \dot{x}^\mu(\tau)\epsilon(\tau)\ , \quad \delta L = \frac{d}{d\tau}[\epsilon(t)L]\ , \tag{2.2}$$

where $\epsilon(\tau)$ is a time dependent parameter. An equivalent Lagrangian function, which is adapted to the $m \to 0$ limit, contains an additional variable e and is of the form:

$$L = -\frac{\dot{x}^2}{2e} - e\frac{m^2}{2}\ . \tag{2.3}$$

Here the infinitesimal form of the reparametrizations is:

$$\delta x^\mu(\tau) = \dot{x}^\mu(\tau)\epsilon(\tau)\ , \quad \delta e(\tau) = \dot{e}(\tau)\epsilon(\tau) + e(\tau)\dot{\epsilon}(\tau)\ , \quad \delta L = \frac{d}{d\tau}[\epsilon(\tau)L]\ . \tag{2.4}$$

String theory is of the same nature, its action is invariant under the reparametrizations of two variables. Gravity is an example of RI field theory. The Einstein action

$$S_E = \int \mathcal{L}_E d^{D+1}x\ , \quad \mathcal{L}_E = -\sqrt{-g}R, \tag{2.5}$$

is invariant under general coordinate transformations $x^\mu \to x'^\mu = x'^\mu(x)$, $g_{\mu\nu}(x) \to g'_{\mu\nu}(x)$, $g'_{\mu\nu}(x') = \frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}g_{\lambda\sigma}(x)$. These are, in fact, reparametrizations of $D+1$ space-time variables. They can be treated as GT,

$$\delta g_{\mu\nu}(x) = D_\mu\epsilon_\nu(x) + D_\nu\epsilon_\mu(x)\ , \quad \delta\mathcal{L}_E = \partial_\mu[\epsilon^\mu(x)\mathcal{L}_E]\ , \tag{2.6}$$

where $\epsilon^\mu(x)$ are GT parameters - arbitrary functions on space-time coordinates.

Any action can be extended to a RI form [4]. Consider, for example, a non-singular action (similar consideration can be made for any singular Lagrangian as well)

$$S = \int_{t_1}^{t_2} L(\mathbf{x}, \dot{\mathbf{x}}, t)dt, \quad \mathbf{x} = (x^i),\ i = 1, ..., D\ , \quad \dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt}\ . \tag{2.7}$$

Let us change t to x^0 and then replace the integration variable x^0,

$$x^0 = f(t), \quad f(t_1) = t_1 \quad f(t_2) = t_2\ . \tag{2.8}$$

Thus, we get

$$S_R = \int_{t_1}^{t_2} L_R(x, \dot{x})dt\ , \quad L_R(x, \dot{x}) = L(\mathbf{x}, \frac{\dot{\mathbf{x}}}{\dot{x}^0}, x^0)\dot{x}^0\ . \tag{2.9}$$

As long as we keep in mind the relations (2.8), the action (2.9) is completely equivalent to the initial one (2.7). On the other hand, one can now treat (2.9) in a new way, namely, one can forget about (2.8) and treat x^0 as a new independent variable, so that the total set of variables of the theory is $x = (x^\mu) = (x^0, \mathbf{x})$.

Let us analyse the relation between the theory with the action (2.9) and (2.7), in particular, in the Hamiltonian formulation. For the non-singular theory (2.7) one can always solve the equations, which define the momenta, with respect to all velocities:

$$\boldsymbol{\pi} = \frac{\partial L}{\partial\dot{\mathbf{x}}} \Rightarrow \dot{\mathbf{x}} = \boldsymbol{\psi}(\mathbf{x}, \boldsymbol{\pi}, t), \quad \boldsymbol{\pi} = (\pi_i)\ . \tag{2.10}$$

Then the time evolution is generated by the Hamiltonian equations without any constraints,

$$\dot{\boldsymbol{\eta}} = \{\boldsymbol{\eta}, H\}\ , \quad \boldsymbol{\eta} = (\mathbf{x}, \boldsymbol{\pi})\ , \quad H = \left(\frac{\partial L}{\partial\dot{\mathbf{x}}}\dot{\mathbf{x}} - L\right)\Big|_{\dot{\mathbf{x}}=\boldsymbol{\psi}} = H(\mathbf{x}, \boldsymbol{\pi}, t)\ . \tag{2.11}$$

In the theory with the action S_R there appear primary constraints in the Hamiltonian formulation. Indeed, let $\pi_\mu = (\pi_0, \boldsymbol{\pi})$ be momenta conjugated to x^μ,

$$\pi_0 = \frac{\partial L_R}{\partial\dot{x}^0} = -\left(\frac{\partial L}{\partial\dot{\mathbf{x}}}\dot{\mathbf{x}} - L\right)\Big|_{\dot{\mathbf{x}}\to\frac{\dot{\mathbf{x}}}{\dot{x}^0}, t\to x^0}\ , \quad \boldsymbol{\pi} = \frac{\partial L_R}{\partial\dot{\mathbf{x}}} = \frac{\partial L}{\partial\dot{\mathbf{x}}}\Big|_{\dot{\mathbf{x}}\to\frac{\dot{\mathbf{x}}}{\dot{x}^0}, t\to x^0}\ . \tag{2.12}$$

¿From the second equation in (2.12) (taking into account (2.10)) we get: $\dot{\mathbf{x}} = \dot{x}^0\boldsymbol{\psi}(\mathbf{x}, \boldsymbol{\pi}, x^0)$, whereas $\dot{x}^0$ is a primarily unexpressible velocity. Then the first equation (2.12) (taking into account (2.11)) appears to be a primary constraint

$$\phi_1 = \pi_0 + H(\mathbf{x}, \boldsymbol{\pi}, x^0) = 0\ . \tag{2.13}$$

Constructing the total Hamiltonian $H^{(1)}$ according to the standart procedure [1, 2] we get

$$H^{(1)} = \left(\frac{\partial L_R}{\partial\dot{x}^\mu}\dot{x}^\mu - L_R\right)\Big|_{\dot{\mathbf{x}}=\dot{x}^0\boldsymbol{\psi}(\mathbf{x},\boldsymbol{\pi},x^0)} = \lambda\phi_1, \quad \lambda = \dot{x}^0\ . \tag{2.14}$$

Thus, the total Hamiltonian vanishes on the constraint surface (on the equations of motion). No more constraints appear from the consistency conditions. To fix a gauge we have to impose a new constraint $\phi_2 = 0$, so that the matrix $\{\phi_a, \phi_b\}$, $a, b = 1, 2$ is not singular. A natural form of a such a condition is $\phi_2 = x^0 - \varphi(\mathbf{x}, \boldsymbol{\pi}, t) = 0$, where the function $\varphi(\mathbf{x}, \boldsymbol{\pi}, t)$ has an essential t-dependence, introduced in the theory, in spite of the fact, that the Hamiltonian is zero. The simplest choice of the gauge condition is (we will call such a condition - *chronological gauge*),

$$\phi_2 = x^0 - t = 0\ . \tag{2.15}$$

The set of second-class constraints (2.13), (2.15) explicitly depends on time. The general method to deal with non-stationary constraints in the canonical formulation and quantization procedure were first proposed in [2]. Then similar results were obtained by a geometrical approach in [7]. The BRST formulation of the non-stationary constraints case was discussed in [8]. Below we briefly remind on the treatment [2] of systems with non-stationary second–class constraints.

Consider a theory with second-class constraints $\phi_a(\eta, t) = 0$ (where $\eta = (x^i, \pi_i)$ are canonical variables) which may explicitly depend on time t. Then the equation of motion of such a system may be written by means of the Dirac brackets, if one formally introduces a momentum ϵ conjugated to the time t, and defines the Poisson bracket in the extended phase space of canonical variables $(\eta; t, \epsilon)$,

$$\dot{\eta} = \{\eta, H + \epsilon\}_{D(\phi)},\ \phi(\eta, t) = 0\ , \tag{2.16}$$

where H is the Hamiltonian of the system, and $\{A, B\}_{D(\phi)}$ is the notation for the Dirac bracket with respect to the system of second–class constraints ϕ. The Poisson brackets, wherever encountered, are henceforth understood as one in the above mentioned extended space. The quantization procedure in "quasi-Schrödinger" picture can be formulated in that case as follows. The variables η of the theory are assigned the operators $\tilde\eta$, which satisfy the following relations

$$[\tilde\eta, \tilde\eta'] = i\{\eta, \eta'\}_{D(\phi)}|_{\eta=\tilde\eta}\;, \quad \phi(\tilde\eta, t) = 0, \tag{2.17}$$

and equations of evolution

$$\dot{\tilde\eta} \;=\; \{\eta, \epsilon\}_{D(\phi)}|_{\eta=\tilde\eta} = -\,\{\eta, \phi_a\}C_{ab}\left.\frac{\partial\phi_b}{\partial t}\right|_{\eta=\tilde\eta}\;, \quad C_{ac}\{\phi_c, \phi_b\} = \delta_{ab}\;. \tag{2.18}$$

One can demonstrate that (2.17) and (2.18) are consistent. To each physical quantity A given in the Hamiltonian formalism by the function $A(\eta, t)$, corresponds a "quasi-Schrödinger" operator $\tilde A$ by the rule $\tilde A = A(\tilde\eta, t)$; in the same manner one constructs the quantum Hamiltonian $\tilde H$, according to the classical one $H(\eta, t)$. The time evolution of the state vectors Ψ in this picture is determined by the Schrödinger equation with the Hamiltonian $\tilde H = H(\tilde\eta, t)$. The total time evolution results both from the evolution of the state vectors and from one of the operators. It is convenient to analyse such an evolution in the Heisenberg picture whose operators $\check\eta$ are related to the operators $\tilde\eta$ as $\check\eta = U^{-1}\tilde\eta U$, where U is the evolution operator related to the Hamiltonian $\tilde H$. Such operators satisfy the equations

$$\begin{aligned} &\dot{\check\eta} = \{\eta, H + \epsilon\}_{D(\phi)}\Big|_{\eta=\check\eta}\,, \\ &[\check\eta, \check\eta'] = i\,\{\eta, \eta'\}_{D(\phi)}\Big|_{\eta=\check\eta}\,, \quad \phi(\check\eta, t) = 0\,. \end{aligned} \tag{2.19}$$

All the relations (2.19) together may be considered as a prescription for quantization in the Heisenberg picture for theories with non-stationary second-class constraints. The total time evolution is controled only by the first set of the equations (2.19) since the state vectors do not depend on time in the Heisenberg picture. In the general case such an evolution is not unitary. Suppose, however, that a part of the set of second-class constraints consists of supplementary gauge conditions, the choice of which is in our hands. In this case one may try to select these gauge conditions in a special form to obtain unitary evolution. The evolution is unitary if there exists an effective Hamiltonian $H_{eff}(\eta)$ in the initial phase space of the variables η so that the right side of the equations of motion (2.16) may be written as follows

$$\dot\eta = \{\eta, H + \epsilon\}_{D(\phi)} = \{\eta, H_{eff}\}_{D(\phi)}\;. \tag{2.20}$$

In this case, (due to the commutation relations (2.19)) the quantum operators $\check\eta$ obey the equations (we disregard here problems connected with operator ordering)

$$\dot{\check\eta} = -i[\check\eta, \check H_{eff}], \quad \check H_{eff} = H_{eff}(\check\eta)\;. \tag{2.21}$$

The latter allows one to introduce the real Schrödinger picture where operators do not depend on time but the evolution is controlled by the Schrödinger equation with the Hamiltonian H_{eff}. We may call the gauge conditions which imply the existence of the effective Hamiltonians as *unitary gauges*. Remember that in the stationary constraint case all gauge conditions are unitary [2]. As it is known [2], the set of second-class constraints can always be solved explicitly with respect to part of the variables $\eta_* = \Psi(\eta^*)$, $\eta = (\eta_*, \eta^*)$, so that η_* and η^*

are sets of pairs of canonically conjugated variables $\eta_* = (q_*, p_*)$, $\eta^* = (q^*, p^*)$. We may call η^* as independent variables and η_* as dependent ones. In fact $\eta_* - \Psi(\eta^*) = 0$ is an equivalent to $\phi(\eta) = 0$ set of second-class constraints. One can easily demonstrate that it is enough to verify the existence of the effective Hamiltonian (the validity of relation (2.21)) for the independent variables only. Then the evolution of the dependent variables which is controlled by the constraint equations is also unitary.

In the situation of our main interest here, when the Hamiltonian is proportional to the constraints, one can put $H = 0$ in the equations (2.19). Thus, the "quasi-Schrödinger" picture and the Heisenberg one coincide. The time evolution is unitary in this case if the following equations hold

$$\dot\eta = \{\eta, \epsilon\}_{D(\phi)} = -\{\eta, \phi_a\}C_{ab}\frac{\partial\phi_b}{\partial t} = \{\eta, H_{eff}(\eta)\}_{D(\phi)}. \tag{2.22}$$

Let us analyse the theory (2.9) in the gauge (2.15) using the above consideration. The matrix $\{\phi_a, \phi_b\}$ is simple in this case: $\{\phi_a, \phi_b\} = \text{antidiag}(-1, 1)$, $C_{ab} = \{\phi_b, \phi_a\}$. The Dirac brackets between the independent variables $\mathbf{x}, \boldsymbol{\pi}$ are reduced to the Poisson ones,

$$\{x^i, x^j\}_D = \{\pi_i, \pi_j\}_D = 0\;, \quad \{x^i, \pi_j\}_D = \delta^i_j\;. \tag{2.23}$$

The time evolution of these variables is given by the equations

$$\dot{\mathbf{x}} = -\{\mathbf{x}, \phi_a\}C_{ab}\dot\phi_b = \{\mathbf{x}, H\}\;, \quad \dot{\boldsymbol{\pi}} = -\{\boldsymbol{\pi}, \phi_a\}C_{ab}\dot\phi_b = \{\boldsymbol{\pi}, H\}\;, \tag{2.24}$$

where H is the Hamiltonian of the theory (2.7) and at the same time it is the effective Hamiltonian in our definition. This means that in the chronological gauge the dynamics of the original non-singular theory is reproduced.

Let us consider instead of (2.15) a more general gauge fixing $\phi_2 = x^0 - \varphi(\mathbf{x}, \boldsymbol{\pi}, t) = 0$. To get conditions on the function φ, which make the gauge unitary, we restrict ourselves to the free particle case, where H from (2.11) is $\mathbf{p}^2/2m$. In this case $\{\phi_a, \phi_b\} = \text{antidiag}(-K, K)$, $C_{ab} = K^{-2}\{\phi_b, \phi_a\}$, $K = \left(1 - \frac{\pi_i}{m}\partial_i\varphi\right)$. The non-zero Dirac brackets between the independent variables $\mathbf{x}, \boldsymbol{\pi}$ are:

$$\{x^i, x^j\}_D = (mK)^{-1}\left(\frac{\partial\varphi}{\partial\pi_i}\pi_j - \frac{\partial\varphi}{\partial\pi_j}\pi_i\right), \quad \{x^i, \pi_j\}_D = \delta^i_j + (mK)^{-1}\pi_i\partial_j\varphi. \tag{2.25}$$

According to (2.22) these variables obey the following equations:

$$\dot{\mathbf{x}} = -\{\mathbf{x}, \phi_a\}C_{ab}\dot\phi_b = (mK)^{-1}\boldsymbol{\pi}\dot\varphi\;, \quad \dot{\boldsymbol{\pi}} = -\{\boldsymbol{\pi}, \phi_a\}C_{ab}\dot\phi_b = 0\;. \tag{2.26}$$

On the other hand, if the effective Hamiltonian H_{eff} does exist (unitary gauge), one can write

$$\begin{aligned} \dot x^i \;&=\; \{x^i, H_{eff}\}_D = (mK)^{-1}\left(\frac{\partial\varphi}{\partial\pi_i}\pi_j - \frac{\partial\varphi}{\partial\pi_j}\pi_i\right)\partial_j H_{eff} + \left[\delta^i_j + (mK)^{-1}\pi_i\partial_j\varphi\right]\frac{\partial H_{eff}}{\partial\pi_j}\,, \\ \dot\pi_i \;&=\; \{\pi_i, H_{eff}\}_D = -\left(\delta^j_i + (mK)^{-1}\pi_j\partial_i\varphi\right)\partial_j H_{eff}\;. \end{aligned} \tag{2.27}$$

Comparing (2.26) with (2.27) we get the following conditions on H_{eff}:

$$\partial_j H_{eff} = 0, \quad \frac{\partial H_{eff}}{\partial \pi_i} = \pi_i (mK)^{-1}\left(\dot\varphi - \frac{\partial H_{eff}}{\partial \pi_i}\partial_i \varphi\right) . \tag{2.28}$$

The first eq. (2.28) means that H_{eff} does not depend on $\mathbf{x}$ and the second one results in the condition

$$\left(\pi_j \frac{\partial}{\partial \pi_i} - \pi_i \frac{\partial}{\partial \pi_j}\right) H_{eff} = 0 , \tag{2.29}$$

which means that H_{eff} depends only on $\boldsymbol{\pi}^2$. Thus, $H_{eff} = H_{eff}(\boldsymbol{\pi}^2, t)$. Using this information in the second equation (2.28), we get:

$$2m \frac{\partial H_{eff}}{\partial \boldsymbol{\pi}^2} = \dot\varphi . \tag{2.30}$$

Thus, $\dot\varphi$ is a function on $\boldsymbol{\pi}^2$ and t only. That leads to the following structure:

$$\varphi(\mathbf{x}, \boldsymbol{\pi}, t) = \chi(\mathbf{x}, \boldsymbol{\pi}) + \psi(\boldsymbol{\pi}^2, t) , \tag{2.31}$$

where χ and ψ are arbitrary functions on the indicated arguments. The effective Hamiltonian in this case can be expressed via the function $\psi(\boldsymbol{\pi}^2, t)$ only:

$$H_{eff} = \frac{1}{2m}\int \psi(\boldsymbol{\pi}^2, t) d\boldsymbol{\pi}^2 . \tag{2.32}$$

As an example of a nonlinear in time t gauge condition we consider here the following

$$\phi_2 = x^0 - t - \frac{ma}{2\pi} t^2 = 0, \tag{2.33}$$

where for simplicity we have selected one-dimensional case, i.e. the Hamiltonian of the initial nonsingular theory is $H = \pi^2/2m$. The previous consideration is valid in this case, thus, (2.33) is an unitary gauge which generates the effective Hamiltonian of the form

$$H_{eff} = \frac{\pi^2}{2m} + \pi a t . \tag{2.34}$$

If we suppose that the initial non-singular action (2.7) corresponds to a theory in an inertial reference frame, then the chronological gauge (2.15) returns us to the description in such a frame, whereas the gauge (2.33) corresponds to the description from the point of view of an accelerating (with acceleration a) frame.

Let us turn to the question about physical quantities in the RI theory under consideration. It is known [1, 2] that in conventional gauge theories physical quantities, which are defined by functions on the phase space, have to commute with first-class constraints on the mass shell (Dirac's criterion). What kind of restrictions does this criterion impose on the physical quantities in our case? Due to the constraint (2.13), the physical quantities, which are given by functions on the phase space of variables x^μ, π_μ, always may be expressed via functions of the form $A = A(x^0, \boldsymbol{\eta})$, $\boldsymbol{\eta} = (\mathbf{x}, \boldsymbol{\pi})$. The condition of commutativity of such functions with the first-class constraint (2.13) on the mass shell results then in

$$\{A, \phi_1\} = \frac{\partial A}{\partial x^0} + \frac{\partial A}{\partial \boldsymbol{\eta}}\{\boldsymbol{\eta}, H\} \approx 0 . \tag{2.35}$$

Remembering, that the equations of motion in the theory under consideration have the form

$$\dot{\boldsymbol{\eta}} = \{\boldsymbol{\eta}, H^{(1)}\} = \lambda\{\boldsymbol{\eta}, H\} , \quad \dot x^0 = \{x^0, H^{(1)}\} = \lambda , \tag{2.36}$$

we may rewrite (2.35) as

$$\frac{\partial A}{\partial x^0}\dot x^0 + \frac{\partial A}{\partial \boldsymbol{\eta}}\dot{\boldsymbol{\eta}} = \frac{dA}{dt} \approx 0 . \tag{2.37}$$

Thus, the Dirac's criterion admits as physical functions only those which present integrals of motion. We believe that the RI theory under consideration in the chronogical gauge (2.15) has to coincide with the initial non-singular theory (2.7), in which all the functions of the form $A = A(t, \eta)$ are physical. Thus, if one accepts the Dirac's criteria then an essential part of real physical quantities of the initial non–singular theory (2.7) are lost and the RI version is not equivalent to the initial theory.

The above consideration looks even more transparent in the case of the field theory. Let us consider, for example, a theory of a scalar field in a flat space-time. The action of the theory being written in an inertial RF has the form:

$$S = \int \mathcal{L} d^{D+1}x = \int \left[\frac{1}{2}\eta^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} + F(\varphi)\right] d^{D+1}x , \tag{2.38}$$

where $\eta_{\mu\nu} = \mathrm{diag}(1, -1, \ldots, -1)$, $F(\varphi)$ are some terms independent of the derivatives of φ, and $\varphi_{,\mu} = \partial\varphi/\partial x^\mu$. Let us change in (2.38) x^μ to y^μ and then let us rewrite the integral in the RHS (2.38) doing the substitution $y^\mu = y^\mu(x)$. Thus, we get

$$S_R = \int \mathcal{L}_R d^{D+1}x = \int \left[\frac{1}{2}g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} + F(\varphi)\right]\sqrt{-g} d^{D+1}x , \tag{2.39}$$

where

$$g^{\mu\nu} = a^\mu_\alpha a^\nu_\beta \eta^{\alpha\beta} , \quad a^\mu_\alpha y^\alpha_{,\nu} = \delta^\mu_\nu , \quad g = det \parallel g_{\mu\nu} \parallel = -e^2 , \quad e = det \parallel y^\mu_{,\nu} \parallel , \tag{2.40}$$

and $g_{\mu\nu}$ is the inverse of $g^{\mu\nu}$. If one treats the y^μ as four new scalar fields, then the theory becomes a gauge one, with the corresponding gauge transformations having the form:

$$\delta y^\mu = y^\mu_{,\alpha}\delta\zeta^\alpha , \quad \delta\varphi = \partial_\alpha\varphi\delta\zeta^\alpha , \tag{2.41}$$

where $\delta\zeta(x)$ are $D+1$ x-dependent parameters of the gauge transformations. To see the relation between the theories (2.38) and (2.39) we construct their Hamiltonian versions as in the previous finite-dimensional case. Let us start with the gauge theory (2.39). Using the relations

$$\frac{\partial e}{\partial \dot y^\alpha} = e a^0_\mu , \quad \frac{\partial g^{\mu\nu}}{\partial \dot y^\alpha} = -2g^{0\mu}a^\nu_\alpha , \tag{2.42}$$

we introduce the canonical momenta:

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot\varphi} = e a^0_\mu a^0_\nu g^{\mu\nu}\dot\varphi + e a^0_\mu a^i_\nu g^{\mu\nu}\varphi_{,i} , \quad \pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot y^\mu} = -\frac{1}{2}e a^0_\mu a^0_\nu a^0_\rho g^{\nu\rho}\dot\varphi^2 -$$

$$- e a^i_\mu a^0_\nu a^0_\rho g^{\nu\rho}\dot\varphi\varphi_{,i} + e\left[\frac{1}{2}a^0_\mu a^i_\nu a^j_\rho - a^i_\mu a^j_\nu a^0_\rho\right] g^{\nu\rho}\varphi_{,i}\varphi_{,j} + e a^0_\mu F(\varphi) . \tag{2.43}$$

Equations (2.43) allow one to express only the velocity $\dot{\varphi}$ via fields and momenta, velocities $\dot{y}^\mu$ remain unexpressible,

$$\dot{\varphi} = \frac{\pi - e a^0_\mu a^i_\nu g^{\mu\nu}\varphi_{,i}}{e a^0_\mu a^0_\nu g^{\mu\nu}} . \tag{2.44}$$

Thus, the primary constraint $\phi_1 = 0$ appear:

$$\phi_{1\mu} = \pi_\mu + a^\alpha_\mu(y)\mathcal{H}_\alpha(y) , \tag{2.45}$$

where

$$\mathcal{H}_0(y) = \frac{\pi^2}{2eg^{00}} - \frac{g^{oi}}{g^{00}}\varphi_{,i}\pi - \frac{e}{2}\frac{\gamma^{ij}}{g^{00}}\varphi_{,i}\varphi_{,j} - eF(\varphi) , \tag{2.46}$$

$$\mathcal{H}_i(y) = \varphi_{,i}\pi , \qquad \gamma^{ij} = -\frac{g^{0i}g^{0j}}{g^{00}} + g^{ij} . \tag{2.47}$$

The density of the total Hamiltonian is

$$\mathcal{H}^{(1)} = \lambda^\mu \phi_{1\mu} , \quad \lambda^\mu = \dot{y}^\mu , \tag{2.48}$$

where the unexpressible velocities $\dot{y}^\mu$ appear as Lagrange multipliers. No more constraints appear and ϕ_1 are the first-class constraints. A possible form of the gauge conditions is

$$\phi^\mu_2 = y^\mu - f^\mu(x) = 0 , \quad \left|\frac{\partial f}{\partial x}\right| \neq 0 . \tag{2.49}$$

Together with the primary constraints they form a set of second-class constraints, which can be written in the following equivalent form $\Phi = 0$, where

$$\Phi = \begin{cases} \pi_\mu + a^\alpha_\mu(f(x))\mathcal{H}_\alpha(f(x)) = 0 , \\ y^\mu - f^\mu(x) = 0 . \end{cases} \tag{2.50}$$

One can select $Q = (\varphi, \pi)$ as independent variables. The Dirac brackets between them are:

$$\{\varphi, \pi\}_{D(\Phi)} = \{\varphi, \pi\} = 1 , \quad \{\varphi, \varphi\}_{D(\Phi)} = \{\varphi, \varphi\} = 0 , \quad \{\pi, \pi\}_{D(\varphi)} = \{\pi, \pi\} = 0 . \tag{2.51}$$

The time evolution is given by an effective Hamiltonian,

$$\dot{Q} = -\{Q, \Phi_A\}C_{AB}\dot{\Phi}_B = \{Q, H_{eff}\} , \quad C_{AB} = \{\Phi, \Phi\}^{-1}_{AB} ,$$
$$H_{eff} = \int \mathcal{H}_0(f(x))d\mathbf{x} . \tag{2.52}$$

Thus, the gauge (2.49) is unitary. One can easily see that the equations of motion (2.52) reproduce the dynamics of the initial theory of scalar field in flat space, but in a curvilinear RF, the coordinates x of which being related to the coordinates y of the inertial RF by the transformation (2.49). If $f^\mu(x) = x^\mu$ (an analog of the chronological gauge (2.15) of the finite-dimensional case), or $f^\mu(x) = \Lambda^\mu_\nu x^\nu$, $(\Lambda^T\eta\Lambda = \eta)$, then we get back to the initial theory in an inertial RF. In this case the effective Hamiltonian (2.52) takes the familiar form:

$$H_{eff} = \int \mathcal{H}d\mathbf{x} = \int \left[\frac{1}{2}(\pi^2 + \varphi_{,i}^2) - F(\varphi)\right] d\mathbf{x} . \tag{2.53}$$

What are physical quantities in the theory (2.39)? The Dirac's criterion admits only those ones which commute with all first–class constraints. In our case, that would mean:

$$\{A, \phi_{1\mu}\} \approx 0 , \tag{2.54}$$

where ϕ_1 is given in (2.45). Due to the same constraint (2.45) the physical quantities, which are functions on the phase space, may always be taken in the form $A = A(y, \boldsymbol{\eta})$, $\boldsymbol{\eta} = (\varphi, \pi)$. For such functions the condition (2.54) results in:

$$\{A, \phi_{1\mu}\} = \frac{\partial A}{\partial y^\mu} + \frac{\partial A}{\partial \boldsymbol{\eta}} a^\alpha_\mu \{\boldsymbol{\eta}, H_\alpha\} \approx 0 . \tag{2.55}$$

Multiplying this equation by the nonsingular matrix $y,^\mu_\beta$ one obtains the following relation:

$$\frac{dA}{dx^\mu} \approx 0 , \tag{2.56}$$

which is the generalization of the finite-dimensional equation (2.37). Equation (2.56) means that the above criterion admits as physical only functions that do not depend on space-time.

Similar to the finite–dimensional case we meet here the following situation. If we accept the Dirac's criterion then we can not identify the RI version of the scalar field theory with the initial formulation in flat space time even in the "chronological" gauge. That circumstance indicates us that the above criterion has to be critically reconsidered in the situation under consideration (see for detailed discussion the next section).

3 Relativistic particle theory. RI and time inversion

In this Section we are going to discuss theory of a relativistic particle as an instructive example of RI system. Such a theory is interesting by itself and has attracted attention already for a long time, in particular, due to the fact that it can serve as a prototype for a string theory (now one can consider it as 0-brane theory). On this example we are going to study different possibilities of time dependent gauge fixing and a relation between reparametrizations and time-inversion symmetry.

Let us restrict ourselves for simplicity to spinless particles moving in an external electromagnetic field with the potentials $A^\mu = (0, \mathbf{A}(\mathbf{x}))$, which corresponds to the case of a constant magnetic field. The theory of such a particle is described by the action [9]:

$$S = \int \left[-m\sqrt{1 - (\dot{\mathbf{x}}^2)} + g\dot{\mathbf{x}}\mathbf{A}\right] dt , \tag{3.57}$$

where $\mathbf{x} = (x^i)$ are spatial coordinates of some inertial reference frame and t is the time of the same frame, g is the algebraic charge of the particle and m its mass. The action (3.57) is non–singular, so that hamiltonianization and quantization can be done directly. The three–dimensional momentum vector $\boldsymbol{\pi}$ is defined by the relation:

$$\boldsymbol{\pi} = \frac{\partial L}{\partial \dot{\mathbf{x}}} = \frac{m\dot{\mathbf{x}}}{\sqrt{1 - (\dot{\mathbf{x}}^2)}} + g\mathbf{A} , \quad \boldsymbol{\pi} = (\pi_i) . \tag{3.58}$$

The classical equations of motion are:

$$\dot{\boldsymbol{\eta}} = \{\boldsymbol{\eta}, \omega\}\ , \quad \boldsymbol{\eta} = (\mathbf{x}, \boldsymbol{\pi})\ , \quad \omega = \sqrt{m^2 + (\boldsymbol{\pi} - g\mathbf{A})^2}\ . \tag{3.59}$$

They describe the motion of a particle with charge g in the constant magnetic field. Going over to the quantum theory we get the commutation relations between the operators $\hat{\mathbf{x}}, \hat{\boldsymbol{\pi}}$: $[\hat{x}^i, \hat{\pi}_k] = i\{x^i, \pi_k\} = i\delta^i_k$. In the coordinate representation $\hat{\mathbf{x}}$ is a multiplication operator, whereas $\hat{\boldsymbol{\pi}} = -i\frac{\partial}{\partial \mathbf{x}}$. The state vectors ψ obey the Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = \hat{\omega}\psi, \quad \hat{\omega} = \sqrt{m^2 + (i\nabla + g\mathbf{A})^2}\ . \tag{3.60}$$

The quantum theory constructed in this way describes only one particle with charge g. Such a theory is not equivalent to the theory which is based on the Klein-Gordon equation. Indeed, the latter describes states of charged particles with positive and negative energies or states of particles and antiparticles (charge $(-g)$) with positive energies.

Let us consider a RI formulation of the system in question. The corresponding action has the form

$$S = \int [-m\sqrt{\dot{x}^2} - g\dot{x}^\mu A_\mu] d\tau\ , \quad \dot{x}^\mu = \frac{dx^\mu}{d\tau}\ , \tag{3.61}$$

where now four $x^\mu = (x^0, \mathbf{x})$ are dynamical variables dependent on a new time τ. The action (3.61) similar to the one (2.1) obeys the reparametrization gauge symmetry (2.2). Hamiltonianization and quantization of the theory is more complicated than in the previous case. Let π_μ be the generalized momenta related to the variables x^μ,

$$\pi_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = -\frac{m\dot{x}_\mu}{\sqrt{\dot{x}^2}} - gA_\mu\ . \tag{3.62}$$

Then there is a constraint $(\pi + gA)^2 = m^2$, which can be written in the following equivalent form, which is convenient for our purposes:

$$\phi_1 = \pi_0 + \zeta\omega = 0, \quad \zeta = -\text{sign}\ \pi_0\ . \tag{3.63}$$

One can express from (3.62) three velocities $\dot{\mathbf{x}}$ as well as the sign of $\dot{x}^0$ in terms of the coordinates, momenta, and one unexpressible velocity, which is here $\lambda = |\ \dot{x}^0\ |$,

$$\dot{\mathbf{x}} = \lambda\omega^{-1}(\boldsymbol{\pi} - g\mathbf{A})\ , \quad \text{sign}\ \dot{x}^0 = \zeta\ , \quad \sqrt{\dot{x}^2} = m\lambda\omega^{-1}\ . \tag{3.64}$$

Thus, one can construct the total Hamiltonian $H^{(1)}$ by substituting (3.64) in the expression $\pi_\mu \dot{x}^\mu - L$,

$$H^{(1)} = \lambda\zeta\phi_1, \tag{3.65}$$

where λ is a Lagrange multiplier subjected, however, to the condition of positivity. The Hamiltonian equations of motion of the form

$$\dot{x}^\mu = \{x^\mu, H^{(1)}\}, \quad \dot{\pi}_\mu = \{\pi_\mu, H^{(1)}\}, \quad \phi_1 = 0, \quad \lambda \geq 0\ , \tag{3.66}$$

are equivalent to the Lagrangian ones. No secondary constraints arise from the consistency conditions and λ remains undetermined. This indicates that we are dealing with a gauge

theory. The total Hamiltonian is proportional to the constraints, as one can expect for a RI theory. Below we are going to discuss some possible gauges and quantization in these gauges.

First, let us consider the case of a neutral ($g = 0$) particles. In this case the action (3.61) is invariant under the time inversion $\tau \to -\tau$. Since the gauge symmetry in the case under consideration is related to the invariance of the action under the changes of the variables τ, there appears two possibilities: namely, to include or not to include the above discrete symmetry in the gauge group together with continuous reparametrizations. Let us first study the former possibility and include the time inversion in the gauge group. Then the gauge conditions have to fix the gauge freedom which corresponds to both kind of symmetries, namely, to fix the variable $\lambda = |\dot{x}^0|$, which is related to the reparametrizations, and to fix the variable $\zeta = \text{sign}\,\dot{x}^0$, which is related to the time inversion. To this end we may select the chronological gauge of the form

$$\phi_2 = x^0 - \tau = 0\ . \tag{3.67}$$

The consistency condition $\dot{\phi}_2 = 0$ leads on the constraint surface to the equation

$$\dot{\phi}_2 = \frac{\partial \phi_2}{\partial \tau} + \{\phi_2, H^{(1)}\} = -1 + \lambda\zeta = 0\ , \tag{3.68}$$

which results in the condition $\zeta\lambda = 1$. Remembering that $\lambda \geq 0$, we get $\zeta = 1, \lambda = 1$. That reduces the constraint surface to the following form: $\phi_a = 0,\ a = 1, 2$,

$$\phi_1 = \pi_0 + \omega\ , \quad \phi_2 = x^0 - \tau\ . \tag{3.69}$$

It is easy to calculate that $\{\phi_a, \phi_b\} = \text{antidiag}\,(-1, 1)$ and $C_{ab} = -\{\phi_a, \phi_b\}$, $C_{ab}\{\phi_b, \phi_c\} = \delta_{ac}$. One can select $\boldsymbol{\eta} = (\mathbf{x}, \boldsymbol{\pi})$ as independent variables. Their Dirac brackets coincide with the Poisson ones,

$$\{\boldsymbol{\eta}, \boldsymbol{\eta}'\}_D = \{\boldsymbol{\eta}, \boldsymbol{\eta}'\}\ . \tag{3.70}$$

The quantum operators $\check{\boldsymbol{\eta}}$ obey the equation (2.19), which in this particular case takes the following form

$$\dot{\check{\boldsymbol{\eta}}} = -\{\boldsymbol{\eta}, \phi_a\} C_{ab} \left.\frac{\partial \phi_b}{\partial \tau}\right|_{\boldsymbol{\eta} = \check{\boldsymbol{\eta}}} = \{\boldsymbol{\eta}, \omega\}|_{\boldsymbol{\eta} = \check{\boldsymbol{\eta}}} = -i[\check{\eta}, \check{\omega}]\ , \tag{3.71}$$
$$[\check{\boldsymbol{\eta}}, \check{\boldsymbol{\eta}}'] = i\{\boldsymbol{\eta}, \boldsymbol{\eta}'\}.$$

Thus, the evolution is unitary and is governed by the effective Hamiltonian ω (3.59). One can consider time independent Schrödinger operators $\hat{\boldsymbol{\eta}} = e^{-i\check{\omega}\tau}\check{\boldsymbol{\eta}}(\tau)e^{i\check{\omega}\tau}$ and time dependent state vectors. The operators $\hat{\boldsymbol{\eta}}$ obey the same commutation relations (3.71) and can be realized as in the non-reparametrization invariant case. Thus, we get the Schrödinger equation (3.60) if one identifies τ with t.

Suppose we do not include the time inversion in the gauge group. That is espesially natural when $g \neq 0$, $A_\mu \neq 0$, because in this case the time inversion is not anymore a symmetry of the action. Thus, one may now consider more general situation of the charged particle moving in the external magnetic field. Under the above supposition the condition

(3.67) is not anymore a gauge, it fixes not only the reparametrization gauge freedom (fixes λ) but it fixes also the variable ζ which is now physical. A possible gauge condition has the form [11]:

$$\phi_2 = x^0 - \zeta\tau = 0 \ . \tag{3.72}$$

The consistency condition $\dot{\phi}_2 = 0$ leads to the equation

$$\dot{\phi}_2 = \frac{\partial\phi_2}{\partial\tau} + \{\phi_2, H^{(1)}\} = -\zeta + \lambda\zeta = 0 \ , \tag{3.73}$$

which fixes only $\lambda = 1$ and retains ζ as a physical variable. Trajectories with $\zeta = +1$ correspond to particles, while trajectories with $\zeta = -1$ to antiparticles [11]. Two second-class constraints

$$\phi_1 = \pi_0 + \zeta\omega \ , \quad \phi_2 = x^0 - \zeta\tau \ , \tag{3.74}$$

form the same algebra like in the previous case. One has only to add the relation $\{\zeta, \boldsymbol{\eta}\}_D = 0$ to the Dirac brackets (3.70). However, we get here an additional operator $\hat{\zeta}$, which has to be realized in the Hilbert space of state vectors. We assume the operator $\hat{\zeta}$ to have the eigenvalues $\zeta = \pm 1$ by analogy with the classical theory. Such an operator can be realized in a Hilbert space whose elements are two-component columns

$$\Psi = \begin{pmatrix} \Psi_1(\mathbf{x}) \\ \Psi_2(\mathbf{x}) \end{pmatrix} , \tag{3.75}$$

if we chose the operator $\hat{\zeta}$ as the matrix $\hat{\zeta} = \mathrm{diag}(1, -1)$. The time independent operators $\hat{\eta}$ can be realized as follows

$$\hat{x}^i = x^i \mathbf{I} \ , \quad \hat{\pi}_j = -i\partial_j \mathbf{I} \ , \tag{3.76}$$

where $\mathbf{I}$ is a unit 2×2 matrix. The time evolution of the state vectors is described by the Schrödinger equation

$$i\frac{\partial\Psi}{\partial\tau} = \hat{\omega}\Psi, \tag{3.77}$$

where $\hat{\omega}$ is given by eq.(3.60). The equation (3.77) differs from the similar equation (3.60) due to the structure of the Hilbert space, which now allows one to describe states for both particles and antiparticles.

As an example of gauge conditions which lead to the description from the point of view some non-inertial reference frames we consider here the gauge (a =const)

$$\phi_2 = x^0 + \frac{\pi_0}{m}\tau + a = 0 \tag{3.78}$$

in case when the time inversion is not included in the gauge group and the gauge

$$\phi_2 = x^0 - \frac{|\pi_0|}{m}\tau + a = 0 \tag{3.79}$$

when it does.

One can demonstrate first that the gauge condition (3.78) corresponds (at any a) to the proper-time gauge $\dot{x}^2 = 1$ in the Lagrangian formulation. Indeed the consistency condition

$$\dot{\phi}_2 = \frac{\partial\phi_2}{\partial\tau} + \{\phi_2, H^{(1)}\} = \frac{\pi_0}{m} + \lambda\zeta = 0 \ , \tag{3.80}$$

defines $\lambda = \frac{|\pi_0|}{m}$. Remembering the last relation (3.64) and the constraint (3.63) we can see that (3.78) at any a is equivalent to the condition $\dot{x}^2 = 1$. Thus, (3.78) may be called proper–time gauge in Hamiltonian formulation. The proper–time gauge, similarly to the chronological gauge (3.72), does not fix the variable ζ, and leaves the possibility to describe particles and antiparticles at the same time. The gauge condition (3.79) similar to one (3.67) fixes the variables ζ, thus it is acceptable only when the time inversion (3.78)is included in the gauge group.

The constraint algebra in both gauges (3.72) and (3.78) is the same, the commutation relations and the realization for the independent operators are also the same, however the effective Hamiltonian in the proper–time gauge is different,

$$H_{eff} = \frac{\omega^2}{2m} \ . \tag{3.81}$$

Thus, the Schrödinger equation has the form

$$i\frac{\partial\Psi}{\partial\tau} = \frac{\hat{\omega}^2}{2m}\Psi \ . \tag{3.82}$$

One can establish a formal relation between the gauges (3.72) and (3.78). Namely, one can present a canonical transformation, which connects both gauges on the classical level. The generating function of a such transformation has the form:

$$W = x^\mu\pi'_\mu + \tau \mid \pi'_0 \mid - \tau\frac{\pi_0'^2}{2m} \ , \tag{3.83}$$

if the phase space variables without the primes are related to the chronological gauge (3.72) and the primed ones to the proper–time gauge (3.78). The transformation does not change the variables x^i and π_μ. It changes only x^0, $x'^0 = x^0 - \zeta\tau - \frac{\pi_0}{m}\tau$. Thus, it transforms the constraint surface of the first gauge into the one of the second gauge. One can also see that this transformation connects both Hamiltonians

$$H = H' + \frac{\partial W}{\partial\tau} = \frac{\pi_0'^2}{2m} + \mid \pi'_0 \mid - \frac{\pi_0'^2}{2m} = \mid \pi'_0 \mid = \mid \pi_0 \mid = \omega \ . \tag{3.84}$$

On the quantum level the state vectors in both gauges are connected by means of a quantum canonical transformation

$$\Psi = e^{-i\hat{W}}\Psi' \ , \quad \hat{W} = \tau\hat{\omega} - \tau\frac{\hat{\omega}^2}{2m} \ . \tag{3.85}$$

In the spirit of the interpretation given in Sec.3 we may say that the chronological gauges (3.67) and (3.72) lead us to the inertial RF, whereas the proper–time gauges (3.78) and (3.79) correspond to the description from the point of view of non–inertial (at $A \neq 0$) RF. A formal possibility to connect these two gauges by means of a canonical transformation does not mean their physical equivalence since such a transformation depends explicitly on time.

4 Possible interpretation

Results of the consideration which was presented in two previous Sections may be summarized in the following generalizing interpretation. Let us turn first to the non-RI actions (2.7), (2.38), and (3.1).It is natural to believe that such actions give descriptions of the corresponding physical systems in certain RF. For example, actions (2.38) and (3.1) provide a description from the point of view of an inertial RF with a Cartesian base. Constructing RI versions of the above mentioned actions we see that a possibility appears to describe the same physical system from the point of view of a more wide class of RF. The theories become gauge ones, they contain additional non-physical variables. The corresponding gauge symmetry - RI leads always to zero Hamiltonian phenomenon. To introduce a dynamics we fix a gauge by means of supplementary conditions which depend on time (or space-time variables) explicitly. It turns out that such a gauge fixing looks literally like a certain choice of a RF. In particular, the chronological gauges correspond to the RF in which initial non-RI actions are formulated. More complicated gauges reproduce in general non-inertial curvilinear RF. Based on the experience that was derived from the simple example consideration we believe that any fixation of the reparametrization gauge freedom corresponds always to a certain choice of the space-time RF. Here we have especially emphasized the origin of the RF which is fixed. The point is that the fixation of the gauge freedom of any kind may be trreated as a choice of some RF. In this sense the reparametrization symmetry is similar to gauge symmetries of different nature, let us call them internal gauge symmetries (one may define the latter symmetries as ones which do not involve the space-time coordinate transformations). The principle distinction between the reparametrizations and internal gauge symmetries are related with the distinction between the corresponding RF. Whereas one believes that the RF for the internal gauge symmetries may not be realized physically (at least until now), the choice of RF to measure space-time coordinates may be physically realized. If in the former case the physical quantities do not depend on the choice of the gauge, in the latter case this may be not true. To describe local physical quantities it is natural to use space-time dependent functions which depend explicitly on the choice of RF and are transformed in a certain way under the RF change. Thus, we have to admit gauge non-invariant objects to decribe physics. As it is known [1, 2], when the gauge transformations do not involve a transformation of space-time coordinates, gauge invariant functions on the phase space have to commute with first-class constraints on the mass shell (Dirac's criterion). The previous reasonings mean that the "local" point of view, which is, in fact, advocated here, abrogate the Dirrac's criterion with respect to the first-class consraints which generate the reparametrizations. Rejection of the Dirac's criterion in the case of the reparametrization gauge symmetry admits, thus, any functions (which are physical with respect to the internal gauge symmetries) as physical ones. The choice of them is dictated by concrete conditions of the problem. Let us, for example, return to the theory of scalar field studied in Sect.II. Let us have a Lorentz tensor in the initial non-RI formulation, let say the vector $\varphi_{,\mu}(x)$. The question is: what kind of physical quantity corresponds to it in the RI formulation? One may present two naturally constructed quantities, the general coordinate vector $\varphi_{,\mu}(x)$ and the scalar $a^{\mu}_{\alpha}\varphi_{,\mu}(x)$. Both of them coincide with the initial physical quantity in the chronological gauge (in the inertial RF). In the literature one may often meet some arguments in favor of the latter choice (see for example [12]).

We know that gauges which fix an internal gauge symmetry may always be selected in time (space-time) independent form (canonical gauges). Such gauges then may be related by means of a time-independent canonical transformation [2]. In such a way, a formal equivalence between descriptions in different gauges may be established. As we have seen from the examples in Sect.II and Sect.III the time-dependent gauges in RI theories may also be connected by means of canonical trransformations (such a possibility cerrtainly follows from generral theorems [2]). However, such transformations necessary depend on time (space-time variables). Thus, in this case a formal possibility to connect different gauges does not mean their literal physical equivalence. The canonical transformations in such a case establish only a relation between descriptions of one and the same system in different RF.

5 RI in general and the zero Hamiltonian phenomenon

Above we have considered several examples of RI systems. The explicit form of the corresponding GT depends on the structure of the theory (compare (2.2) and (2.3)). At the same time, in all known examples the total Hamiltonian vanishes on the constraint surface of the theory. Is it possible to discover some specific structure of RST in general and a relation of the latter with the zero-Hamiltonian phenomenon? Below we are going to discuss this problem and present such a relation.

Let us have a theory with finite number of degrees of freedom, which is described by an action ($q=q^a$, $a=1,...,D$ are generalized coordinates and t is time),

$$S=\int L(q,\dot{q},t)dt\ . \tag{5.1}$$

Consider a transformation in the space of trajectories $q^a(t)$,

$$q^a(t)\to q'^a(t)=G^a_t(q)\ , \tag{5.2}$$

where $G^a_t(q)$ are some functionals on $q^a(t)$, depending parametrically on time. We will call (5.2) a symmetry transformation (ST) of the theory if the Lagrangian function $L(q,\dot{q},t)$ is changed under such a transformation only by a total derivative of some function,

$$L'(q,\dot{q},t)=L(G_t(q),\dot{G}_t(q),t)=L(q,\dot{q},t)+\frac{dF}{dt}\ . \tag{5.3}$$

One can see that the Lagrangians $L(q,\dot{q},t)$ and $L'(q,\dot{q},t)$ have the same extremals. That can be regarded as an argument in favor of the proposed definition of the ST.

The ST can be discrete, continuous global and gauge ones. Continuous global ST are parametrized by a set of parameters ϵ_α, $\alpha=1,...,r$. It is convenient to define the point $\epsilon_\alpha=0$ as the one that corresponds to the identical transformation. In this case (5.2) can be presented in the form

$$q'^a(t)=G^a_t(q|\epsilon),\quad G^a_t(q|0)=q^a(t), \tag{5.4}$$

where the ϵ-dependence is indicated explicitly. The infinitesimal form of a global continuous ST is:

$$q'^a(t)=q^a(t)+\delta q^a(t),\quad \delta q^a(t)=\rho^a_\alpha(t)\epsilon_\alpha,\quad \rho^a_\alpha(t)=\left.\frac{\partial G^a_t(q|\epsilon)}{\partial\epsilon_\alpha}\right|_{\epsilon=0}, \tag{5.5}$$

where $\rho^a_\alpha(t)$ are the generators of the transformations. Continuous ST are GT (or local ST) if they are parametrized by some arbitrary functions on time (or in the case of field theories by functions of space-time variables). They can be presented in the form (5.4) where, however, $G^a_i(q|\epsilon)$ may depend not only on ϵ but on its derivatives over time. In this case

$$\delta q^a(t) = \int R^a_\alpha(t,t')\epsilon_\alpha(t')dt', \quad R^a_\alpha(t,t') = \left.\frac{\delta G^a_i(q|\epsilon)}{\delta\epsilon_\alpha(t')}\right|_{\epsilon=0} . \tag{5.6}$$

As it was demonstrated in [2] the generators $R^a_\alpha(t,t')$ are local in time (in the case of ordinary bosonic variables) i.e. they have the following structure

$$R^a_\alpha(t,t') = \sum_{k=0}^{M} \rho^a_{\alpha(k)}(t)\partial^k_t\delta(t-t') , \tag{5.7}$$

where M is finite. Thus, one can write in this case:

$$\delta q^a(t) = \sum_{k=0}^{M} \rho^a_{\alpha(k)}(t)\epsilon^{(k)}_\alpha(t), \quad \epsilon^{(k)}_\alpha(t) = \frac{d^k\epsilon_\alpha(t)}{dt^k} . \tag{5.8}$$

The presence of r-parametrical continuous global ST indicates that there exist r conserved charges. Indeed, in this case $\delta L = \frac{d}{dt}\delta F$, which is an infinitesimal form of (5.3). The variations δL and δF can be presented as follows:

$$\delta L = \frac{\delta S}{\delta q^a}\delta q^a + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\delta q^a\right) = \left[\frac{\delta S}{\delta q^a}\rho^a_\alpha + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q^a}\rho^a_\alpha\right)\right]\epsilon_\alpha, \quad \delta F = f_\alpha\epsilon_\alpha, \tag{5.9}$$

where

$$\frac{\delta S}{\delta q^a} = \frac{\partial L}{\partial q^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q^a}\right) ,$$

so that $\frac{\delta S}{\delta q^a} = 0$ are the Euler-Lagrange equations of motion. Thus, we get

$$\frac{dQ_\alpha}{dt} = -\rho^a_\alpha\frac{\delta S}{\delta q^a}, \quad Q_\alpha = \frac{\partial L}{\partial \dot q^a}\rho^a_\alpha - f_\alpha , \tag{5.10}$$

and therefore Q_α are the above mentioned conserved charges. An analogous statement is valid for GT as well. Moreover, in this case one can make some conclusions about the structure of the corresponding conserved charges. Below we are going to formulate and prove some statements, which are useful for our purposes.

Let an action obey a gauge ST. In the infinitesimal form that results in the condition

$$\delta L = \frac{\delta S}{\delta q^a}\delta q^a + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q^a}\delta q^a\right) = \frac{d}{dt}\delta F, \tag{5.11}$$

where δq^a are given by eq.(5.8) and δF is a function. Similarly to the derivation (5.9), (5.10) that implies the conservation law

$$\frac{dQ}{dt} = -\frac{\delta S}{\delta q^a}\delta q^a, \quad Q = \left(\frac{\partial L}{\partial \dot q^a}\delta q^a - \delta F\right) . \tag{5.12}$$

The conserved charge Q may be presented in the form:

$$Q = \sum_{k=0}^{M'} Q_{\alpha(k)}(t)\epsilon^{(k)}_\alpha(t). \tag{5.13}$$

Substituting (5.8) and (5.13) into (5.12) we get

$$\sum_{k=0}^{M'}\left[\dot Q_{\alpha(k)}\epsilon^{(k)}_\alpha(t) + Q_{\alpha(k)}\epsilon^{(k+1)}_\alpha(t)\right] = -\frac{\delta S}{\delta q^a}\sum_{k=0}^{M}\rho^a_{\alpha(k)}\epsilon^{(k)}_\alpha(t) . \tag{5.14}$$

It is clear that $M' = M-1$. Due to the arbitrariness of $\epsilon_\alpha(t)$, one can consider the derivatives $\epsilon^{(k)}_\alpha(t)$ as independent arbitrary functions and compare the terms on the left and right hand side of (5.14) with the same $\epsilon^{(k)}_\alpha(t)$. Thus one gets:

$$Q_{\alpha(M-1)} = -\frac{\delta S}{\delta q^a}\rho^a_{\alpha(M)} , \quad \dot Q_{\alpha(M-1)} + Q_{\alpha(M-2)} = -\frac{\delta S}{\delta q^a}\rho^a_{\alpha(M-1)} , \quad \dots ,$$
$$\dot Q_{\alpha(k)} + Q_{\alpha(k-1)} = -\frac{\delta S}{\delta q^a}\rho^a_{\alpha(k)} , \quad \dots . \tag{5.15}$$

It follows from the system (5.15) that

$$Q_{\alpha(k)} = \Lambda^a_{\alpha(k)}\frac{\delta S}{\delta q^a} , \quad \text{or} \quad Q = \Lambda^a\frac{\delta S}{\delta q^a}, \quad \Lambda^a = \sum_{k=0}^{M-1}\epsilon^{(k)}_\alpha(t)\Lambda^a_{\alpha(k)} , \tag{5.16}$$

where $\Lambda^a_{\alpha(k)}$ contain operators of the differentiation in time up to the order $(M-k-1)$. Thus, one may say that[1]:

The conserved charge (5.13) which corresponds to any GT and its components $Q_{\alpha(k)}$ vanish on the equations of motion.

Let a global ST be the reduction of a GT to constant values of the parameters $\epsilon_\alpha(t)$. In this case the generators $\rho^a_\alpha(t)$ from equation (5.5) are just $\rho^a_{\alpha(0)}(t)$ from equation (5.8), and therefore $\delta q^a(t) = \rho^a_{\alpha(0)}(t)\epsilon_\alpha$. The corresponding conserved charges Q_α from (5.10) coincide with $Q_{\alpha(0)}$ from (5.13) and vanish on the equation of motion according to (5.16). An inverse statement is also valid, namely:

If some global continuous ST of an action , $\delta q^a(t) = \rho^a(t)\epsilon$, generates a conserved charge, which vanishes on the equation of motion, then this action obeys a gauge symmetry.

Let us prove this. Similar to eq.(5.9) one can get:

$$\frac{\partial L}{\partial q^a}\rho^a + \frac{\partial L}{\partial \dot q^a}\dot\rho^a = \frac{d}{dt}f . \tag{5.17}$$

We can use this equation to write the following relation:

$$\frac{\partial L}{\partial q^a}\rho^a\epsilon(t) + \frac{\partial L}{\partial \dot q^a}\frac{d}{dt}\left[\rho^a\epsilon(t)\right] = \frac{d}{dt}\left[f\epsilon(t)\right] + \dot\epsilon(t)Q_{(0)} , \tag{5.18}$$

[1]This statement was, in fact, known to Noether [13]

where $Q_{(0)} = \frac{\partial L}{\partial \dot{q}^a}\rho^a - f$ is the conserved charge related to the global continuous ST (see (5.10), and $\epsilon(t)$ an arbitrary function of t. Let this charge vanish on the equations of motion, that is

$$Q_{(0)} = \Lambda^a_{(0)}\frac{\delta S}{\delta q^a}, \tag{5.19}$$

where $\Lambda^a_{(0)}$ may contain operators of differentiation with respect to time up to a finite order. Thus, the last term in the right hand side of (5.18) has the form $\dot{\epsilon}(t)\Lambda^a_{(0)}\frac{\delta S}{\delta q^a}$. One can always write this term in the different form:

$$\dot{\epsilon}(t)\Lambda^a_{(0)}\frac{\delta S}{\delta q^a} = -\frac{\delta S}{\delta q^a}\Lambda^a\dot{\epsilon}(t) + \frac{d\varphi}{dt}, \tag{5.20}$$

where Λ^a is an operator symmetric to $\Lambda^a_{(0)}$, and φ some function. On the other hand

$$\frac{\delta S}{\delta q^a}\Lambda^a\dot{\epsilon}(t) = \frac{\partial L}{\partial q^a}\Lambda^a\dot{\epsilon}(t) + \frac{\partial L}{\partial \dot{q}^a}\frac{d}{dt}\left[\Lambda^a\dot{\epsilon}(t)\right] - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^a}\Lambda^a\dot{\epsilon}(t)\right]. \tag{5.21}$$

Gathering (5.18), (5.20) and (5.21) we get

$$\delta L = \frac{\partial L}{\partial q^a}\delta q^a(t) + \frac{\partial L}{\partial \dot{q}^a}\delta\dot{q}^a(t) = \frac{d}{dt}\left[f\epsilon(t) + \varphi + \frac{\partial L}{\partial \dot{q}^a}\Lambda^a\dot{\epsilon}(t)\right],$$

where $\delta q^a(t)$ is a GT,

$$\delta q^a(t) = \rho^a\epsilon(t) + \Lambda^a\dot{\epsilon}(t). \tag{5.22}$$

Based on the two statements proved above we may define what can be called reparametrization ST in general. To this end let us first discover what is a global representative of such a symmetry. One can remember that in all known examples the existence of the reparametrization invariance leads to the zero-Hamiltonian phenomenon. More exactly, the total Hamiltonian [1, 2] appears to be proportional to constraints of the theory, or it vanishes on the equations of motion. Such a Hamiltonian can be derived from the expression for the Lagrangian energy, if one replaces there all the primary-expressible velocities as functions on phase space variables and denotes the primary unexpressible velocities by λ, which play then the role of Lagrange multipliers. Thus, in this case one can write

$$\mathcal{E} = \frac{\partial L}{\partial \dot{q}^a}\dot{q}^a - L = \Lambda^a_{(0)}\frac{\delta S}{\delta q^a}. \tag{5.23}$$

Another observation is that in all known examples, where RI takes place, the corresponding Lagrangians do not depend explicitly on time. Thus, we have the conservation law:

$$\frac{d\mathcal{E}}{dt} = -\dot{q}^a\frac{\delta S}{\delta q^a}. \tag{5.24}$$

On the other hand, one can interpret energy $\mathcal{E}$ as a conserved charge related to the global ST, which are translations in time, $q^a(t) \to q^a(t+\epsilon)$ or in the infinitesimal form:

$$\delta q^a(t) = \dot{q}^a(t)\epsilon. \tag{5.25}$$

Indeed, in this case

$$\delta L = \frac{\partial L}{\partial q^a}\dot{q}^a\epsilon + \frac{\partial L}{\partial \dot{q}^a}\ddot{q}^a\epsilon = \epsilon\frac{dL}{dt}, \tag{5.26}$$

so that (5.25) is a symmetry and, at the same time, (5.24) follows also from (5.26). Taking all said into account it is natural to regard translations in time as global representatives of the reparametrization GT. Then, one can define the latter GT as a possible extension of the translations in time to GT, in the manner which was used in the proof of the inverse statement. Thus, such GT have the form (5.22) with $\rho^a = \dot{q}^a$,

$$\delta q^a(t) = \dot{q}^a(t)\epsilon(t) + \Lambda^a\dot{\epsilon}(t), \tag{5.27}$$

where the operators Λ^a are defined by the explicit form of the Lagrangian of the theory (see for example the transformations (2.2) and (2.4)).

Considering the above finite-dimensional case, we have seen that the conserved charge Q (5.13) related to any GT and all its components $Q_{\alpha(\kappa)}$ vanish on the equations of motion. In particular, the components $Q_{\alpha(0)}$, which are the conserved charges related to the corresponding global ST (global representatives of the GT), with $\epsilon_\alpha(t) = \epsilon_\alpha = const$, also vanish on the equations of motion. However, such a conclusion may be wrong in the case of field theory. As an example, let us take electrodynamics coupled to a scalar field $\varphi(x)$,

$$S = \int \mathcal{L}d^{D+1}x, \quad \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu + ieA_\mu)\varphi^\dagger(\partial^\mu - ieA^\mu)\varphi - V(\varphi^\dagger\varphi). \tag{5.28}$$

The conserved charge, (an analog of (5.10)), related to GT $\delta A_\mu(x) = \partial_\mu\epsilon(x)$, $\delta\varphi(x) = ie\varphi(x)\epsilon(x)$, $\delta\varphi^\dagger(x) = -ie\varphi^\dagger(x)\epsilon(x)$, where $\epsilon(x)$ are parameters of the GT, is

$$Q = \int\left(\frac{\partial L}{\partial \dot{A}_\mu}\delta A_\mu + \frac{\partial L}{\partial \dot{\varphi}}\delta\varphi + \frac{\partial L}{\partial \dot{\varphi}^\dagger}\delta\varphi^\dagger\right)d^Dx = \int\left[F_{0k}\partial_k\epsilon(x) - j_0\epsilon(x)\right]d^Dx, \tag{5.29}$$

$$j_0 = \varphi^\dagger(\partial_0 - ieA_0)\varphi - \varphi(\partial_0 + ieA_0)\varphi^\dagger. \tag{5.30}$$

This expression can be transformed on the equation of motion $\partial_k F_{0k} + iej_0 = 0$, to the following form

$$Q = \int \partial_k\left[F_{0k}\epsilon(x)\right]d^Dx. \tag{5.31}$$

In the case of GT with $\epsilon(x)$ decreasing rapidly enough in the limit $|\mathbf{x}| \to \infty$, the charge (5.31) is zero. In the case of global ST with $\epsilon(t) = \epsilon = const$, we have

$$Q = \epsilon\int \partial_k F_{0k}d^Dx = -ie\epsilon\int j_0 d^Dx. \tag{5.32}$$

This expression may differ from zero. In the Coulomb phase F_{0k} behaves at large r as $r^{-(D-1)}$, so that the integral in (5.32) is proportional to the total electrical charge of the system, which is in general not zero. However, if a spontaneous symmetry braking takes place (Higgs phase) the vector field becomes massive and F_{0k} decreases exponentially, resulting in $Q = 0$. (The total charge of any state is zero.)

One meets a similar situation in the theory of gravity. Let us select the action of the gravitational field of the form, which was first proposed by Dirac [5] (for a detailed treatment see [9, 2]),

$$S = \int L d^4x, \quad L = A + \partial_i q^i , \tag{5.33}$$

where

$$A = \sqrt{-g^{00} g_{(3)}} \left[\frac{z_{ik}}{4} \left(e^{il} e^{km} - e^{ik} e^{lm} \right) z_{lm} - -\frac{R_{(3)}}{g^{00}} \right] , \quad g_{(3)} = \mid g_{ik} \mid , \quad e^{ik} g_{kl} = \delta^i_l ,$$

$$z_{ik} = \dot{g}_{ik} - g_{0i,k} - g_{0i,k} + 2\gamma^l_{ik} g_{0l}, \quad q^i = \sqrt{-g_{(3)}} g_{lm,k} \left(e^{il} e^{km} - e^{ik} e^{lm} \right) ,$$

and γ^l_{ik}, $R_{(3)}$ are the Christoffel symbols and the scalar curvature constructed for the three-dimensional metric g_{ik}. This action is equivalent to the Einstein-Hilbert one under certain assumptions about the global structure of the theory. The Lagrangian L contains neither higher (second) order derivatives of the metric, nor velocities $\dot{g}_{0\mu}$. The variation of L under the GT (2.6) has the form $\delta L = \partial_\mu [L \epsilon^\mu (x)]$. The corresponding conserved charge is

$$Q = \int \left(\frac{\partial L}{\partial \dot{g}_{ik}} \delta g_{ik} - L \epsilon^0 \right) d^3x. \tag{5.34}$$

If $\epsilon^\mu (x) \to 0$ when $\mid \mathbf{x} \mid \to \infty$, then one can see that it vanishes on the equations of motion. For example, if $\epsilon^i(x) \equiv 0$

$$Q = \int \epsilon^0 \left[g_{\mu\nu} \frac{\delta S}{\delta g_{\mu\nu}} + g_{00} \frac{\delta S}{\delta g_{00}} \right] d^3x . \tag{5.35}$$

In the case of $\epsilon_0(x) = \epsilon_0 = const$, $\epsilon_i(x) \equiv 0$, the charge (5.34) is proportional to the total energy and has the form:

$$Q = -\epsilon_0 \int \partial_i q^i d^3x . \tag{5.36}$$

The integral on the right hand side of (5.36) is generally non-zero. In particular, in an asymptotically flat space [10] for the system with the total mass M

$$g_{ik} = -\delta^i_k \left(1 + \frac{M}{8\pi r} \right) + O\left(\frac{1}{r^2} \right) . \tag{5.37}$$

Then $Q = \epsilon_0 M$ is not zero. One can remark, considering for example the theory of gravity, that in spite of the fact that four-dimensional divergency terms in the Lagrangian do not affect the form of the equations of motion, they can affect the form of the corresponding conserved charges. That may serve as an additional argument in favor of a certain form of the selected Lagrangian.

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