

UNIVERSIDADE DE SÃO PAULO

PUBLICAÇÕES

INSTITUTO DE FÍSICA
CAIXA POSTAL 66318
05315-970 SÃO PAULO - SP
BRASIL

IFUSP/P-1265

BRST-co-BRST QUANTIZATION OF
REPARAMETRIZATION INVARIANT
THEORIES

Géza Fülöp
Instituto de Física, Universidade de São Paulo

Reg. 1-22

Abril/1997

São Paulo

April, 1997

IFUSP/P-1265

hep-th/yymmnnn

April 25, 1997

BRST-co-BRST Quantization of Reparametrization Invariant Theories

Géza Fülöp¹

Institute of Mathematical Physics

USP

Brazil

Abstract

We study some reparametrization invariant theories in context of the BRST-co-BRST quantization method. The method imposes restrictions on the possible gauge fixing conditions and leads to well defined inner product states through a gauge regularization procedure. Two explicit examples are also treated in detail.

1 Introduction

There exists a large class of theories that are invariant under reparametrization. In order to quantize these theories one has to introduce a gauge fixing that relates the initial time-like variable of the theory to an internal parameter. This parameter is understood as the physical time of the system. It means that we obtain a time dependent gauge fixing condition. The standard Dirac quantization procedure [1] can not be applied for this case. There are several methods developed to handle time dependent constraints. One of the methods by Gitman and Tyutin [2] decomposes time evolution into two quite natural components. Part of time evolution in this treatment is generated by the Hamiltonian as is the case for the ordinary theories with time independent constraints. The other part is given by the time variation of the constraint surface. This part might not lead to a unitary time evolution in the quantum theory. However, in cases when it does, one can find an effective Hamiltonian defined on the physical phase space that generates the same time evolution for the physical variables as the one found by the above mentioned treatment. For some details of this method see [3] and [4].

A purely geometrical method of quantization of time dependent constraints also exists [5] and [6]. This method has the advantage to be able to find possible topological obstructions in the way of constructing a globally defined effective Hamiltonian. It does not give however a recipe on how to find the physical variables of the theory.

Both these methods are generalizations of the Dirac quantization procedure. We have however, another very powerful quantization method, namely BRST-quantization [7]. This method has the advantage that together with a co-BRST operator it leads to a well defined inner product space [8], [9], [10].

The BRST-charge Q (or operator in the quantum case) is a conserved nil-potent quantity, built on a linear combination of the first class constraints of the theory:

$$Q = \varphi_i \eta^i + \dots \quad (1.1)$$

where φ_i are the first class constraints and the η^i are the so called ghost variables, which have different Grassman parity from the constraints: if a constraint is Grassmann-even, the corresponding ghost is odd, and vice-versa.

¹E-mail: fulop@fma.if.usp.br

Co-BRST *Q is built on the combination of the gauge fixing conditions in a way quite similar to the one used in the case of the BRST charge. In our case, where the gauge fixing is necessarily time dependent, co-BRST will depend on time too: it will contain the information on how the physical time t is related to one of the configurations space variables (x^0). We shall see that there are some conditions that arise from the demand that there exist physical states (or functions in the classical case) in the theory.

There exists an important difference between reparametrization invariant theories and other models with time dependent constraints. In the case of the latter ones the effective Hamiltonian generates the physical time evolution. In our case however, we may choose two different interpretations. In one of them t is the physical time and the effective Hamiltonian, being the physical Hamiltonian, generates the evolution along it. In the other interpretation however, t is only an internal parameter and the physical Hamiltonian has to be found separately. It will then only depend on the physical variables.

The second interpretation is the natural one, when one has an externally given time. However, for generic reparametrization invariant theories (like gravitation) the time variable is not a priori given. For these models the first interpretation is valid. This is also the interpretation we follow in this paper. For a more thorough discussion see [11].

There are many interesting reparametrization invariant models. As shown in [4], all models with vanishing Hamiltonian are reparametrization invariant. At the same time, any other model (both with regular and singular Lagrangians) can be transformed into a reparametrization invariant form by making the time variable depend on a parameter $x^0 \rightarrow x^0(t)$:

$$S = \int L(q, \frac{dq}{dx^0}, x^0) dx^0 \rightarrow S' = \int L(q, \frac{\dot{q}}{x^0}, x^0) x^0 d\tau, \quad (1.2)$$

where $\dot{x}^0 = \frac{dx^0}{d\tau} > 0, \forall t$. The action S' is invariant under the reparametrization of x^0 .

The plan of the paper is the following: In the next Section we introduce the two important charges (operators) we are going to use later: the BRST and co-BRST quantization. The in Sections 3 we show the BRST-co-BRST quantization of a reparametrization invariant theory. In Sections 4 and 5 we treat two simple examples, those of the non-relativistic and the relativistic particle, written in a reparametrization invariant form. We show here how the restrictions on the possible gauge-fixing conditions appear as a consequence of

demanding the existence of a consistent co-BRST charge. We conclude the paper with a discussion in the last Section.

Throughout the paper we use the following conventions: in the classical case all variables (even the ghost ones) come in canonical pairs:

$$\{x, p\} = 1, \quad \{\eta, \mathcal{P}\} = 1. \quad (1.3)$$

In the quantum case we have:

$$[x, p] = i \quad [\eta, \mathcal{P}] = i. \quad \eta^\dagger = \eta. \quad \mathcal{P}^\dagger = -\mathcal{P} \quad (1.4)$$

2 BRST and co-BRST

BRST-quantization is based on a symmetry of the gauge-fixed Lagrangian [7]. In the Hamiltonian picture this is expressed in the definition of the physical states, namely, that the physical states are those that get annihilated by the BRST-operator Q . However, the existence of this symmetry means that there is freedom left in the states. The freedom connected to the BRST-symmetry is manifested through the nilpotency of the BRST operator ($Q^2 = 0$), which is in some sense its most important property. Nilpotency means that all BRST-exact states are also BRST-invariant:

$$Q(Q | i) = 0, \quad \forall | i). \quad (2.1)$$

This means that our previous definition of the physical states is too vague: it contains too many states. The obvious definition of a physical states should be the following: *physical states* are the representatives of the cohomology classes of the BRST operator. There is of course some freedom in choosing a representative, the same way, as there exists freedom in choosing a gauge condition. In simpler words: states that are annihilated by the BRST operator, but that can not be produced by Q acting on any other states, should be considered the physical ones:

$$Q | p\hbar) = 0, \quad | p\hbar) \neq Q | i). \quad (2.2)$$

The question is then, how to find the cohomology classes of Q . The method followed in this paper is to find an operator, that can fix the BRST-freedom. This is done by the

so called co-BRST operator. Its definition was given in [8]. The whole BRST-co-BRST structure resembles closely the Hodge structure known in geometry. Using that language we can say, that we are looking for the BRST-harmonic states. The explicit form of the co-BRST operator can be found as in [9]. The idea is based on the fact that the state space is not entirely positive definite: half of the states are of negative norm. We can then find those metric operators that "turn" these states too into positive definite norm states. The metric operators ($\hat{\gamma}$) found this way are used to produce the co-BRST operator:

$${}^*Q = \gamma Q \gamma . \quad (2.3)$$

The previous definition of the physical states is identical to the following: $|s\rangle$ is a physical state if it satisfies the following equations:

$$Q |s\rangle = {}^*Q |s\rangle = 0 . \quad (2.4)$$

What is the advantage of this quantization scheme as compared to other methods? A first thing is its elegance: the common solutions of two equations contain all the physics of the model. A more important feature of this method is, that it automatically leads to well defined inner-product states. The darker side of the coin is that it is not always entirely clear what should be the exact form of the gauge fixing conditions that enter in *Q . The method gives some clue, though, through an $SL(2, R)$ algebra existing between the terms of *Q . This algebra exists on the operator level in the case of Abelian models, but only in an effective way for non-Abelian models, when using the simplest possible forms of the gauge fixing conditions.

The BRST-co-BRST method can also give some information on the time evolution of the system (see e.g. [12]). However, there are many details yet to be understood.

Co-BRST, as well as BRST, is also well defined on the classical level, as a function on the phase space. They can even give some information about what kind of gauge fixing conditions we might use: we have to demand that the formalism is such that both Q and *Q are conserved in time. To show how this works, we shall treat two simple classical examples in the Sections 4 and 5.

3 BRST-co-BRST Quantization of a Reparametrization Invariant Model

Let us consider a regular Lagrangian $L(q, \dot{q})$, where q represents all the configuration space variables. To make this theory reparametrization invariant we change the variables $t \rightarrow x^0$ and we make the new variable x^0 depend on a parameter t . The new action has the form shown in (1.2). The only constraint is

$$\varphi_2 = p_0 + H = 0 , \quad (3.1)$$

where H is the Hamiltonian of the original non-reparametrization invariant theory. The total Hamiltonian is

$$H_t = \lambda \varphi_2 = 0 . \quad (3.2)$$

The standard form of the phase-space Lagrangian can be written as:

$$L = p\dot{q} + p_0\dot{q}^0 - \lambda\varphi . \quad (3.3)$$

Now we have a primary constraint, the momentum conjugated to λ being constrained to vanish:

$$\varphi_1 \equiv \pi_\lambda = 0 . \quad (3.4)$$

The secondary constraint is just the one given by (3.1). To describe physics, we need gauge fixing conditions for both constraints, since they are of the first class. The natural form of these conditions is:

$$\psi_1 = \lambda = 0 , \quad \psi_2 = x^0 - f(t; q, p, p_0) = 0 , \quad (3.5)$$

where the function f can depend on all variables except x^0 . As proven in Appendix A of [2], it is always allowed to make such a choice. As we shall see in the two specific examples (the non-relativistic and the relativistic particle), the gauge-fixing function f can depend in those cases on the phase space variables only through the reparametrization constraint. That is why we let f be even in the general case: $f = f(t; \varphi_2)$.

The BRST-co-BRST quantization of this model can be done along the line followed in [9]. The idea is to find first some simple BRST-invariant states, which can be written as a direct product of a ghost and a matter state. These states are not going to be well defined

inner product states: the inner product between a such of a state with itself is typically ∞ . We can then apply a gauge regulator that leads us to inner product states, which is also co-BRST invariant.

In this Section we use a coordinate representation:

$$\Psi(q, x^0, \lambda, \eta, \bar{\eta}) = \langle q, x^0, \lambda | \langle \eta \bar{\eta} | \Psi \rangle. \quad (3.6)$$

The operators corresponding to the variables in the classical theory are defined in the standard way: the coordinates are multiplicative, while the momenta are derivative operators, e.g.:

$$p_0 = i \frac{\partial}{\partial x^0}, \quad \mathcal{P} = i \frac{\partial}{\partial \eta} \quad \text{etc.} \quad (3.7)$$

The BRST operator is

$$Q = -i\pi_\lambda \bar{\mathcal{P}} + (p_0 + H)\eta. \quad (3.8)$$

We look for BRST-invariant states of the form

$$|\varphi_i\rangle = |M_i\rangle |Gh_i\rangle, \quad (3.9)$$

where $|M_i\rangle$ is a purely matter state, while $|Gh_i\rangle$ is a purely ghost state. Two simple solutions of this form can be found, $|\varphi_i\rangle$, $i = 1, 2$, which satisfy the following equations:

$$\pi_\lambda |\varphi_1\rangle = \eta |\varphi_1\rangle = 0 \quad \Rightarrow \quad \pi_\lambda |M_1\rangle = \eta |Gh_1\rangle = 0, \quad (3.10)$$

$$(p_0 + H) |\varphi_2\rangle = \bar{\mathcal{P}} |\varphi_2\rangle = 0 \quad \Rightarrow \quad (p_0 + H) |M_2\rangle = \bar{\mathcal{P}} |Gh_2\rangle = 0, \quad (3.11)$$

We know from [13], that one can always impose some conditions on the BRST-invariant states without changing their physical content. One of the possible restrictions is the demand that the states should also be anti-BRST invariant. The anti-BRST operator in this model is

$$\bar{Q} = i\pi_\lambda \mathcal{P} + (p + H_0)\bar{\eta}. \quad (3.12)$$

Anti-BRST invariance means the following conditions on our states:

$$\bar{\eta} |\varphi_1\rangle = 0 \quad \Rightarrow \quad \bar{\eta} |Gh_1\rangle = 0, \quad (3.13)$$

$$\mathcal{P} |\varphi_2\rangle = 0 \quad \Rightarrow \quad \mathcal{P} |Gh_2\rangle = 0. \quad (3.14)$$

There is one more freedom left in both states, which can be fixed by a gauge fixing condition:

$$(x^0 - f(t)) |\varphi_1\rangle = (x^0 - f(t)) |M_1\rangle = 0, \quad (3.15)$$

$$\lambda |\varphi_2\rangle = \lambda |M_2\rangle = 0. \quad (3.16)$$

We note that these two BRST-invariant states are not co-BRST invariant, thus, they are not the physical states of the theory. One more problem with them is that they are not well defined inner product states either. As it can be controlled easily, the inner product of a state $|\varphi_i\rangle$ with itself is not a finite number:

$$\langle \varphi_i | \varphi_i \rangle = \langle M_i | M_i \rangle \langle Gh_i | Gh_i \rangle = \infty, \quad i = 1, 2. \quad (3.17)$$

The inner product between a ghost state and itself vanishes because of its Grassmann parity:

$$\langle \eta | \eta' \rangle = \delta(\eta - \eta') = i(\eta - \eta'). \quad (3.18)$$

Following the recipe given in [9], one can define the gauge regulators, that take care of both problems related to the $|\varphi_i\rangle$ states. Let us define the two singlet states in the following way:

$$|s_i\rangle = e^{\alpha K_i} |\varphi_i\rangle, \quad i = 1, 2, \quad (3.19)$$

where

$$K_i = [Q, \rho_i], \quad i = 1, 2, \quad (3.20)$$

and the operators ρ_i are the following

$$\rho_1 = i\lambda \mathcal{P}, \quad \rho_2 = (x^0 - f(t))\bar{\eta}. \quad (3.21)$$

This means that

$$K_1 = -\lambda(p_0 + H) - i\bar{\mathcal{P}}\mathcal{P}, \quad (3.22)$$

$$K_2 = \pi_\lambda(x^0 - f(t)) - i\eta\bar{\eta}. \quad (3.23)$$

One can prove that the two K_i operators and their commutator satisfy an $SL(2, \mathbf{R})$ algebra:

$$[K_1, K_2] = K_3, \quad [K_1, K_3] = -2K_1, \quad [K_2, K_3] = 2K_2. \quad (3.24)$$

The linear combinations of these generators $X_1 = \frac{1}{2}(K_1 + K_2)$, $X_2 = \frac{i}{2}(K_1 - K_2)$, $X_3 = \frac{i}{2}K_3$ obey the usual commutation rules for $SL(2, \mathbf{R})$.

As one defined the $|\varphi_i\rangle$ states by the operators annihilating them, one can do the same thing here, and define the singlet states by their annihilating operators. Denote the operators annihilating $|\varphi_i\rangle$ by B_{ia} , $a = 1, 4$. Then the operators that annihilate $|s_i\rangle$ are:

$$D_{ia} = e^{\alpha_i K_i} B_{ia} e^{-\alpha_i K_i}, \quad (3.25)$$

where α_i , $i = 1, 2$, are real, finite, non-zero constants. The operators D_{ia} can be found using

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \dots \quad (3.26)$$

This way we find

$$(\pi_\lambda - i\alpha_1(p_0 + H)) |s_1\rangle = (x^0 - f(t) + i\alpha_1 \lambda) |s_1\rangle = (\eta + \alpha_1 \bar{\mathcal{P}}) |s_1\rangle = (\bar{\eta} - \alpha_1 \mathcal{P}) |s_1\rangle = 0, \quad (3.27)$$

$$(p_0 + H + i\alpha_2 \pi_\lambda) |s_2\rangle = (\lambda - i\alpha_2(x^0 - f(t))) |s_2\rangle = (\bar{\mathcal{P}} + \alpha_2 \eta) |s_2\rangle = (\mathcal{P} - \alpha_2 \bar{\eta}) |s_2\rangle = 0. \quad (3.28)$$

The same set of states $|s_i\rangle$ are obtained both ways, as can be seen by setting $\alpha_1 \alpha_2 = 1$. This means that there is no reason to talk about two different sets of singlet states. There exists one set of singlets, whose members can be obtained from both initial BRST-invariant states $|\varphi_i\rangle$. For this reason we will drop the indices i from the states and the corresponding constants α .

There exists a relatively simple form of the BRST and the co-BRST operators based on the operators annihilating the singlets (3.27) or on some linear combinations of them that lead to a Fock-space representation. The advantage of this latter representation is that it makes easier to find the metric operator leading to the explicit form of the co-BRST operator. Let us first introduce a new notation for the operators in (3.27):

$$\phi = \alpha(p_0 + H) + i\pi_\lambda, \quad \xi = \frac{1}{2}\lambda - \frac{i}{2\alpha}(x^0 - f(t)), \quad (3.29)$$

$$\rho = \eta + \alpha \bar{\mathcal{P}}, \quad \kappa = \frac{i}{2\alpha} \bar{\eta} - \frac{i}{2} \mathcal{P}. \quad (3.30)$$

The normalization is chosen here to give the commutation relations:

$$[\phi, \xi^\dagger] = 1, \quad [\kappa, \rho^\dagger] = 1, \quad (3.31)$$

the other ones being zero. Using these operators, the BRST has the following form:

$$Q = \frac{1}{\alpha} [\phi \rho^\dagger + \phi^\dagger \rho]. \quad (3.32)$$

One can now go over to the Fock representation by defining the following operators:

$$a = \xi + \frac{1}{2}\phi, \quad b = \xi - \frac{1}{2}\phi, \quad A = \rho + \frac{1}{2}\kappa, \quad B = \rho - \frac{1}{2}\kappa. \quad (3.33)$$

The non-vanishing commutators between these operators are:

$$[a, a^\dagger] = 1, \quad [b, b^\dagger] = -1, \quad [A, A^\dagger] = 1, \quad [B, B^\dagger] = -1. \quad (3.34)$$

These commutation relations show, that the operators $a, b, A, B; a^\dagger, b^\dagger, A^\dagger, B^\dagger$ can be regarded as annihilation resp. creation operators. Half of them (b, B) correspond to states with negative definite norms. (We have to note here, that the definition of the operators a, b, A, B can be made more general by using complex numbers instead of $\frac{1}{2}$ in (3.33). For a discussion about the general case see [9].) Now we can define the metric operators as the ones, that change all states created by $a^\dagger, b^\dagger, A^\dagger, B^\dagger$ into positive normed states:

$$\hat{\gamma} = e^{i\pi b^\dagger b} e^{-i\pi B^\dagger B}. \quad (3.35)$$

Through the relation given in [8] ${}^*Q = \hat{\gamma} Q \hat{\gamma}$ we find:

$$Q = \frac{1}{2\alpha} [(a - b)(A^\dagger + B^\dagger) + (a^\dagger - b^\dagger)(A + B)], \quad (3.36)$$

$${}^*Q = \frac{1}{2\alpha} [(a + b)(A^\dagger - B^\dagger) + (a^\dagger + b^\dagger)(A - B)].$$

Returning to the previous notation the co-BRST operator has the form:

$${}^*Q = \frac{1}{\alpha} [\xi \kappa^\dagger + \xi^\dagger \kappa] = -\frac{1}{4} \left[i\lambda \mathcal{P} + \frac{1}{\alpha^2} \bar{\eta} (x^0 - f(t)) \right]. \quad (3.37)$$

This form is equivalent to the one used in the previous sections. Now we can write the condition for the physical states in the simplest possible form. Physical states are those that are annihilated by both the BRST and the co-BRST operators (2.4), or equivalently

$$\phi |s\rangle = \xi |s\rangle = \rho |s\rangle = \kappa |s\rangle = 0. \quad (3.38)$$

It is clear from here, that in this case the ghost part and the matter part of the states can be separated: $|s\rangle = |s_M\rangle |s_{Gh}\rangle$. To obtain the matter part one has to allow the

eigenvalues of one of the operators become imaginary. It was shown by Pauli [14] that hermitian operators in indefinite metric spaces must have imaginary eigenvalues. Now the solution of the matter part of (4.15) can be written as:

$$\Psi_M(\lambda, ix^0) \equiv \langle \lambda | \langle ix^0 | s_M \rangle = \delta(\lambda - \frac{1}{\alpha}(x^0 - f(t))), \quad (3.39)$$

where the vectors $|ix^0\rangle$ have the following properties:

$$x^0 |ix^0\rangle = ix^0 |ix^0\rangle, \quad \int dx^0 |ix^0\rangle \langle x^0| = 1, \quad |ix^0\rangle^\dagger = \langle -ix^0|. \quad (3.40)$$

The norm of the Ψ_M state is $\frac{2}{\alpha}$. The case of the ghost part is slightly more complicated. Since α is real, the eigenstates of ρ and κ corresponding to zero eigenvalue are of the form $e^{\frac{i}{\alpha}\bar{\eta}\eta}$ which are of an imaginary norm. The only way to change this without altering the commutators between the ghost operators and their momenta is to use a ghost-eigenstate with an imaginary eigenvalue. That means that we change one of the ghost equations in (3.7) into:

$$\hat{\eta} |i\eta\rangle = i\eta |i\eta\rangle, \quad \hat{P} |i\eta\rangle = \frac{\partial}{\partial \eta} |i\eta\rangle. \quad (3.41)$$

The equation for the antighost remains the same. The ghost states then have the norm: $\frac{\alpha}{2}$, meaning that the singlets have a unit norm.

From the point of view of the physical states, the singlets defined before correspond to the vacuum. The other physical states have to be defined by acting with physical creation operators on $|s\rangle$. These operators have to commute both with the BRST and the co-BRST operator and are typically of the form $p \pm iq$.

4 The 1-Dimensional Particle in a Potential

As a first example let us consider the Lagrangian of a particle in 1+1 dimensions and make it reparametrization invariant:

$$\int dt L(x, \dot{x}) \rightarrow \int dt L(x, \frac{\dot{x}}{\dot{x}^0}) \dot{x}^0. \quad (4.1)$$

We follow now the procedure in the previous Section with the explicit form of the constraint $\varphi_2 = p_0 + H = 0$, $H = \frac{p^2}{2m} + V(x)$. The natural form of the gauge conditions is:

$$\psi_1 = \lambda = 0, \quad \psi_2 = x^0 - f(t; x, p, p_0) = 0, \quad (4.2)$$

where the function f can depend on all variables except x^0 . The non-vanishing part of the constraint algebra is

$$\{\varphi_1, \psi_1\} = -1, \quad \{\varphi_2, \psi_2\} = -1 + \{f(t), H\}. \quad (4.3)$$

The BRST charge and the co-BRST charge are of the following form:

$$Q = \varphi_2 \eta - i\varphi_1 \bar{P}, \quad {}^*Q = i\psi_1 P + \psi_2 \bar{\eta}. \quad (4.4)$$

The factors i appear to ensure that both Q and *Q are real in the classical and Hermitian in the quantum case. The Poisson bracket between them:

$$\{Q, {}^*Q\} = i\lambda(p_0 + H) - i\pi_\lambda [x^0 - f(t; x, p, p_0)] + \eta \bar{\eta} [-1 + \{f(t), H\}] - \bar{P} P \quad (4.5)$$

The time evolution of the variables can be computed following the Gitman-Tyutin scheme [2], which is, as mentioned before, a generalization of the Dirac-bracket for the case of time dependent constraints. The total time derivative of a variable q is given by:

$$\frac{d}{dt} q = \{q, H_t\}_D - \{q, \varphi_i\} [\{\varphi, \varphi\}^{-1}]^j \frac{\partial \varphi_j}{\partial t} \quad (4.6)$$

where we denoted all constraints (the gauge fixing conditions included) by φ_i , $i = 1, 4$. The gauge fixing conditions are denoted here as $\psi_1 \equiv \varphi_3$, $\psi_2 \equiv \varphi_4$. The time evolution of our basic variables is then:

$$\frac{d}{dt} x = \frac{p}{m} \frac{\partial f}{1 - \{f, H\}}, \quad (4.7)$$

$$\frac{d}{dt} x^0 = \frac{\partial f}{1 - \{f, H\}}, \quad (4.8)$$

$$\frac{d}{dt} p = -\partial_x V(x) \frac{\partial f}{1 - \{f, H\}}, \quad (4.9)$$

$$\frac{d}{dt} p_0 = 0. \quad (4.10)$$

We see that both Q and *Q are conserved in t . The physical variables are the ones in the BRST-cohomology, thus those that have vanishing Poisson brackets both with the BRST and the co-BRST charge. We see that neither x nor p satisfy these conditions.

$$\{x, Q\} = \frac{p}{m} \eta, \quad \{x, {}^*Q\} = -\{x, f\} \bar{\eta}, \quad (4.11)$$

$$\{x^0, Q\} = \eta, \quad \{x^0, {}^*Q\} = -\{x^0, f\} \bar{\eta}, \quad (4.12)$$

$$\{p, Q\} = -\partial_x V(x) \eta, \quad \{p, {}^*Q\} = -\{p, f\} \bar{\eta}, \quad (4.13)$$

$$\{p_0, Q\} = 0, \quad \{p_0, {}^*Q\} = -\bar{\eta}. \quad (4.14)$$

We can however find some combinations of the variables we have, to obtain the physical ones. We would like to obtain physical variables that are canonically conjugated to each other and that do not depend on the ghost variables. In the light of the previous equations we see that the BRST-invariance of a variable $y(x, x^0, p, p_0)$ can be written as:

$$\partial_0 y = \hat{O} y, \quad (4.15)$$

where

$$\partial_0 \equiv \frac{\partial}{\partial x^0} \quad \text{and} \quad \hat{O} \equiv \frac{dV}{dx} \frac{\partial}{\partial p} - \frac{p}{m} \frac{\partial}{\partial x}. \quad (4.16)$$

The effect of operator \hat{O} is identical with taking the Poisson bracket with the original Hamiltonian:

$$\hat{O} \equiv \left\{ \frac{p^2}{2m} + V, \right\}. \quad (4.17)$$

We have to note here that this treatment is entirely classical and the operator \hat{O} has nothing to do with the quantum theory. The solution for equation (4.15) can be written formally as:

$$y = e^{x^0 \hat{O}} x. \quad (4.18)$$

The explicit form of this solution can be very complicated and it depends on the form of the potential V . The other BRST-invariant variable can be "built" on p and it is:

$$\pi = e^{x^0 \hat{O}} p. \quad (4.19)$$

These two variables y and π are canonically conjugated to each other, as proved in the Appendix. One can also prove that there are no more independent BRST-invariant variables in the theory.

Two simple examples might give us a better feeling about these variables. If the potential is a constant, the two physical variables are:

$$y = x - \frac{p}{m} x^0, \quad \pi = p, \quad (4.20)$$

while for a potential of the form $V = -Ax$:

$$y = x - \frac{p}{m} x^0 + \frac{A(x^0)^2}{2}, \quad \pi = p - Ax^0 \quad (4.21)$$

It is particularly simple to understand the first example. The physical variable there, is y which can be understood as x which evolves in x^0 as:

$$\frac{dx}{dx^0} = \frac{p}{m} \quad (4.22)$$

and the momentum $\pi = p$ conjugated to it. The momentum is conserved in x^0 . This corresponds entirely to what we expect to be physical. The second example gives us the variables of a particle accelerated by a constant force.

For these variables y and π to be physical they have to be co-BRST invariant as well. This demand gives a condition on the possible gauge fixing conditions f .

$$\{y, {}^*Q\} = 0 \Rightarrow -\partial_x y \partial_p f + \partial_p y \partial_x f + \partial_0 y \partial_{p_0} f = 0 \quad (4.23)$$

By using (4.15) we obtain:

$$\partial_x y [-\partial_p f + \frac{p}{m} \partial_{p_0} f] + \partial_p y [\partial_x f - V' \partial_{p_0} f] = 0 \quad (4.24)$$

The solution of this equation can be found if we decompose it as follows:

$$-\partial_p f + \frac{p}{m} \partial_{p_0} f = \alpha(x, p, p_0) \partial_p y \quad (4.25)$$

$$\partial_x f - V' \partial_{p_0} f = -\alpha(x, p, p_0) \partial_x y \quad (4.26)$$

where $\alpha(x, p, p_0)$ can be any function of these variables. We know that for all the possible potentials $\partial_p y = -\frac{x^0}{m} + \dots$. From the structure of the gauge fixing condition it follows that f does not depend on x^0 . This means that equation (4.25) can only be satisfied if α vanishes. This in turn means that the only x, p, p_0 dependence of the gauge fixing function f is through the constraint:

$$f(t; x, p, p_0) = f(t; \phi_2). \quad (4.27)$$

This result implies that there can be no further physical information hidden in the gauge fixing conditions related to the variables of the theory and in most cases we can set the gauge fixing condition to:

$$\psi_2 = x^0 - f(t). \quad (4.28)$$

The implication of (4.27) is that the constraint algebra simplifies and (4.3) becomes:

$$\{\psi_2, \psi_2\} = -1. \quad (4.29)$$

simplifying the t-evolution of all the variables (4.7)-(4.10). The time-evolution (along t) of the physical variables is

$$\dot{y} = \dot{\pi} = 0. \quad (4.30)$$

which is not surprising since they by definition commute with all the constraints and gauge fixing conditions.

Under these conditions one can find an effective Hamiltonian that generates the same t-evolution for all the variables that we have here.

$$\frac{dx}{dt} = \frac{p}{m} \partial_t f = \{x, H_{eff}\} = \partial_p H_{eff} \quad (4.31)$$

$$\frac{dp}{dt} = -\partial_x V \partial_t f = \{p, H_{eff}\} = -\partial_x H_{eff} \quad (4.32)$$

$$\frac{dx^0}{dt} = \partial_t f \partial_{p_0} H_{eff} \quad (4.33)$$

It is important to note again, that unlike in other cases of systems with time dependent constraints, in the case of reparametrization invariant models, the effective Hamiltonian is not the physical Hamiltonian. The effective Hamiltonian here is defined on the whole phase space, while the physical one will be only defined on the physical phase space.

The conditions for the existence of H_{eff} are fulfilled:

$$\partial_p \partial_x H_{eff} = \partial_x \partial_p H_{eff} \Leftrightarrow \partial_p (\partial_x V \partial_t f) = \partial_x \left(\frac{p}{m} \partial_t f \right) = \frac{p}{m} \partial_t \partial_{\varphi_2} f, \quad (4.34)$$

$$\partial_p \partial_{p_0} H_{eff} = \partial_{p_0} \partial_p H_{eff} \Leftrightarrow \partial_{p_0} \left(\frac{p}{m} \partial_t f \right) = \partial_p \partial_t f = \frac{p}{m} \partial_t \partial_{\varphi_2} f, \quad (4.35)$$

$$\partial_x \partial_{p_0} H_{eff} = \partial_{p_0} \partial_x H_{eff} \Leftrightarrow \partial_x \partial_t f = \partial_{p_0} (\partial_x V \partial_t f) = \partial_x V \partial_t \partial_{\varphi_2} f. \quad (4.36)$$

In other words, the conditions for the existence of the effective Hamiltonian are the same as the conditions for the existence of a consistent co-BRST charge. From the equations (4.31)-(4.33) one can find the Hamiltonian by integration. In the particular case when f depends only on t (4.28) the effective Hamiltonian is:

$$H_{eff} = \left(p_0 + \frac{p^2}{2m} + V \right) \partial_t f(t). \quad (4.37)$$

The question is now whether there exists any preferred choice for the time variable of the non-reparametrization invariant theory. Is there anything that would point out the original x^0 as an observable time? In some sense the answer is yes. One can interpret x^0

as the physical time of the system. In that case the phase space is reduced to x and p , since x^0 becomes a parameter and p_0 loses any physical meaning. The physical variables y and π in the extended phase space can be interpreted as the ones in the reduced phase space, which evolve along x^0 as

$$\frac{dx}{dx^0} = \frac{p}{m} \quad \frac{dp}{dx^0} = -\partial_x V(x). \quad (4.38)$$

This time evolution is independent (as expected) of the gauge fixing, as long as the gauge fixing condition respects the restrictions above. The physical Hamiltonian is now the original one. There is a problem however, with this interpretation: it can be valid only for a restricted class of theories, where there exists the externally defined time, thus for theories, where reparametrization invariance is not a generic property.

This is why one can make a quite different interpretation, namely, to consider t as the physical time of the system. In that case H_{eff} is elevated to be the physical Hamiltonian. The choice of the gauge fixing $f(t)$ has a physical meaning. The choice $f(t) = t$ leads us back to the inertial frame description and to equations of motion similar to (4.38). Another interesting choice is

$$x^0 = t + \frac{ma}{2p} t^2. \quad (4.39)$$

This choice in case of vanishing potential energy leads us to an accelerating frame, in consistency with the results of [4].

5 The Mass-less Relativistic Particle

The most convenient action to start from in the description of a relativistic particle is the one involving the "ein-bein":

$$S = \int dt \frac{1}{2e(t)} \eta_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}, \quad (5.1)$$

where the metric is $\eta = \text{diag}(-1, 1, 1, 1)$. The momenta are conjugated to the x^μ variables are:

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{x_\mu}{e}. \quad (5.2)$$

The momentum conjugated to the e variable gives us the primary constraint:

$$\varphi_1 = p_e = 0 \quad (5.3)$$

The Hamiltonian becomes:

$$H = \frac{\epsilon}{2} p_\mu p^\mu + p_c \lambda \quad (5.4)$$

and the consistency condition given by the time evolution of the first constraint gives the expected:

$$\varphi_2 = \frac{1}{2} p_\mu p^\mu \quad (5.5)$$

Thus the BRST-charge is

$$Q = \varphi_1 \bar{P} + \varphi_2 \eta. \quad (5.6)$$

The gauge fixing conditions can be written as

$$\varphi_3 = \epsilon - f(t; x^i, p_\mu) = 0 \quad (5.7)$$

$$\varphi_4 = x^0 - g(t; x^i, p_\mu) = 0 \quad (5.8)$$

The t-evolution given by the Gitman-Tyutin method is consistent with these gauge fixing conditions, and there are no restrictions upon them coming from t-evolution. The physical variables are those that have vanishing Poisson brackets with both the BRST- and the co-BRST-charge. Using the same method as in the previous Section one finds that the physical variables can be written as:

$$y^j = e^{\frac{x^0 p_i \theta^i}{p_0}} x^j = x^j + \frac{x^0}{p_0} p^j \quad (5.9)$$

and their conjugated momenta

$$\pi_j = e^{\frac{x^0 p_i \theta^i}{p_0}} p_j = p_j. \quad (5.10)$$

The demand, that the Poisson bracket between these variables and the co-BRST should vanish, imposes some restrictions on the possible form of the gauge fixing conditions.

Because of the simple form of *Q we can treat the two gauge-fixing functions separately.

$$\{\pi_i, ^*Q\} = 0 \Rightarrow \partial_i f(t; x^j, p_j) = \partial_i g(t; x^j, p_j) = 0 \quad (5.11)$$

thus none of the gauge fixing conditions depend on x^μ . This in turn leads to:

$$\{y^i, f(t; p_\mu)\} = \partial_{p_i} f + \frac{p^i}{p_0} \partial_{p_0} f = 0 \quad (5.12)$$

which means that f can only depend on the momenta through the second constraint:

$$f(t; p_\mu) = f(t; \varphi_2). \quad (5.13)$$

The situation with the second gauge-fixing condition is somewhat more complicated, since there exist no possible functions $g(t; p_\mu)$ that would give φ_4 a vanishing Poisson bracket with the expectedly physical variable y^i . The only possibility is to find a gauge-fixing condition whose Poisson bracket with y^i is proportional to the gauge-fixing condition itself.

$$\{x^j + \frac{x^0}{p_0} p^j, x^0 - g(t; p_\mu)\} = \alpha(p_\mu)(x^0 - g(t; p_\mu)) \quad (5.14)$$

where $\alpha(p_\mu)$ is a function. There exists only one class of solutions to (5.14): $\varphi_4 = x^0 + g_0 p^0 t$. A special case of this is the standard proper time gauge with $g_0 = \pm \frac{1}{m}$:

$$\varphi_4 = x^0 \pm p^0 t \quad (5.15)$$

We note here that this classical formulation is not very sensitive to the introduction of a mass term into the Lagrangian:

$$\Delta L = -\frac{\epsilon}{2} m^2. \quad (5.16)$$

The secondary constraint (5.5) changes to become:

$$\varphi_2' = \frac{1}{2} (p_\mu p^\mu + m^2) \quad (5.17)$$

but since the mass is a scalar, the Poisson bracket algebra does not change. Neither do the physical variables. *

The t -time evolution of the physical variables vanishes and this means that the physical variables in the extended space can be understood as those variables (x^i, p_i) in the physical subspace that evolve along x^0 as:

$$\frac{dx^i}{dx^0} = \frac{p^i}{\sqrt{p^i p_i + m^2}}, \quad \frac{dp_i}{dx^0} = 0. \quad (5.18)$$

The Hamiltonian that would generate this evolution is:

$$H_{ph} = \sqrt{p^i p_i + m^2} \quad (5.19)$$

We have come now to the important question: does the BRST-co-BRST formalism distinguish a specific time variable? Does it give information about which is the physical

time variable? The answer is again: yes, there can exist an interpretation, where it does. First of all, the physical variables as chosen in equation (5.9) could not have been chosen in any other way. It is only the p_0 which is certain not to vanish. In the particle's own reference frame all the other momenta are null, but there exist no reference frame, where p_0 would vanish. The same thing can also be said about the light-cone coordinates: one can always find a frame, where $p_0 \pm p_1$ vanish. This means that equation (5.9) is the unique choice of the physical configuration space variable. From here also follows that in this interpretation x^0 plays the role of the physical time.

For the reasons specified in the previous Section, the interpretation with more generality is the one, which considers t as the physical time of the system. The effective Hamiltonian is:

$$H_{eff} = g \frac{p_\mu p^\mu}{2} + p_e \partial_t f. \quad (5.20)$$

The freedom obtained this way (we can freely choose the value of g) lets us include both the particles and the antiparticles in the same formalism.

The BRST-co-BRST quantization of the relativistic particle model can be done as in the general case of Section 3. The only thing one has to explicitly note is the form of the physical creation and annihilation operators. These are of the standard form $p_i \pm x_i$, and these operators commute with both the BRST and the co-BRST operators.

6 Discussion

Since there are many important reparametrization invariant theories, it was interesting to see what can we learn about them by the BRST-co-BRST method. On the treated examples we could notice that the BRST-co-BRST mechanism imposed some restrictions on the possible gauge choices, though it did not give a definite answer on what gauge fixing conditions should be used, that is: it did not show what the time variable should be. In the two examples we treated, the conditions for the existence of a consistent and conserved co-BRST charge, showed that the gauge-fixing of the original time variable (x^0) to the time-parameter (t) could only be done through functions $f(t, \varphi_i)$, that is: functions depending on the phase space variables only through the reparametrization constraint.

Time evolution in reparametrization invariant systems can be found in two ways. One

way is to follow the Gitman-Tyutin procedure, and define time evolution of the physical variables directly through the time-dependence of the gauge-fixing conditions. The expressions one gets, are relatively simple, because the total Hamiltonian vanishes, thus all time-evolution is defined by the evolution of the constraint surface. One can also define an effective Hamiltonian, which generates the same evolution by Poisson brackets (or commutators). This way one "translates" the time evolution of the system to a more familiar language. The conditions for the existence of the effective Hamiltonian were the same as the conditions for the existence of the co-BRST charge. This is one aspect of the relations between the BRST-co-BRST scheme and time evolution.

By fixing reparametrization invariance, one fixes to some extent the reference frame. For one-dimensional problems one can choose inertial or accelerating frames by choosing a gauge fixing for reparametrization.

The BRST-co-BRST quantization of reparametrization invariant systems leads to well defined inner product states. This, in concordance with earlier result also demanded that one introduces some imaginary eigenstates to half of the operators. We started with two classes of BRST-invariant states that were of the simple form of a matter state times a ghost state. These states were neither inner product states, nor co-BRST invariant. Then by a gauge regularization we obtained the physical states, that exhibited both these important properties. We also noticed the $SL(2, R)$ algebra between the regulators.

We only studied time-reparametrization invariant theories and even between those, such theories that could be understood as reparametrization invariant extensions of non-singular theories. It is also possible to generalize the method to theories with singular Lagrangians. This gives us hope that the BRST-co-BRST method might give us some insight even in the case of theories with several reparametrization invariances, e.g. string theory or gravity.

Acknowledgement This work was supported by the FAPESP-grant No 96/03108-0. I am very grateful to Prof. D.M. Gitman for many useful discussions.

Appendix A

The proof of y and π being canonically conjugated

The two BRST-invariant variables in the case of the non-relativistic particle are given by (4.18) and (4.19):

$$y = e^{x^0} \hat{O} x \quad (\text{A.1})$$

$$\pi = e^{x^0} \hat{O} p \quad (\text{A.2})$$

We notice that

$$p = -m \hat{O} x. \quad (\text{A.3})$$

Using this fact and the Jacobi identities we find that

$$\left\{ y, -\frac{\pi}{m} \right\} = \sum_{n=0}^{\infty} (x^0)^n W_n \quad (\text{A.4})$$

where

$$W_n = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n+1-2j}{(n+1-j)! j!} \{ \hat{O}^j x, \hat{O}^{n+1-j} x \} \quad (\text{A.5})$$

and $\lfloor \frac{n}{2} \rfloor$ is the integer part of $\frac{n}{2}$. One can use the Jacobi identities to prove that

$$\hat{O} \{ \hat{O}^j x, \hat{O}^i x \} = \{ \hat{O}^{j+1} x, \hat{O}^i x \} + \{ \hat{O}^j x, \hat{O}^{i+1} x \} \quad (\text{A.6})$$

Using this relation we obtain:

$$\hat{O} W_n = (n+1) W_{n+1}. \quad (\text{A.7})$$

We prove this only for odd n but the proof for even n is very similar.

$$W_n = \sum_{j=0}^{\frac{n-1}{2}} \frac{n+1-2j}{(n+1-j)! j!} \{ \hat{O}^j x, \hat{O}^{n+1-j} x \} \quad (\text{A.8})$$

$$\hat{O} W_n = \sum_{j=0}^{\frac{n-1}{2}} \frac{n+1-2j}{(n+1-j)! j!} \{ \hat{O}^{j+1} x, \hat{O}^{n+1-j} x \} + \sum_{j=0}^{\frac{n-1}{2}} \frac{n+1-2j}{(n+1-j)! j!} \{ \hat{O}^j x, \hat{O}^{n+2-j} x \} \quad (\text{A.9})$$

If in the first term we introduce a new index notation

$$k = j + 1. \quad (\text{A.10})$$

we have

$$\sum_{k=1}^{\frac{n+1}{2}} \frac{n+1-2(k-1)}{(n+1-(k-1))! (k-1)!} \{ \hat{O}^k x, \hat{O}^{n+2-k} x \} \quad (\text{A.11})$$

changing the notation back $k \rightarrow j$ the whole sum becomes:

$$\begin{aligned} \hat{O} W_n &= \frac{n+1}{(n+1)! 0!} \{ \hat{O}^0 x, \hat{O}^{n+2} x \} + \sum_{j=1}^{\frac{n-1}{2}} \frac{(n+1)(n-2j+2)}{(n+2-j)! j!} \{ \hat{O}^j x, \hat{O}^{n+2-j} x \} + \\ &+ \frac{n+1}{(n+2-\frac{n+1}{2})! (\frac{n+1}{2})!} \{ \hat{O}^{(\frac{n+1}{2})} x, \hat{O}^{n+2-\frac{n+1}{2}} x \} \end{aligned} \quad (\text{A.12})$$

which can be added together to obtain:

$$\hat{O} W_n = (n+1) \sum_{j=0}^{\frac{n+1}{2}} \frac{n-2j+2}{(n+2-j)! j!} \{ \hat{O}^j x, \hat{O}^{n+2-j} x \} \equiv (n+1) W_{n+1}. \quad (\text{A.13})$$

This equation is exactly what we wanted to prove. It means that it is enough to find W_0 because the other ones are of the form:

$$W_n = \frac{\hat{O}^n}{n!} W_0 \quad (\text{A.14})$$

Since $W_0 = -\frac{1}{m}$ which is a constant, all the other W_n vanish. So from (A.4) we have

$$\{ y, \pi \} = 1 \quad (\text{A.15})$$

thus they are canonically conjugated.

References

- [1] P.A.M. Dirac, *Lectures on Quantum Mechanics* (Befer Graduate School of Science, Yeshiva University, New York 1964)
- [2] D.M. Gitman, I.V. Tyutin, *Quantization of Fields with Constraints* (Springer-Verlag, 1990)
- [3] S.P. Gavrilov, D.M. Gitman, *Class. Q. Grav.* 10, (1993), 57
- [4] G. Fülöp, D.M. Gitman and I.V. Tyutin, preprint IFUSP/P-1263
- [5] J.M. Evans and Ph.A. Tuckey, *Int. J. Mod. Phys. A* 8 (1993), 4055

- [6] J.M. Evans and Ph.A. Tuckey. Proc. Conf. Geom.and Constr.Syst., Isaac Newton Inst, Cambridge, 1994
- [7] C. Becchi, A. Rouet and R. Stora. Phys. Lett. **B52** (1974) 344, I.V. Tyutin, Lebedev preprint FIAN No 39. (1975)
- [8] W. Kalau and J.W. van Holten. Nucl. Phys. B **361** (1991) 233
- [9] G. Fülöp and R. Marnelius. Nucl. Phys. B **456** (1995) 442
- [10] G. Fülöp. PhD Thesis. 1996. Göteborg, Sweden
- [11] M. Henneaux and C. Teitelboim. *Quantization of Gauge Systems*. Princeton University Press, Princeton, New Jersey. 1992
- [12] G. Fülöp. Int. J. Mod. Phys. **A11** (1996). 4785
- [13] R. Marnelius. Nucl. Phys. **B372** (1992) 218
- [14] W. Pauli, Rev. Mod. Phys. **15** (1943) 175