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**A NOTE ON CONVERGENCE UNDER DYNAMICAL
THRESHOLDS WITH DELAYS**

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A Note on Convergence under Dynamical Thresholds with Delays

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Abstract We complement the study of the asymptotic behavior of the dynamical threshold neuron model with delay, introduced in [1], by providing a description of the dynamics of the system in the remaining parameters range. We characterize regions of "harmless" delays and those in which delay-induced oscillations appear.

It was shown in [1] that after appropriate transformations the behavior of the dynamical threshold neuron model with a single delay, denoted by τ , can be described by a delay-differential equation similar to:

$$\frac{dx}{dt}(t) = -x(t) + A\sigma(x(t)) + B\sigma(x(t-\tau)). \quad (1)$$

DDE (1) is defined over the infinite dimensional phase space $S = \mathcal{C}[-\tau, 0]$ of continuous functions on the interval $[-\tau, 0]$. The function σ is smooth, odd and $\sigma(x) \rightarrow +1$ as $x \rightarrow +\infty$, its first derivative satisfies $\sigma'(0) = 1$ and its second derivative $\sigma''(x)$ is strictly negative for $x > 0$. In [1] it was considered $\sigma(x) = \tanh(x)$, and the parameters were $A > 0$, and $B < 0$ (in their notation $A = a$, $B = -ab$, where $a > 0$ and $b > 0$).

In [1], the parameter set such that the neuron exerts instantaneous self-excitation ($A > 0$), delayed self-inhibition ($B < 0$), and displays global asymptotic stability independent of the value of the delay τ was determined as the region enclosed within the triangle delimited by the three lines $A = 0$, $B = 0$ and $A - B = 1$ in the A - B parameter-plane (shaded region in Fig 1). Studying the dynamics of DDE (1) is important for understanding the behavior of larger neural networks in the presence of delays, so that in the following, we examine other parameter regions and determine those where the delay does not affect the system. Our analysis combines the theoretical results on delay-differential equations that have been established in recent years to provide a full picture of the effect of delays in the dynamics of the neural network model considered.

Asymptotic dynamics

Depending on the parameters A and B , DDE (1) has either one or three equilibria. More precisely, for $A+B \leq 1$, the origin is the only equilibrium point of (1), whereas for $A+B > 1$, this equation has three equilibrium points, denoted by $x_1 = -a$, $x_2 = 0$ and $x_3 = a$, where a is the unique strictly positive real number satisfying $a = (A+B)\sigma(a)$.

The region corresponding to $A+B < 1$, in which the origin is the unique equilibrium of DDE (1), is divided into regions Ia and Ib by the line $A-B=1$. In Region Ia, global asymptotic stability is preserved for all delays:

Delay-independent global asymptotic stability. *If $A+B < 1$ and $A-B < 1$ (i.e. $|B| < 1-A$) then the origin is globally asymptotically stable (GAS) for all $\tau \geq 0$.*

This result stems from the fact that in this range of parameters the system is *contractive* i.e. dissipation exerted by the first two terms in the right hand side of (1) dominates the perturbation (the delayed term) so that solutions, whether oscillating or not, are damped to the origin.

In Region Ib ($B < -|A-1|$), the system is *frustrated* i.e. it possesses a delayed negative feedback loop. Since the gain of this loop ($|B|$) is larger than the dissipation ($|A-1|$), the origin loses its stability through a Hopf bifurcation at

$$\tau_H = \frac{1}{\sqrt{B^2 - (1-A)^2}} \arccos\left(\frac{1-A}{B}\right)$$

as the delay is increased. So that global asymptotic stability subsists only for delays shorter than a critical value τ_c (with $0 < \tau_c \leq \tau_H$, the strict inequality $\tau_c < \tau_H$ can occur, for example, when the Hopf bifurcation is subcritical). For delays larger than τ_c undamped oscillations appear. In fact, for $\tau > \tau_H$, only solutions in the stable manifold of the origin

are convergent and most solutions display undamped oscillations. The monotonicity of σ restricts the complexity of undamped oscillations so that no asymptotically aperiodic oscillations including chaos can occur. Indeed, thanks to the Poincaré-Bendixson theorem [2] and the non-existence of homoclinic orbits [3, 4] it can be shown that undamped oscillations are asymptotically periodic. More on their organization in the phase space can be stated by remarking that along any oscillating solution the number of sign changes decreases in time [5, 6, 7].

In the same way as for the region with a unique equilibrium point, the region where system (1) has three equilibria ($A+B > 1$) can be divided into a region of "harmless" delays (Region IIa) and one in which stable undamped oscillations occur when the delay is increased beyond a critical value (Region IIb). These dynamics are described below.

Delay-independent almost convergence. *If $B > 0$ and $A+B > 1$ then for all $\tau \geq 0$*

1- x_1 and x_3 are locally asymptotically stable while x_2 is unstable,

2- the union of the basins of attraction of x_1 and x_3 is an open dense subset of S ,

3- the complement of the union of the two basins is the boundary separating them. It is a codimension-one locally Lipschitz manifold \mathcal{M} containing x_2 . This manifold divides S into two regions in the same way a line separates the plane into two half-planes.

This result is similar to those in [8, 9] where the case $A=0$ and $B > 0$ is studied. The proof relies on the fact that in Region IIa, system (1) is *cooperative* i.e. it possesses a delayed positive feedback loop ($B > 0$), so that it generates an eventually strongly monotone semi-flow [10, 11]. This result shows that most solutions tend to either one of the stable equilibria independent of the delay. Thus, in Region IIa bistability is preserved in the presence of

delays.

For short delays, DDE (1) is convergent in Region IIa as the boundary separating the two basins is exactly the stable manifold of the origin. However, as the delay is increased, the origin undergoes successive Hopf bifurcations leading to the generation of periodic orbits [12]. Thus, for large delays, there are solutions that do not converge to any equilibria. In the same way as for the oscillating solutions in Region Ib, these are asymptotically periodic. The important difference between Regions Ib and IIa lies in the fact that in the latter the periodic solutions are unstable, and, together with their stable manifold, are contained in the “narrow band” forming the boundary separating the basins of attraction of the two stable equilibria (hence the “almost convergence” denomination attributed to this region). Therefore, as far as practical applications are concerned, such oscillatory solutions are unlikely to occur and the system behaves like a bistable convergent network.

In Region IIb, DDE (1) is again frustrated with a negative feedback gain that dominates the dissipative terms so that convergence is not preserved for all delays. Indeed, the instabilities that occur by increasing the delay when $1 - A < B < 0$ (Region IIb) are similar to those in Region Ib and will not be detailed here. We only remark that again for small delays the system is convergent, with two stable equilibria, namely x_1 and x_3 , and one unstable one (x_2), but convergence is lost as the delay is increased as both stable equilibria lose their stability through a Hopf bifurcation giving rise to periodic oscillations. So that for large delays most solutions display asymptotically periodic oscillations.

In summary, we have shown that a typical solution of DDE (1) converges to the origin (resp. to either x_1 or x_3) when the system is contractive (resp. cooperative), i.e. for A and B in Region Ia (resp. in Region IIa) no matter what value the delay τ takes. These

two regions define thus the harmless delay regimes. Conversely, in Regions Ib and IIb, the system is frustrated with a strong delayed negative feedback loop so that convergence is preserved only for delays shorter than a critical value. For delays larger than this value stable undamped oscillations occur, and in fact for large delays, a typical solution of DDE (1) displays asymptotically periodic oscillations.

Transient dynamics

The delay also affects the transient behavior of solutions of (1). To clarify this point it is more appropriate to rescale the time to the delay, and rewrite (1) as:

$$\epsilon \frac{dx}{dt}(t) = -x(t) + A\sigma(x(t)) + B\sigma(x(t-1)) \quad (2)$$

where $\epsilon = 1/\tau$. Then for ϵ small, it can be shown [13, 14] that the solutions of (2) follow transiently those of the difference equation:

$$x(t) = f(B\sigma(x(t-1))) \quad (3)$$

where f is the inverse function of $g(x) = x - A\sigma(x)$. Note that for $A > 1$, the function f is two-valued in the neighborhood of the origin, leading to hysteresis in the evaluation of (3). The difference equation (3) has attracting periodic orbits in Region IIa so that some solutions of DDE (3) display long-lasting transient oscillations in this range of parameters [8, 9].

Conclusion

The A - B parameter plane of the neuron with dynamical threshold in the presence of delay was partitioned into three regions, namely,

i) Region Ia in which the system is contractive, leading to global asymptotic stability independently of the delay;

ii) Region IIa in which the system is cooperative, implying almost convergence for all delays;

iii) Regions Ib and IIb in which the system is frustrated with strong delayed negative feedback, and delay-induced oscillations occur.

Finally, we would like to point out that similar results apply to networks of arbitrary size, as contractive networks display delay-independent global asymptotic stability [15], cooperative networks are almost convergent for all delays [16, 17] and frustration is necessary for delay-induced oscillations [18].

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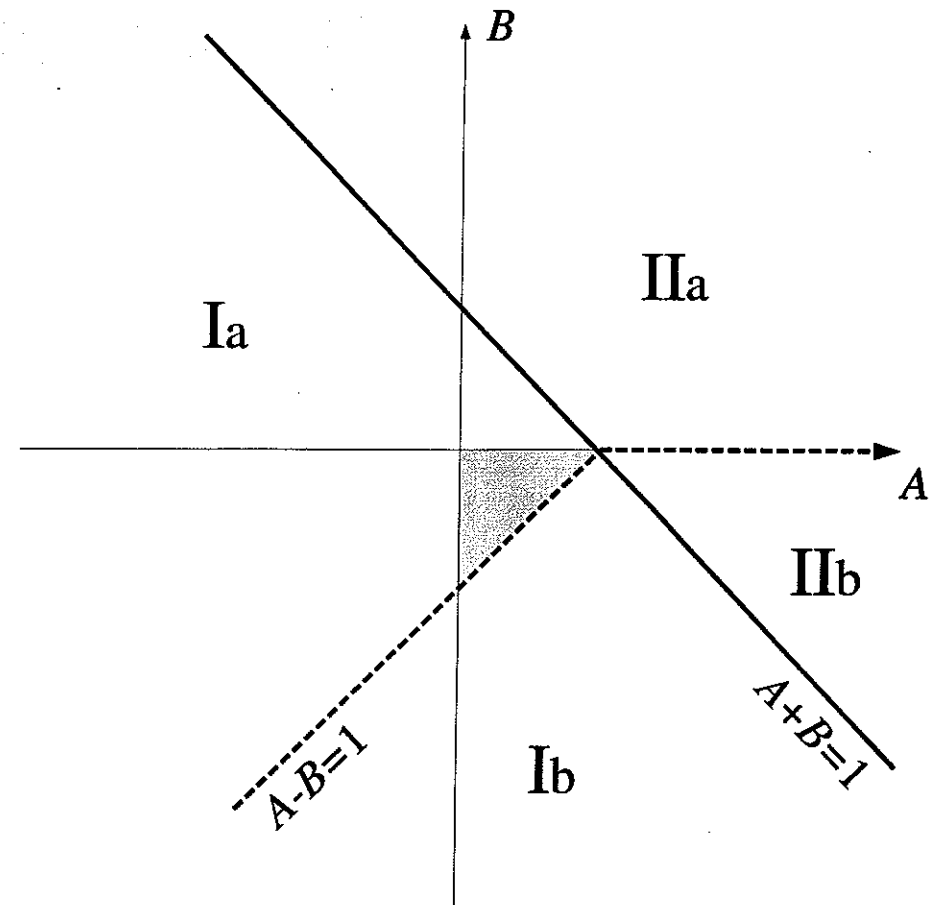


Figure 1: *Convergence and oscillations in A-B parameter plane.*

Regions Ia and Ib represent parameter values where the system has a unique equilibrium point, while Regions IIa and IIb correspond to those where the system has three equilibrium points. In Region Ia the system is globally asymptotically stable independent of the delay. The shaded triangle contained in Ia corresponds to the region of GAS obtained in [1]. In IIa it is almost convergent for all delay values. In Ib and IIb delay-induced instabilities occur leading to stable periodic oscillations.