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BRASIL

PUBLICAÇÕES

IFUSP/P-1274

SEMICLASSICAL FORM OF THE RELATIVISTIC PARTICLE PROPAGATOR

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Junho/1997

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Semiclassical Form of the Relativistic Particle Propagator

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(June 20, 1997)

Abstract

A detailed derivation of the prefactor for the relativistic particle propagator in arbitrary external electromagnetic field is presented. To this end a path-integral representation is used. The final formula is a generalization of the Van Vleck–Pauli–Morette semiclassical representation in the nonrelativistic quantum mechanics. We demonstrate the efficiency of the former in the case of an arbitrary constant electromagnetic field.

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In this paper we derive the so called prefactor in the relativistic particle propagator which is a generalization of the prefactor in the semiclassical expansion of the transition amplitude in the non-relativistic quantum mechanics. As it is known the latter can be expressed [1,2] via the so called Van Vleck determinant [3] by means of the Feynman path integral representation [4]. Applying by analogy the final formula to the relativistic case, one can get the propagator in a constant electric field [2]. We find it interesting to present a detailed derivation of the expression for the prefactor in the relativistic case, namely for a relativistic spinless particle propagator in an arbitrary external field. To this end we use also the path integral method. The expression obtained is a relativistic generalization of the Pauli-Morette [1,2] formula. We demonstrate its efficiency in the case of an arbitrary constant external electromagnetic field. The derivation presented seems to be non-trivial and contains a set of instructive technical tricks which can be useful in path-integral calculatins in relativistic quantum mechanics.

Consider in D-dimensional space-time a spinless particle of mass m and electric charge g placed into an arbitrary external electromagnetic field with potentials A_μ $(\mu = 0, \cdots, D-1)$. The particle propagator $D^c(x,y)$ is the causal Green's function for Klein – Gordon operator[1],

$$\left[\eta^{\mu\nu}\left(i\hbar\frac{\partial}{\partial x^\mu}-gA_\mu(x)\right)\left(i\hbar\frac{\partial}{\partial x^\nu}-gA_\nu(x)\right)-m^2\right]D^c(x,y)=-\delta^D(x-y), \quad (1)$$

where $\eta^{\mu\nu} = \mathrm{diag}(1,-1,\cdots,-1)$. First its path-integral representation was derived by Feynman [5]. We will use it in the convenient for our purposes form [6]

$$D^c(x_{\rm out},x_{\rm in}) = \frac{i}{2\hbar}\int_0^\infty f(x_{\rm out},x_{\rm in},e)de, \quad (2)$$

$$f(x_{\rm out},x_{\rm in},e) = \int_{x_{\rm in}}^{x_{\rm out}} DxM(e)\exp\left\{\frac{i}{\hbar}S[x,e]\right\}, \quad (3)$$

where[2]

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[1] We retain $\hbar$, which is important in our problem explicitly, whereas $c = 1$ for simplicity.

[2] Lorentz indices will be oftenly omitted, also in case of evident covariant contraction.

$$S[x,e] = -\int_0^1 \left[\frac{\dot{x}^2}{2e} + \frac{e}{2}m^2 + g\dot{x}A(x)\right] d\tau, \qquad \dot{x}^\mu = \frac{d}{d\tau}x^\mu \tag{4}$$

will be refered to as "action" for a spinless particle (however in the real reparametrization-invariant action the einbein e must be treated as a τ-dependent variable), $M(e)$ is a normalization factor,

$$M(e) = \int D\frac{p}{2\pi\hbar} \exp\left\{\frac{ie}{2\hbar}\int_0^1 p^2 d\tau\right\} \tag{5}$$

and the integration in (3) is over the trajectories $x(\tau)$ obeying the boundary conditions

$$x(0) = x_{\text{in}}, \qquad x(1) = x_{\text{out}}. \tag{6}$$

The integral over s in (2) is, in fact, the Schwinger proper-time integral and the "transformation function" f admits an interpretation as a quantum-mechanical transition amplitude in the proper-time picture. The semiclassical expansion of the amplitude f can be obtained as usual by means of the shift $x(\tau) \to x_{\text{cl}}(\tau) + \hbar^{1/2}y(\tau)$ in the path integral (3), where $x_{\text{cl}}(\tau)$ is the solution of the classical equation of motion

$$\frac{\delta S}{\delta x} = 0 \quad \Leftrightarrow \quad \ddot{x}^\mu_{\text{cl}} - geF^\mu{}_\nu(x_{\text{cl}})\dot{x}^\nu_{\text{cl}} = 0, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{7}$$

obeying the conditions (6). We are going to study here the semiclassical approximation, which is given by the Gaussian integral only

$$f(x_{\text{out}}, x_{\text{in}}, e) = I e^{\frac{i}{\hbar}S_{\text{cl}}(e)}, \qquad S_{\text{cl}}(e) = S[x_{\text{cl}}, e] \tag{8}$$

where I is the prefactor

$$I = \int_E DyM(e)\exp\left\{\frac{i}{2e}\int_0^1 d\tau \int_0^1 d\tau' y^\mu(\tau)K_{\mu\nu}(\tau,\tau')y^\nu(\tau')\right\}. \tag{9}$$

The integration in Eq. (9) is over the linear space E of trajectories obeying the boundary conditions $y(0) = y(1) = 0$, and

$$\begin{aligned} K_{\mu\nu}(\tau,\tau') &= \frac{\delta^2 S}{\delta x^\mu(\tau)\delta x^\nu(\tau')} \\ &= \left(\eta_{\mu\nu}\frac{d^2}{d\tau^2} - \frac{ge}{2}[F_{\mu\nu}(x_{\text{cl}}), \frac{d}{d\tau}]_+ - ge\dot{x}^\kappa_{\text{cl}}\partial_\mu\partial_\nu A_\kappa(x_{\text{cl}})\right)\delta(\tau-\tau'). \end{aligned} \tag{10}$$

The Gaussian path integral in (9) can be expressed in terms of a simpler one and a functional determinant,

$$I = \left(\text{Det}\hat{K}\hat{K}_0^{-1}\right)^{-1/2}\int_E DyM(e)\exp\left\{\frac{i}{2e}\int_0^1 \dot{y}^2 d\tau\right\}, \tag{11}$$

where $\hat{K}_0 = \frac{d^2}{d\tau^2}$ and $\hat{K}$ is defined by

$$(\hat{K}y)^\mu(\tau) = \int_0^1 K^\mu{}_\nu(\tau,\tau')y^\nu(\tau')d\tau', \qquad y \in E. \tag{12}$$

It is interesting to remark that the path integral in Eq. (11), can be expressed in terms of a Van Vleck determinant of the free (without the field A_μ) particle action (4)

$$\int_E DyM(e)\exp\left\{-\frac{i}{2e}\int_0^1 \dot{y}^2 d\tau\right\} = \left[\det\frac{i}{2\pi\hbar}\frac{\partial^2 S^{(0)}_{\text{cl}}(e)}{\partial x^\mu_{\text{out}}\partial x^\nu_{\text{in}}}\bigg|_{x_{\text{out}}=x_{\text{in}}=0}\right]^{1/2} = \left(\det\frac{i}{2\pi\hbar e}\eta_{\mu\nu}\right)^{1/2}, \tag{13}$$

where $S^{(0)}_{\text{cl}}(e)$ stays for the action on a solution of the classical equation of motion (7) in the absence of field, $A_\mu = 0$. In fact, the main problem is to calculate the determinant $\text{Det}\hat{K}\hat{K}_0^{-1}$ of the operators defined in the infinite-dimensional space. Moreover, we would like to express it in terms of corresponding classical solutions. Similar determinants, understood as products of ratios of the eigenvalues of the corresponding differential operators, have been studied in [7,8] and expressed in terms of finite-dimensional determinants which are related to the classical solutions. However, those results cannot be directly applied to the case under consideration due to the specific structure of the operators $\hat{K}$ and $\hat{K}_0$. That is why we are going to use another way of the transformation of the determinant under consideration to a finite-dinensional determinant.

The operator $\hat{K}$ in Eq. (11) can be substituted by an operator $\hat{K}_1 = \hat{K}_0 + \hat{R}$, the second term being an operator of multiplication by some simmetric matrix $R(\tau)$. Let us define the kernel $K_1(\tau,\tau')$ of $\hat{K}_1$ by

$$K(\tau,\tau') = V(\tau)K_1(\tau,\tau')V^{-1}(\tau') \tag{14}$$

where the matrix $V(\tau)$ satisfies the equation $\dot{V}(\tau) = \frac{ge}{2}F(x_{\text{cl}})V(\tau)$. The chronolological exponent

$$V(\tau) = T\exp\left\{\frac{ge}{2}\int_0^\tau F(x_{cl})d\tau'\right\} \tag{15}$$

is a suitable choice. An expression for $R(\tau)$ can be easily obtained, but we do not need its explicit form here. The spectra of $\hat{K}$ and $\hat{K}_1$ coincide, therefore

$$\mathrm{Det}\hat{K}\hat{K}_0^{-1} = \mathrm{Det}\hat{K}_1\hat{K}_0^{-1} = \exp\mathrm{Tr}\ln\hat{K}_1\hat{K}_0^{-1} = \exp\left\{\mathrm{Tr}\int_0^1 \hat{K}_s^{-1}\hat{R}\,ds\right\}, \tag{16}$$

where $\hat{K}_s = \hat{K}_0 + s\hat{R}$. Introducing the Green function $G_s(\tau,\tau')$,

$$\hat{K}_s G_s(\tau,\tau') = \delta(\tau-\tau'), \tag{17}$$

which obeys the boundary conditions

$$G_s(1,\tau) = G_s(0,\tau) = G_s(\tau,0) = G_s(\tau,1) = 0, \quad 0<\tau<1, \tag{18}$$

one can present the determinant (16) in the form

$$\mathrm{Det}\hat{K}\hat{K}_0^{-1} = \exp\left\{\mathrm{tr}\int_0^1 d\tau\int_0^1 ds G_s(\tau,\tau)R(\tau)\right\}. \tag{19}$$

The Green function $G_s(\tau,\tau')$ can be expressed in terms of $2D$ linearly independent solutions of the homogeneous equation

$$\hat{K}_s u(\tau,s) = 0. \tag{20}$$

It is convenient to choose these solutions $u_{i\alpha}(\tau,s)$ $(i=1,2;\ \alpha=0,\cdots,D-1)$ to satisfy the conditions

$$u^\mu_{1\alpha}(0,s)=0,\quad \dot{u}^\mu_{1\alpha}(0,s)=\delta^\mu_\alpha,\quad u^\mu_{2\alpha}(1,s)=0,\quad \dot{u}^\mu_{2\alpha}(1,s)=\delta^\mu_\alpha. \tag{21}$$

One can define an antisymmetric bilinear form on the space of the solutions by

$$w(u,v) = \eta_{\mu\nu}\left(u^\mu(\tau,s)\dot{v}^\nu(\tau,s) - \dot{u}^\mu(\tau,s)v^\nu(\tau,s)\right). \tag{22}$$

It does not depend on τ. This can be checked using Eq. (20) and the symmetry of $R(\tau)$. Setting $\tau=1$ and using Eqs. (21) one obtains

$$w(u_{i\alpha},u_{i\beta}) = 0, \quad w_{\alpha\beta}(s) \equiv w(u_{1\alpha},u_{2\beta}) = (u_{1\alpha})_\beta(1,s). \tag{23}$$

The matrix $w_{\alpha\beta}(s)$ is invertible provided zero modes (solutions of Eq. (20) obeying zero boundary conditions) are absent. Then the Green function $G_s(\tau,\tau')$ can be presented in the form

$$G^{\mu\nu}_s(\tau,\tau') = \sum_{\alpha,\beta}\left(w^{-1}\right)^{\alpha\beta}(s)\left[\Theta(\tau-\tau')u^\mu_{2\alpha}(\tau,s)u^\nu_{1\beta}(\tau',s) + \Theta(\tau'-\tau)u^\mu_{1\beta}(\tau,s)u^\nu_{2\alpha}(\tau',s)\right], \tag{24}$$

where $\Theta(\tau)$ is the step function. Such a Green function was used in [9] in attempts to calculate the higher-order tems in the semiclassical expansion of the nonrelativistic quantum-mechanical amplitude. One can check that it obeys Eqs. (17) and (18) using the following relations

$$h^{\mu\nu}_s(\tau,\tau) = 0, \quad \frac{\partial}{\partial\tau}h^{\mu\nu}_s(\tau,\tau')\bigg|_{\tau'=\tau} = -\frac{\partial}{\partial\tau'}h^{\mu\nu}_\Lambda(\tau,\tau')\bigg|_{\tau'=\tau} = \eta^{\mu\nu}, \tag{25}$$

where

$$h^{\mu\nu}_s(\tau,\tau') = \sum_{\alpha,\beta}\left(w^{-1}\right)^{\alpha\beta}(s)\left[u^\mu_{2\alpha}(\tau,s)u^\nu_{1\beta}(\tau',s) - u^\mu_{1\beta}(\tau,s)u^\nu_{2\alpha}(\tau',s)\right]. \tag{26}$$

They, in fact, express the completeness of the set of solutions $u_{i\alpha}(\tau,s)$. Substituting expression (24) into Eq. (19) and taking into account that $G_s(\tau,\tau')$ is continuous at $\tau=\tau'$, one finds

$$\mathrm{Det}\hat{K}\hat{K}_0^{-1} = \exp\left\{\sum_{\alpha,\beta}\int_0^1 ds\left(w^{-1}\right)^{\alpha\beta}(s)\int_0^1 d\tau u^\mu_{2\alpha}(\tau,s)u^\nu_{1\beta}(\tau,s)R_{\mu\nu}(\tau)\right\}. \tag{27}$$

Starting with the obvious identity

$$u^\mu_{2\alpha}(\tau,s)\frac{\partial}{\partial s}\left(\hat{K}_s u_{1\beta}\right)_\mu(\tau,s) - \left(\hat{K}_s u_{2\alpha}\right)_\mu(\tau,s)\frac{\partial}{\partial s}u^\mu_{1\beta}(\tau,s) = 0, \tag{28}$$

using Eq. (20) and the symmetry of the matrix $R(\tau)$ one can derive

$$u^\mu_{2\alpha}(\tau,s)R_{\mu\nu}(\tau)u^\nu_{1\beta}(\tau,s) = \eta_{\mu\nu}\frac{\partial}{\partial\tau}\left[\dot{u}^\mu_{2\alpha}(\tau,s)\frac{\partial}{\partial s}u^\nu_{1\beta}(\tau,s) - u^\mu_{2\alpha}(\tau,s)\frac{\partial}{\partial s}\dot{u}^\nu_{1\beta}(\tau,s)\right]. \tag{29}$$

Substituting (29) into (27), performing the integration over τ and using (21) and (23) one obtains

$$\int_0^1 u^{\mu}_{2\alpha}(\tau,s)u^{\nu}_{1\beta}(\tau,s)R_{\mu\nu}(\tau)d\tau = \frac{\partial}{\partial s}w_{\beta\alpha}(s). \tag{30}$$

Substituting (30) into (27) and taking into account that $w_\alpha{}^\beta(0)=\delta^\beta_\alpha$, one gets the desired result, namely, the determinant in question is reduced to a finite-dimensional one:

$$\mathrm{Det}\hat{K}\hat{K}_0^{-1} = \exp\left\{\sum_{\alpha,\beta}\int_0^1 ds\left(w^{-1}\right)_\alpha{}^\beta(s)\frac{\partial}{\partial s}w_\beta{}^\alpha ds\right\} = \det w_\alpha{}^\beta(1) = \det u^\beta_{1\alpha}(1,1). \tag{31}$$

One can express $u^\beta_{1\alpha}(1)$ in terms of solutions of the equation

$$\hat{K}\tilde{u}=0. \tag{32}$$

The latter is the Jacobi equation with respect to the classical equation of motion (7). Making use of (14) we find that

$$\tilde{u}^\mu_{1\alpha}(\tau) = V^\mu{}_\nu(\tau)u^\nu_{1\alpha}(\tau,1) \tag{33}$$

solve Eq. (32) and obey the initial conditions $\tilde{u}_{1\alpha}(0)=0$, $\dot{\tilde{u}}^\mu_{1\alpha}(0)=\delta^\mu_\alpha$. Taking into account the form (15) of $V(\tau)$ and the antisymmetry of $F_{\mu\nu}$ one finds that $\det V^\mu{}_\nu(\tau) = \det(V^{-1})^\mu{}_\nu(\tau)=1$. Using this property and Eqs. (31), (23) we can replace $u^\beta_{1\alpha}(1,1)$ in (31) by means of (33) to get

$$\mathrm{Det}\hat{K}\hat{K}_0^{-1} = \tilde{u}^\beta_{1\alpha}(1). \tag{34}$$

It is easy to see that the derivatives of the general solution $x_{\rm cl}(\tau)$ of the equation of motion (7) with respect to its parameters (integrals of motion) solve the Jacobi equation (32). Choosing the parameters to be $x_{\rm in}$ and the initial value $p_{\rm in}=p(0)$ of the momentum on the trajectory

$$p_\mu(\tau) = -\frac{1}{e}\dot{x}_{\rm cl}(\tau)_\mu - gA_\mu(x_{\rm cl}(\tau)), \tag{35}$$

one finds

$$\left.\frac{\partial}{\partial p^\alpha_{\rm in}}x^\mu_{\rm cl}(\tau)\right|_{\tau=0}=0, \qquad \left.\frac{\partial}{\partial p^\alpha_{\rm in}}\dot{x}^\mu_{\rm cl}(\tau)\right|_{\tau=0} = -e\delta^\mu_\alpha.$$

Remembering the initial conditions obeyed by $\tilde{u}_{1\alpha}(\tau)$ one identifies these solutions as

$$\tilde{u}^\mu_{1\alpha}(\tau) = -\frac{1}{e}\frac{\partial}{\partial p^\alpha_{\rm in}}x^\mu_{\rm cl}(\tau). \tag{36}$$

Taking into account that $p_\mu(\tau)$ is, in fact, the value on the trajectory $x_{\rm cl}(\tau)$ of the canonical momentum calculated from the the action (4), one finds

$$(p_{\rm in})_\alpha = -\frac{\partial S_{\rm cl}}{\partial x^\alpha_{\rm in}}, \tag{37}$$

and, therefore,

$$\det\tilde{u}^\mu_{1\alpha}(1) = \det\left(-\frac{1}{e}\frac{\partial x^\mu_{\rm out}}{\partial p^\alpha_{\rm in}}\right) = \left[\det\left(-e\frac{\partial p^\alpha_{\rm in}}{\partial x^\mu_{\rm out}}\right)\right]^{-1} = \left[\det e\frac{\partial^2 S_{\rm cl}(e)}{\partial x^\mu_{\rm out}\partial(x_{\rm in})_\alpha}\right]^{-1}. \tag{38}$$

Substituting expression (38) into Eq. (34) and then into Eq. (11) and (2) one obtains

$$D^c(x_{\rm out},x_{\rm in}) = \frac{i}{2\hbar}\int_0^\infty e^{\frac{i}{\hbar}S_{\rm cl}(e)}\left(\det\frac{i}{2\pi\hbar}\frac{\partial^2 S_{\rm cl}(e)}{\partial x^\mu_{\rm out}\partial x^\nu_{\rm in}}\right)^{1/2}de. \tag{39}$$

As an example, consider the propagator in a constant uniform electromagnetic field $F=$ const. In the gauge $A_\mu = -\frac{1}{2}F_{\mu\nu}x^\nu$ the x-dependent part of the action functional (4) becomes bilinear and the semiclassical approximation for the transformation function f is exact. One finds

$$S_{\rm cl}(e) = \frac{g}{2}x_{out}Fx_{in} - \frac{g}{4}(x_{out}-x_{in})F\coth\frac{geF}{2}(x_{out}-x_{in}) - \frac{e}{2}m^2\,. \tag{40}$$

Calculating the Van Vleck matrix

$$\frac{\partial^2\overline{S}}{\partial x^\mu_{out}\partial x^\nu_{in}} = \left[F\frac{\exp\left(\frac{geF}{2}\right)}{2\sinh\frac{geF}{2}}\right]_{\mu\nu}, \tag{41}$$

and taking into account that $\det\exp\frac{geF}{2}=1$ we obtain

$$\begin{aligned} D^c(x_{\rm out},x_{\rm in}) &= \frac{i}{2\hbar}\int_0^\infty e^{\frac{i}{\hbar}\overline{S}(x_{\rm out},x_{\rm in},e)}\left[\det\left(\frac{i}{4\pi\hbar}\frac{gF}{\sinh\frac{geF}{2}}\right)_{\mu\nu}\right]^{1/2}de \\ &= -\frac{1}{2\hbar(4\pi i\hbar)^{D/2}}\int_0^\infty e^{\frac{i}{\hbar}S_{\rm cl}(e)}\left[\det\left(\frac{\sinh\frac{geF}{2}}{gF}\right)^\mu{}_\nu\right]^{-1/2}de, \end{aligned} \tag{42}$$

which, in the case $D=4$, coincides with the Schwinger formula [10].

In the conclusion we have to stress an important difference between the nonrelativistic and relativistic cases. In the nonrelativistic quantum mechanics the zero-order approximation in stationary phase method in the Feynman path integral for the transition amplitude allows one to separate the nonanalytic in $\hbar$ part (semiclassical approximation) from the analytic in $\hbar$ terms (corrections to the semiclassical approximation). In the relativistic case one can achieve such a result only for the transformation function. The remaining integral over the proper time e may, however, destroy such an exact separation. Namely, it can modify the nonanalytic in $\hbar$ part and, moreover, to create analytical in $\hbar$ terms. One may think that doing the stationary phase method already in the integral over e, such terms can be separated effectively. However, in this case the validity of the stationary phase method depends essentially on the values of x_{in} and x_{out}. That makes it impossible to get an universal (valid for all values of the arguments of the propagator) real semiclassical representation in the relativistic case. Let us demonstrate the above said on the simplest example of the free relativistic particle. Integrating in (39) by means of the stationary point method one gets

$$D^c(x_{\text{out}}, x_{\text{in}}) \approx -\frac{1}{2\hbar(2\pi i\hbar)^{D/2}} \left(\frac{1}{2\pi i\hbar} \left. \frac{\partial^2 S_{\text{cl}}(e)}{\partial e^2} \right|_{e=\bar{e}} \right)^{-1/2} \exp\left\{ \frac{i}{\hbar} S_{\text{cl}}(\bar{e}) \right\}, \tag{43}$$

where $\bar{e}$ is the stationary point, $\left.\frac{\partial S_{\text{cl}}}{\partial e}\right|_{e=\bar{e}} = 0$. For a free particle in the four-dimensional space-time one gets[3]

$$D^c(x) \approx \begin{cases} \left[\frac{m}{32 i\pi^3\hbar^5 (x^2)^{3/2}}\right]^{1/2} \exp\left\{-\frac{i}{\hbar} m\sqrt{x^2}\right\}, & x^2 > 0, \\ i\left[\frac{m}{32\pi^3\hbar^5(-x^2)^{3/2}}\right]^{1/2} \exp\left\{-\frac{1}{\hbar} m\sqrt{-x^2}\right\}, & x^2 < 0, \end{cases} \tag{44}$$

and for $x^2 = 0$ no stationary point exists. The exact propagator (see e.g. [11]) is given by

$$\begin{aligned} D^c(x) = {} & \frac{1}{4\pi\hbar^2}\delta(x^2) - \frac{m}{8\pi\hbar^3\sqrt{x^2}}\Theta(x^2)\left[J_1(\frac{m}{\hbar}\sqrt{x^2}) - iN_1(\frac{m}{\hbar}\sqrt{x^2})\right] \\ & + \frac{im}{4\pi^2\hbar^3\sqrt{-x^2}}\Theta(-x^2)K_1(\frac{m}{\hbar}\sqrt{-x^2}). \end{aligned} \tag{45}$$

Eq. (44) reproduces correctly the asymptotic behavior of the propagator (45) at large $|x^2|$. However, it fails in the vicinity of the light cone $x^2 = 0$. For example, there is no contribution

[3]We set $x_{\text{out}} = x, \quad x_{\text{in}} = 0$ here.

in (44) from the first term in Eq. (45). The latter, being of order $\hbar^{-2}$, could not be restored by the loop corrections.

Acknowledgments

D.M.G thanks Brazilian Foundation CNPq for support. S.I.Z. thanks Departamento de Física Matemática at Universidade de São Paulo for hospitality.

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