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CLASSICAL INTERPRETATION OF THE PAULI-SCHRÖDINGER EQUATION

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ABSTRACT

We derive a Pauli-Schrödinger type equation in configuration space, from the classical Liouville equation for a neutral particle with arbitrary spin and magnetic dipole. We show that the derivation does not apply to an arbitrary classical phase space distribution. However, in certain particular cases, discussed in the paper, there is an equivalence between the classical and the Pauli-Schrödinger descriptions. Consequently, the results of the Stern-Gerlach, and also the Rabi type molecular beam experiments, can be interpreted classically, that is, in such a way that the particles have well-defined and continuous trajectories, and also continuous orientations angles of the spin vector and magnetic dipole.

1. INTRODUCTION

The central idea of this work is based on the fact that the classical and the quantum theories, together, explain an enormous quantity of physical phenomena. Therefore, both are correct and it is possible that, in the future, classical and quantum physics can be put in a form that does not exhibit conflicting concepts.

The first papers along this direction appeared many years ago and are due to Planck (1911), Einstein and Stern (1913) and Nernst (1916). In these works, the authors uses the statistical properties of the classical zero-point eletromagnetic radiation[1, 2], in order to show the equivalence between some classical and quantum theoretical explanations

of the experimental observations. Another very important contribution with the same goal was made by Wigner[3], in 1932. Wigner's proposal, allowed the formulation of quantum mechanics in phase space, and disclosed the similarity between the Liouville and the Schrödinger equations. The two equations are dynamically equivalent for particles subjected to various forces [4–6]. In 1963, Marshall (see[1]) developed even more the same idea, giving a detailed phase space study of a spinless charged harmonic oscillator immersed in the thermal and zero-point radiations.

In our paper we shall apply Wigner's idea to study the classical motion of a neutral particle, with spin and magnetic dipole, in an external magnetic field. In this regard, it should be mentioned the work of Bohm et al.[7], and the more recent approach by Dewdney et al.[8]. These papers gives an objective account of the Stern-Gerlach experiment in which the particle have continuous trajectories and continuous orientation of the spin vector. The concept of quantum potential is used and it is not necessary to introduce any wave packet collapse hypothesis.

Our paper is organized as follows. We first introduce the equations which govern the classical dynamics of the system, namely, Newton's equations and the Larmor equations for the precession. We show (section 2) that the same equations can be obtained from the Heisenberg formalism [9, 10], that is, the quantum dynamical equations of motion are independent of the Planck's constant $\hbar$. Section 3 is devoted to the introduction of the spinorial notation[11] in order to describe the Larmor precession. Within section 4 we obtain a Pauli-Schrödinger type equation, from the Liouville equation, using a new method[5] which is inspired in the Wigner original work[3]. However, since the method is entirely classical, Planck's constant does not appears in the Pauli-Schrödinger type equation. Section 5 is devoted to the application of our method to the analysis of the Stern-Gerlach type experiments[12]. Finally, our conclusions are summarized in section 6.

2. CLASSICAL EQUATIONS OF MOTION ACCORDING TO THE HEISENBERG NOTATION

We shall denote the magnetic moment of the neutral particle (a silver atom for instance) by the vector $\vec{\mu}$. The spin vector is denoted by $\vec{S}$ and these quantities will be related by

$$\vec{\mu} = \frac{eg}{2mc}\vec{S} \quad , \tag{1}$$

where g is the gyromagnetc factor, e is the elementary charge, m is the electron mass and c is the velocity of light. The magnitude $S = |\vec{S}|$ is supposed known but its value is arbitrary. We shall also assume that the particle (rest mass M) is moving with velocity

$$\dot{\vec{r}} = \frac{\vec{p}}{M} \quad , \tag{2}$$

in a non uniform magnetic field $\vec{B}$. Therefore, the rate of variation of $\vec{p}$ is:

$$\dot{\vec{p}} = \vec{\nabla}(\vec{\mu}.\vec{B}) \equiv \vec{F} \quad . \tag{3}$$

The orientation the spin vector $\vec{S}$ also varies with time and is governed by the Larmor equation $\dot{\vec{S}} = \vec{\mu} \times \vec{B}$ or

$$\dot{\vec{\mu}} = \vec{\omega}_L \times \vec{\mu} \quad , \quad \vec{\omega}_L \equiv -\frac{eg\vec{B}}{2mc} \quad . \tag{4}$$

The above equations (2), (3) and (4) are the well known classical dynamical equations. We shall show in the following that these equations are the physical basis for our proposed classical interpretation of the Pauli-Schrödinger equation. In order to give a more clear explanation of the our proposition, we shall present firstly the corresponding Heisenberg equations of motion for the spinning particle. We can easily show that these equations are independent of the Planck's constant $\hbar$ and are identical to the classical equations introduced above.

According to the Heisenberg notation, the vectors $\vec{\mu}$ and $\vec{S}$ are operators related by the equation (1). However, the components of $\vec{S}$ are such that the commutation relation

$$i\hbar\, S_1 = [S_2, S_3] \quad , \tag{5}$$

is postulated in accordance to the quantum theory.

The dynamical evolution of the system is derived from the Hamiltonian operator H such that

$$H = -\frac{\hbar^2}{2M}\nabla^2 - \vec{\mu}\cdot\vec{B} \quad . \tag{6}$$

Therefore, one can show that the rate of variation of $\vec{r}$ is given by the commutator

$$\dot{\vec{r}} = \frac{i}{\hbar}[H, \vec{r}] = \frac{\vec{p}}{M} \quad , \tag{7}$$

whereas the rate of variation of the momentum operator is

$$\dot{\vec{p}} = \frac{i}{\hbar}\left[H, -i\hbar\vec{\nabla}\right] = \vec{\nabla}(\vec{\mu}\cdot\vec{B}) \quad . \tag{8}$$

Moreover, it is also possible to show that

$$\dot{\vec{\mu}} = \frac{i}{\hbar}[H, \vec{\mu}] = \vec{\omega}_L \times \vec{\mu} \quad , \tag{9}$$

where $\vec{\omega}_L$ was defined in (4).

The equations (7), (8) and (9) are independent of $\hbar$ and are identical to the corresponding classical equations (2), (3) and (4). Therefore, the physical content of both descriptions naturally allows the construction of a unified (classical and quantum) interpretation of the experiments. Moreover, the recent[13] recognition of the similarity between both approaches, for the description of the Stern-Gerlach experiment, will help us to understand better the physical content of the Pauli-Schrödinger equation and the corresponding spinorial notation.

3. LARMOR PRECESSION AND NEWTON'S EQUATIONS IN THE SPINORIAL NOTATION

Let us consider firstly the simple case of an uniform magnetic field $\vec{B} = (0, 0, B_0)$. The more general case will be discussed afterward. We shall also assume that the magnetic particle is precessing at the rest in laboratory frame. The orientation of the vectors $\vec{\mu}$ and $\vec{S}$ is such that (see FIG.1)

$$\vec{\mu} = \mu\,(\sin\theta\cos\phi\,, \sin\theta\sin\phi\,, \cos\theta) \quad , \tag{10}$$

where θ is the angle between $\vec{B}$ and $\vec{\mu}$. The azimutal angle ϕ is a linear function of the time and is given by

$$\phi(t) = \frac{\mu B_0}{S}\, t + \phi_0 \quad , \tag{11}$$

in accordance with the equation (4). The angle ϕ_0 is an arbitrary phase. These angles vary continuously whitin the range $0 \le \theta \le \pi$ and $0 \le \phi_0 \le 2\pi$.

The classical equation (4) can be cast in an spinorial notation as was shown by Pauli[14] and many authors in the past (see refs.[7, 8, 11]). We shall give below an exposition based on the paper by Ralph Schiller [11].

Let us to introduce the spinor $\chi(\theta,\phi)$ defined by

$$\chi(\theta,\phi) \equiv \chi_u + \chi_d \equiv \cos\frac{\theta}{2} e^{i\frac{\phi}{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\sin\frac{\theta}{2} e^{-i\frac{\phi}{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad , \tag{12}$$

and also the Pauli[14] matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad . \tag{13}$$

These definitions are very convenient because one can write any component of the vector $\vec{\mu}$ as $(j = 1, 2, 3)$

$$\mu_j = \mu\ \chi^\dagger(\theta,\phi)\,\sigma_j\,\chi(\theta,\phi) \quad . \tag{14}$$

If the magnetic field $\vec{B}$ varies in space, the magnetic force $\vec{F}$ (see (3) or (8)) is such that

$$F_j = \frac{\partial}{\partial x_j}\left[\mu\ \chi^\dagger(\theta,\phi)\,\vec{\sigma}\,\chi(\theta,\phi).\vec{B}\right] = \mu\cos\theta\frac{\partial B}{\partial x_j} \quad , \tag{15}$$

where $B = |\vec{B}|$ is the magnitude of the non uniform magnetic field.

Notice that, according to the spinorial notation, $\vec{F} \equiv \vec{F}_u + \vec{F}_d$ because

$$\vec{F}_u \equiv \mu\left(\chi_u^\dagger \sigma_3 \chi_u\right)\vec{\nabla}B = +\mu\cos^2\frac{\theta}{2}\vec{\nabla}B \quad , \tag{16}$$

and

$$\vec{F}_d \equiv \mu\left(\chi_d^\dagger \sigma_3 \chi_d\right)\vec{\nabla}B = -\mu\sin^2\frac{\theta}{2}\vec{\nabla}B \quad . \tag{17}$$

It should be remarked that $\vec{F}_u$ (and also $\vec{F}_d$) varies continuously $(0 \le |\vec{F}_u| \le \mu|\vec{\nabla}B|)$ because $0 \le \theta \le \pi$. Another important observation is that $\vec{F}_u$ and $\vec{F}_d$ are always opposite in sign. According to Pauli[14] the factors $\cos^2\theta/2$ and $\sin^2\theta/2$ are interpreted, respectively, as the "orientation probabilities", *up* and *down*, with respect to the vector $\vec{\nabla}B$ (see FIG.2).

The classical precession (see eq.(4)) can be written as

$$\dot{\mu}_j = \frac{d}{dt}\left[\chi^\dagger(\theta,\phi)\,\mu\sigma_j\,\chi(\theta,\phi)\right] = \left[\vec{\omega}_L \times \left(\chi^\dagger\,\mu\vec{\sigma}\,\chi\right)\right]_j \quad , \tag{18}$$

or equivalently

$$i\frac{\partial}{\partial t}\chi(\theta,\phi) = -\frac{\mu\vec{B}}{2S}\cdot\vec{\sigma}\,\chi(\theta,\phi) \quad , \tag{19}$$

where we have used well known properties of the Pauli matrices[11].

This classical equation is very interesting. It can be cast in a form which is identical to the Pauli-Schrödinger equation for a magnetic dipole precessing at rest in a magnetic field $\vec{B}$. Multiplying both sides of (19) by $\hbar$ and using equation (1) we get

$$\begin{aligned} i\hbar\frac{\partial\chi(\theta,\phi)}{\partial t} &= -\frac{egS}{2mc}\vec{\sigma}.\vec{B}\,\chi(\theta,\phi)\left(\frac{\hbar}{2S}\right) \\ &= -\frac{eg}{2mc}\left(\frac{\hbar}{2}\vec{\sigma}\right).\vec{B}\,\chi(\theta,\phi) \quad . \end{aligned} \tag{20}$$

It is remarkable that this occurs for an *arbitrary* magnitude of the spin vector $\vec{S}$. Moreover, (20) is independent of $\hbar$ since it was obtained from the classical Larmor precession. It is also possible to show that (20) is valid if $\vec{B}$ is time dependent (see[11]).

4. DERIVATION OF THE PAULI-SCHRÖDINGER EQUATION FROM THE LIOUVILLE EQUATION

According to the classical dynamical equations, the phase space evolution of an ensemble of particles is described by the instantaneous phase space distribution which will be denoted by

$$W = W(\vec{r}, \vec{p}, t) \quad . \tag{21}$$

This function is associated with a particle with momentum $\vec{p} = (p_1, p_2, p_3)$, located at the point $\vec{r} = (x_1, x_2, x_3)$ and with a given spin orientation θ_0 with respect to the local magnetic field $\vec{B}$ (see FIG.2). This magnetic field varies in space and, therefore the particle describes a classical trajectory. The instantaneous variation of $\vec{r}$ and $\vec{p}$ is governed by the equations (2) and (3) or (7) and (8).

The associated Liouville equation will be written as

$$\left[\frac{\partial}{\partial t} + \dot{\vec{r}} \cdot \frac{\partial}{\partial \vec{r}} + \dot{\vec{p}} \cdot \frac{\partial}{\partial \vec{p}}\right] W = 0 \quad , \tag{22}$$

and its evolution can be obtained from the solutions of equations (2) and (3) for $\dot{\vec{r}}$ and $\dot{\vec{p}}$. The equation (22) does not describes the precession (see (4) or (9)).

We shall present a method, for studying the mathematical problem of finding some solutions of (22), which was proposed recently by Dechoum and França[5] for the study of spinless particles. This method is based on a convenient modification of the original proposal introduced by Wigner [3] in 1932.

Let us define a Fourier transform $Q(\vec{r}, \vec{y}, t)$, which is associated with $W(\vec{r}, \vec{p}, t)$, by

$$Q(\vec{r}, \vec{y}, t) \equiv \int d^3p \, W(\vec{r}, \vec{p}, t) \, e^{-2i\vec{p}.\vec{y}/\hbar'} \quad . \tag{23}$$

Here $\vec{y}$ is another point in configuration space, and $\hbar'$ is a free parameter with dimension of action. It is assumed that $\hbar'$ is very small ($\hbar' \ll \hbar$ for instance), and the limit $\hbar' \to 0$ can be taken in the end the calculation, if necessary. Therefore, one can conclude that $Q(\vec{r}, \vec{y}, t) \neq 0$ only for very small values of $|\vec{y}|$. We shall see that in some simple cases the calculation leads to results which are independent of $\hbar'$. It is also important to remark that the initial orientation angles θ_0 and ϕ_0 (see FIG.1 and FIG.2) are being considered as independent variables.

The evolution equation for $Q(\vec{r}, \vec{y}, t)$ can be obtained easily. After substituting (23) into (22), we get

$$\left\{ i\hbar' \frac{\partial}{\partial t} + \frac{(\hbar')^2}{2M} \frac{\partial^2}{\partial \vec{y} \cdot \partial \vec{r}} + 2\vec{y} \cdot \frac{\partial}{\partial \vec{r}} \left[\vec{\mu} \cdot \vec{B}(\vec{r})\right] \right\} Q = 0 \quad , \tag{24}$$

where we have used (2) and (3).

Since in this equation $|\vec{y}|$ has to be considered very small, due to the function Q, it is possible to write

$$2\vec{y} \cdot \frac{\partial}{\partial \vec{r}} [B_j(\vec{r})] = B_j(\vec{r} + \vec{y}) - B_j(\vec{r} - \vec{y}) \quad . \tag{25}$$

Therefore,

$$2\vec{y} \cdot \frac{\partial}{\partial \vec{r}} \left[\vec{\mu} \cdot \vec{B}(\vec{r})\right] = \mu \left[\chi^\dagger(\theta_0, \phi_0) \, \vec{\sigma} \, \chi(\theta_0, \phi_0)\right] \cdot \left[\vec{B}(\vec{r} + \vec{y}) - B(\vec{r} - \vec{y})\right] \quad , \tag{26}$$

when $|\vec{y}| \to 0$. In the last equality we have used our previous equation (14).

For what follows it is convenient to introduce complex spinorial functions $\Psi(\vec{r}, t|\theta_0, \phi_0)$, and an additional hypothesis. We shall consider only phase space distributions $W(\vec{r}, \vec{p}, t)$ such that its Fourier transform (23) can be written as

$$Q(\vec{r}, \vec{y}, t) = \Psi^\dagger(\vec{r} + \vec{y}, t|\theta_0, \phi_0) \Psi(\vec{r} - \vec{y}, t|\theta_0, \phi_0) \quad , \tag{27}$$

where (see (12))

$$\Psi(\vec{r},t|\theta_0,\phi_0) \equiv \chi(\theta_0,\phi_0)\Phi(\vec{r},t) \equiv \Psi_u + \Psi_d \quad , \tag{28}$$

and $\Phi(\vec{r},t)$ is a scalar function.

A more general expression for $Q(\vec{r},\vec{y},t)$ is

$$Q(\vec{r},\vec{y},t) = \sum_k \sum_l C_{kl}(t) G_{kl}(\vec{r},\vec{y}) \quad , \tag{29}$$

where $\{G_{kl}\}$ is a complete set of orthogonal functions (or states) indicated by the indices k and l. A differential equation for the coefficients C_{kl} can be obtained from (22). Therefore, there is no loss of generality in using the hypothesis (27), provided the complete set of "phase space" states $\{G_{kl}\}$ is introduced in a latter stage of the calculation (see refs. [5] and [15]).

Using (26), (27), (28) and the fact that $\chi(\theta,\phi)\chi^{\dagger}(\theta,\phi) \equiv I$, it is straightforward to show that (24) leads to

$$\left[i\hbar'\frac{\partial}{\partial t} + \frac{(\hbar')^2}{2M}\nabla^2 + \mu\vec{\sigma}\cdot\vec{B}(\vec{r})\right]\Psi(\vec{r},t|\theta_0,\phi_0) = 0 \quad . \tag{30}$$

It is interesting to notice that there is a direct correspondence between each term of (22) and (30). For instance, the Schrödinger type operator $[(\hbar')^2/2M]\nabla^2$ has its origin in the convective operator $\dot{\vec{r}}\cdot\frac{\partial}{\partial\vec{r}}$ of the classical Liouville equation. For $\hbar' = \hbar$, the above equation is known as the Pauli-Schrödinger equation.

The statistical interpretation of the function $\Psi(\vec{r},t|\theta_0,\phi_0)$ is also obtained from the phase space distribution $W(\vec{r},\vec{p},t)$ and the normalization condition

$$\begin{aligned}\int d^3r \int d^3p\, W(\vec{r},\vec{p},t) &= \int d^3r |\Psi(\vec{r},t|\theta_0,\phi_0)|^2 = \\ &= \int d^3r \left(|\Psi_u|^2 + |\Psi_d|^2\right) = 1 \quad ,\end{aligned} \tag{31}$$

as it is easy to verify (see also the original paper by Pauli [14]).

A Schrödinger type equation, similar to (30), but for a spinless charged particle, bounded by a harmonic force (frequency ω_0), was obtained by Dechoum and França[5] within the realm of classical stochastic electrodynamics (SED)[1]. The zero-point electric field associated with the vacuum fluctuations was included in their approach. Therefore, it was possible to show that, in the limit $\hbar' \to 0$, the oscillator has an average energy of $\hbar\omega_0/2$. The presence of the Planck's constant in this result is due to the effects of the zero-point background radiation field. The mathematical interpretation of the harmonic oscillator excited states (solutions of the time independent Schrödinger type equation) was also provided by Dechoum and França[5] and by França and Marshall[15].

The equation (30) is valid for a general $\vec{B}(\vec{r})$, provided that the limit $\hbar' \to 0$ is considered. However, we shall see below that, in some simple cases, the physical results obtained from (30) are *independent* of $\hbar'$. Notice that the thermal and zero-point fields are *not* included in (30).

5. CLASSICAL DESCRIPTION OF THE STERN-GERLACH EXPERIMENT

We shall obtain here an aproximate solution of the classical (Pauli-Schrödinger type) equation (30) in the particular case in which the magnetic field $\vec{B}$ is such that (see FIG.2)

$$\vec{B} = (-\beta x, 0, B_0 + \beta z) \tag{32}$$

for $0 \leq y \leq l$. The field is assumed to be zero for $l < y \leq D$, where D is the distance from the magnet to the screen (or detector).

This non uniform magnetic field gives an approximate description of the experimental situation encountered in the Stern-Gerlach type devices [16–18]. Moreover, it is easy to see from (3) that the non uniform field (32) generates different forces on the particles of the beam, depending on their position at the entrance of the Stern-Gerlach magnet, and the orientation of the vector $\vec{\mu}$ (see FIG.2).

According to (15) and (32) the acceleration at the entrance of the magnet is such that

$$M\ddot{z} = \mu \cos\theta_0 \frac{\partial B}{\partial z} = \mu \cos\theta_0 \left(\frac{B_0 + \beta z}{B}\right) \beta \quad , \tag{33}$$

and

$$M\ddot{x} = \mu \cos\theta_0 \frac{\beta^2 x}{B} \quad , \tag{34}$$

where $B = |\vec{B}|$ (see also [19]). The solutions of these non-linear equations, with the appropriate initial conditions characterizing the beam [16–18], will be discussed elsewhere. We shall see in the following that an approximate solution of the Pauli-Schrödinger type equation (30) can be more easely constructed (in comparison with (22)). The main reason for this advantage is that (30) depends on the *energy* $(-\vec{\mu} \cdot \vec{B})$, whereas the original Liouville equation (22) depends on the *non-linear force* with components given by (33) and (34).

5.1 . MOTION INSIDE THE MAGNET ($\Psi = \Psi_1$)

We shall assume that the magnetic particle is heavy (a cesium or a silver atom for instance), and spend a short time l/v_0 inside the magnet (v_0 is the velocity of the particles in the y direction). Therefore, one can neglect the transversal convective contribution in (30), that is, we shall take (see also ref [9])

$$\frac{(\hbar')^2}{2M} \nabla^2 \Psi_1 \simeq 0 \quad . \tag{35}$$

According to (30) one can write

$$i\hbar' \frac{\partial \Psi_1(t)}{\partial t} = -\mu \sigma_3 B \Psi_1(t) \quad , \tag{36}$$

where $B = B(x,z) = |\vec{B}|$ and σ_3 is the Pauli matriz (13).

The above equation can be easely integrated giving

$$\begin{aligned} \Psi_1(t) &= \Phi(x,z)\left\{\cos\frac{\theta_0}{2}\exp\left[\frac{i}{2}\left(\phi_0 + \frac{2\mu B t}{\hbar'}\right)\right]\begin{pmatrix}1\\0\end{pmatrix}\right. \\ &+ \left. i\sin\frac{\theta_0}{2}\exp\left[-\frac{i}{2}\left(\phi_0 + \frac{2\mu B t}{\hbar'}\right)\right]\begin{pmatrix}0\\1\end{pmatrix}\right\} \quad . \end{aligned} \tag{37}$$

The function $\Phi(x,z)$ is related to the cross section of the beam of spinning atoms, conveniently prepared by the experimentalist [12, 16, 17]. Here $0 \le t \le t_1 = l/v_0$. Notice that

$$\phi_0 + \frac{2\mu B}{\hbar'} t \neq \phi(t) \quad , \tag{38}$$

defined in (11). Both expressions coincides only if $\hbar' = 2S$.

5.2. FREE MOTION FROM THE MAGNET TO THE SCREEN ($\Psi = \Psi_2$)

The screen is situated far enough from the Stern-Gerlach type magnet in order to allow the physical spliting of the beam. Notice that, in practice, the spliting already occurs inside the magnet, and it is due to two factors: 1) the initial beam is such that $\langle p_z \rangle = 0$ but there are particles, in the ensemble , with positive and negative values of the momentum p_z; 2) the space variation of $\vec{B}(\vec{r})$ generates different forces on the particles of the ensemble, depending on the sign of F_x and F_z (see (33), (34) and also (15)).

The equation for Ψ_2 is such that

$$i\hbar' \frac{\partial \Psi_2(t)}{\partial t} = -\frac{(\hbar')^2}{2M}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\Psi_2(t) \simeq -\frac{(\hbar')^2}{2M}\frac{\partial^2 \Psi_2(t)}{\partial z^2} \quad , \tag{39}$$

where we have neglected the convective motion in the x direction because, in a typical experiment, the beam is very narrow (see refs, [16] and [17]).

We shall take $\Psi_2(0) = \Psi_1(t_1)$ where $\Psi_1(t)$ is given by (37). We shall also assume that $B_0 \gg \beta|z|$ and $B_0 \gg \beta|x|$, that is, $B \simeq B_0 + \beta z$. Therefore, it is possible to take

$\Phi(x,z) \simeq (2\pi\alpha^2)^{-1/4}\exp(-z^2/4\alpha^2)$ in expression (37). The parameter α is related with the width of the beam. Typical values are described in references [12],[16] and [17].

Using these approximations, the integration of (39) is straightforward. The result is

$$\Psi_2(t) \simeq \frac{\left(1-\frac{i\hbar' t}{2M\xi^2}\right)}{(2\pi\xi^2)^{\frac{1}{4}}}\left\{\cos\frac{\theta_0}{2}exp\left[\frac{i}{2}\left(\phi_0+\frac{2\mu B_0}{\hbar'}t_1\right)-\frac{(z-z_c(t))^2}{4\xi^2}\right]\begin{pmatrix}1\\0\end{pmatrix}\right.$$
$$\left.+\ i\sin\frac{\theta_0}{2}exp\left[-\frac{i}{2}\left(\phi_0+\frac{2\mu B_0}{\hbar'}t_1\right)-\frac{(z+z_c(t))^2}{4\xi^2}\right]\begin{pmatrix}0\\1\end{pmatrix}\right\} \qquad (40)$$

where $z_c(t)=\frac{\mu\beta t_1 t}{M}$ and $\xi^2=\alpha^2\left[1+\left(\frac{\hbar' t}{2M\alpha^2}\right)^2\right]$. We shall take $t=t_2=D/v_0>t_1=l/v_0$.

According to the experiment by J.R. Zacharias, which uses a beam of cesium atoms (see[12]), we have: $\mu\simeq 8.7\times10^{-21}\ erg/Gauss$, $Mv_0^2\simeq 1.1\times10^{-13}\ erg$, $\beta=10^4\ Gauss/cm$, $l=12.5\ cm$, $D=50\ cm$, $\alpha\simeq 0.1\ cm$ and $2z_c=\frac{2\mu\beta l D}{Mv_0^2}=0.37\ cm$. Therefore, taking into account these values, we see that $\xi^2\simeq\alpha^2$. Consequently, (40) depends on $\hbar'$ only through the phase factors $\pm i\mu B_0 t_1/\hbar'$, and $|\Psi_2(t)|^2$ is independent of the parameter $\hbar'$.

One can calculate the distribution of particles on the screen, generate by an unpolarized, beam of cesium atoms. It is given by

$$P(z) \simeq \frac{1}{2}\int_0^\pi d\theta_0\sin\theta_o \mid \Psi_2(t_2)\mid^2$$
$$= \frac{1}{\sqrt{8\pi\alpha^2}}\left\{exp\left[-\frac{(z-z_c)^2}{2\alpha^2}\right]+exp\left[-\frac{(z+z_c)^2}{2\alpha^2}\right]\right\} \quad . \qquad (41)$$

This results, despite the various approximations used to obtain it, is in good agreement with the experiment described by French and Taylor (see [12] and FIG.3A).

An interesting observation is that P(z) does not depend on the parameter $\hbar'$ (introduced in (23)), and also does not depend on the Planck's constant $\hbar$ used in the derivation

of the Heisenberg equations presented within section 2. Therefore, in our opinion the results of the Stern-Gerlach type experiments does not allow us to infer the "directional quantization" in a magnetic field[16]. Moreover, the result (41) is valid for an arbitrary magnitud of the spin vector $\vec{S}$.

6. DISCUSSION

We have shown that it is possible to give a classical interpretation to the Pauli-Schrödinger equation for a neutral, spinning particle. This classical interpretation is valid for any magnitude of the spin $|\vec{S}|$ and magnetic dipole $|\vec{\mu}|$. The Pauli-Schrödinger type equation obtained in section 4, was derived from the classical Liouville equation in phase space. The spin vector $\vec{S}$ and the magnetic dipole $\vec{\mu}$ are not quantized, and exhibit orientation angles θ and ϕ wich vary continously (see section 3). An adequate classical interpretation of the Stern-Gerlach experiment was provided by the equations (30) and (41) derived from the Liouville equation (22).

The classical equation (4), which has the spinorial form (20), is independent of $\hbar$, being both valid for a time dependent magnetic field also. Let us consider that this magnetic field is given by

$$\vec{B}=(B_1\cos\omega t,\ B_1\sin\omega t,\ B_0) \quad , \qquad (42)$$

where B_1 and B_0 are constants. Therefore, using (4) or (20) and defining $\gamma\equiv eg/2mc$, and $\omega_0\equiv\gamma B_0$, it is possible to show that the angle $\theta(t)$ between $\vec{\mu}$ and z axis is given by

$$\frac{\mu_z(t)}{\mu}=\cos\theta(t)=1-\left[\frac{2(\gamma B_1)^2}{(\omega-\omega_0)^2+\gamma^2B_1^2}\right]\sin^2\left(\frac{t}{2}\sqrt{(\omega-\omega_0)^2+\gamma^2B_1^2}\right) \quad . \qquad (43)$$

This result was firstly derived by Rabi[20], and Schwinger[21] using the Pauli equation (20). It was soon recognized, by Rabi, Ramsey and Schwinger[22], that these Rabi resonant *oscillations* are equally obtained from the classical or the quantum mechanical approaches. A good exposition of this equivalence is also provided by Bloembergen[23]. The conclusion is that the magnetic dipole vector (and also the spin vector) have orientation angles which vary *continuously* with respect to the applied magnetic field. The beautiful experiments by Stern (Nobel prize 1943 for the discovery of the proton magnetic moment), by Rabi (Nobel prize 1944 for the discovery of the resonance method to record the magnetic properties of the atomic nuclei), and collaborators are the most striking confirmation of our statement (see refs.[16–18] and [20–23]).

The forces generated by the radiation reaction, and the zero-point (and thermal) fluctuations of the eletromagnetic field, were neglected in our the paper. Their effects only appear in the equilibrium (stationary) regimen. This was shown previously by Boyer[24] and by Barranco et al.[25]. According to these authors the equation of motion (4) is modified to

$$\dot{\vec{S}} = \vec{\mu} \times \vec{B}_0 + \vec{\mu} \times \vec{B}_{VF}(t) + \frac{2}{3c^3}\vec{\mu} \times \ddot{\vec{\mu}} \quad , \tag{44}$$

where $\vec{B}_0$ is a constant magnetic field and $\vec{B}_{VF}$ is the random magnetic field characteristic of SED [1]. The last term in (44) is the self reaction torque. Equation (44) is known as the stochastic Babha equation.

The random magnetic field $\vec{B}_{VF}$ is such that

$$\frac{1}{4\pi}\langle \vec{B}_{VF}(t) \cdot \vec{B}_{VF}(0) \rangle = \int_0^\infty \frac{\hbar\omega^3}{2\pi^2 c^3} \coth\left(\frac{\hbar\omega}{2kT}\right) \cos(\omega t) \quad , \tag{45}$$

where $\hbar$ is the Planck's constant and T is the temperature.

According to Boyer[24], and Barranco et al.[25], the orientation angles of the vector $\vec{\mu}(t)$ vary *continuously*. Therefore, the paramagnetic behaviour of the particle can be calculated according to the classical SED approach. The average value of $\mu_z(t)$ is

$$\begin{aligned} \langle \mu_z \rangle &= \frac{egS}{2mc}\int_0^\pi d\theta \cos\theta R(\theta) = \\ &= \frac{e\hbar}{2mc}\left\{ \frac{S}{\hbar} \coth\left(\frac{2S/\hbar}{\coth\left(\frac{\hbar\omega_0}{2kT}\right)} \right) - \coth\left(\frac{\hbar\omega_0}{2kT}\right) \right\} \quad , \end{aligned} \tag{46}$$

where $\omega_0 = \mu B_0/S$, and $R(\theta)$ is the orientation probability calculated by Boyer[24]. Notice that this result depends crucially on the Planck's constant $\hbar$. Its origin can be traced back to the thermal and zero-point electromagnetic noise, whose spectral distribution is given by (45).

The approach of Boyer[24], and Barranco et al.[25], shows that the quantization of $\vec{S}$ is not necessary to give a good account for the paramagnetic behaviour of the magnetic particles. Therefore, the SED proposal, gives a simple (unified) picture of the spinning particle, were the classical and the quantum approaches merge into the same equations and results (see[1] and [26]). We think, however, that in certain special cases the classical approach based on SED may present some technical advantages [24,25]. One example is the predicted[27] "anomalous" paramagnetic behaviour, which is generated when the paramagnetic sample is influenced by the zero-point current fluctuations of a simple circuit.

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FIGURE CAPTIONS

FIG.1 — The orientation angles of the vector $\vec{\mu}(t)$ which precesses around the magnetic field $\vec{B}$.

FIG.2 — Schematic picture of the precession in a non uniform magnetic field $\vec{B}$. The spinorial notation for the force components acting on the neutral particle are presented in equations (15), (16) and (17). The particles in the beam (shaded area), move with velocity v_0 in the y direction.

FIG.3A — Beam profiles obtained by J.R. Zacharias [12]. Curve (a) shows the spreading of the beam with a low magnetic field. The gradient field β is not great enough to cause separation of the beam. Curve (b) shows separation in the high field gradient $\left(z_c = \frac{\mu\beta l D}{M v_0^2} \simeq 0.2cm\right)$. See our expression (41) for comparison. Notice the range of deflections, due principally to spread of velocities of cesium atoms from oven.

FIG.3B — Beam profiles (intensity in arbitrary units) showing the magnetic deflection of a beam of HD. The experimental data was used to measure the magnetic moment of the proton (see ref.[18]).

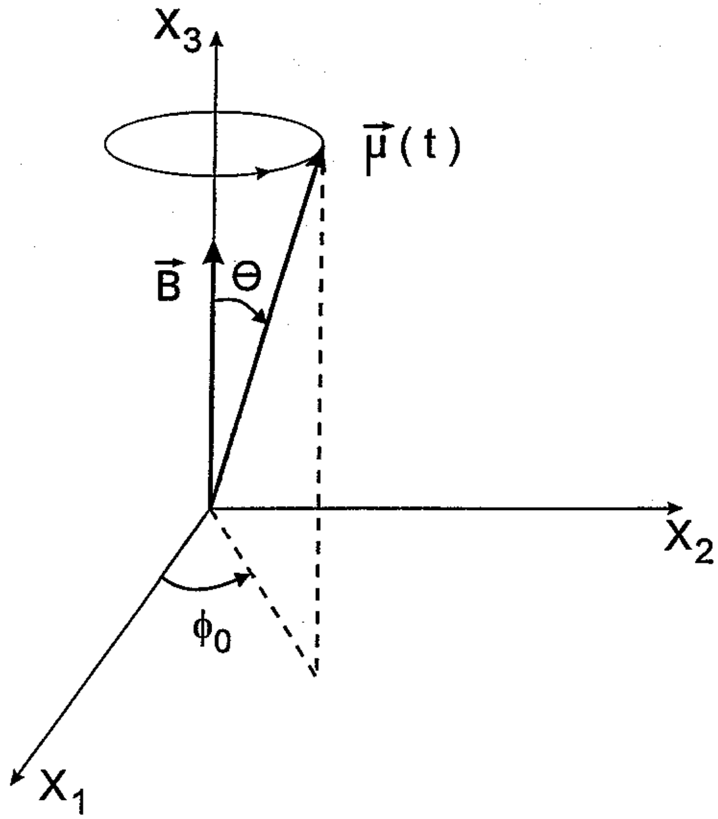
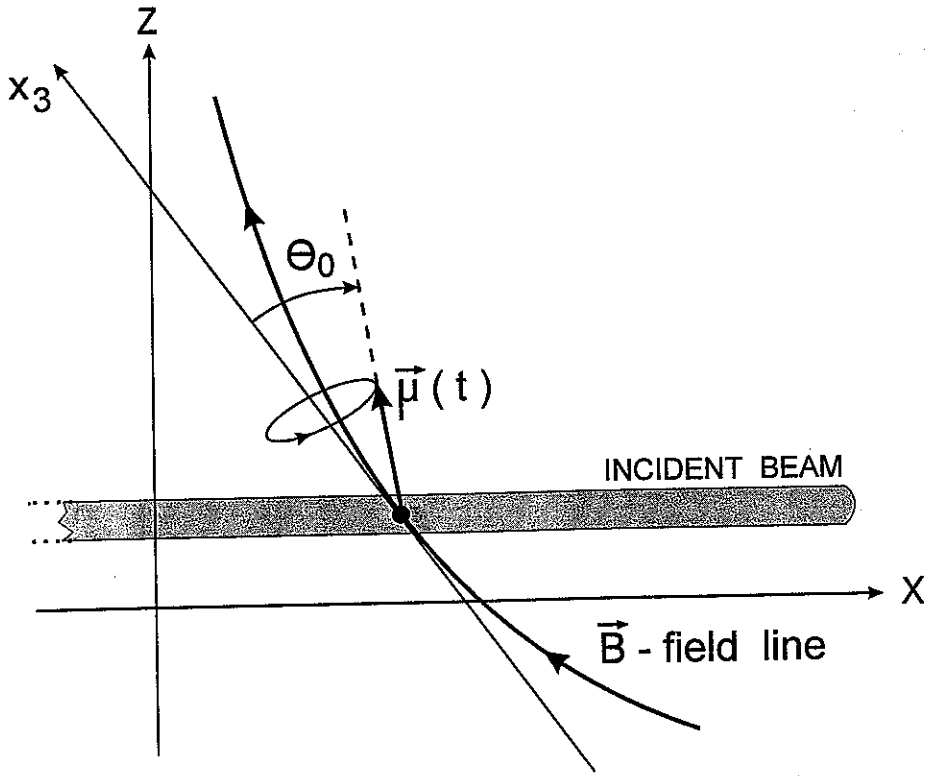
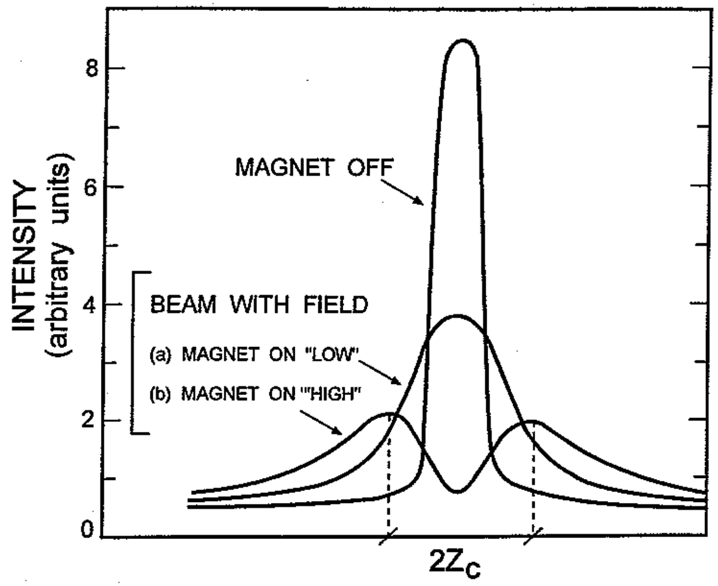


FIG. 1



Z
x_3
Θ_0
$\vec{\mu}(t)$
INCIDENT BEAM
X
$\vec{B}$ - field line

FIG. 2



INTENSITY (arbitrary units)
8
6
4
2
0
MAGNET OFF
BEAM WITH FIELD
(a) MAGNET ON "LOW"
(b) MAGNET ON "HIGH"
$2Z_c$

FIG. 3A

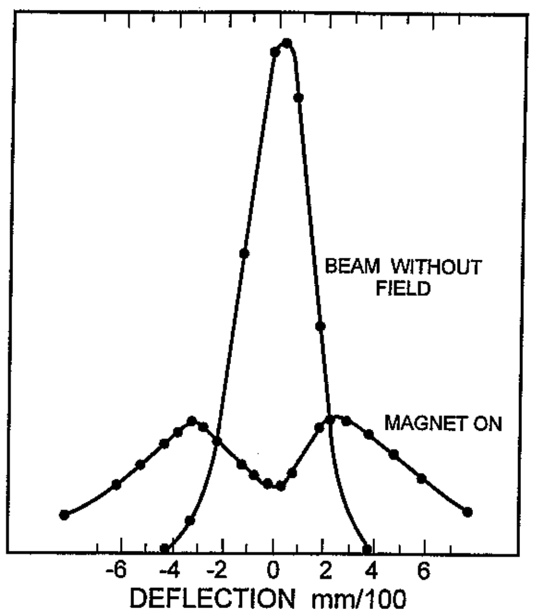


FIG. 3B