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CLASSICAL ASPECTS OF THE PAULI-SCHRÖDINGER EQUATION

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Abstract

We derive a Pauli-Schrödinger type equation in configuration space, from the classical Liouville equation for a neutral particle with *arbitrary* spin and magnetic dipole. Our derivation does not apply to a general classical phase space distribution. However, in certain particular cases, discussed in the paper, there is a correspondence between the classical equations and the Pauli-Schrödinger equation. Consequently, the results of the Stern-Gerlach, and also the Rabi type molecular beam experiments, can be interpreted classically, that is, in such a way that the particles have well-defined and continuous trajectories, and also continuous orientation angles of the spin vector and magnetic dipole. Theoretical and experimental implications of this conclusion are briefly commented.

I. INRODUCTION

The central idea of this work is based on the fact that the classical and the quantum theories, together, explain an enormous quantity of physical phenomena. Therefore, both are correct and it is possible that, in the future, classical and quantum physics can be

put in a form that does not exhibit conflicting concepts, thus revealing a more complete physical theory of the microscopic and macroscopic worlds.

The first papers along this line appeared many years ago and are due to Planck (1911), Einstein and Stern (1913) and Nernst (1916). In these works, the authors use the statistical properties of the classical zero-point eletromagnetic radiation [1,2], in order to show the equivalence between some classical and quantum theoretical explanations of the experimental observations. Another very important contribution with the same goal was made by Wigner [3], in 1932. Wigner's proposal, allowed the formulation of Quantum Mechanics in phase space, and disclosed the similarity between the Liouville and the Schrödinger equations. The two equations are dynamically equivalent for particles subjected to various forces [4–6]. In 1963, Marshall (see [1]) developed even more the same idea, giving a detailed phase space study of a spinless charged harmonic oscillator immersed in the thermal and zero-point radiations. Further improvements were introduced later on mainly by E. Santos, A. M.Cetto, L. de la Peña and T. H. Boyer (see [1]). These more recent theoretical attempts are known as Stochastic Electrodynamics (SED) in the current literature [1,7].

In our paper we shall apply Wigner's idea to study the classical motion of a neutral particle, with spin and magnetic dipole, in an external magnetic field. In this regard, it should be mentioned the work of Bohm, Schiller and Tiomno [8], and the more recent approach by Dewdney, Holland, Kyprianidis and Vigier [9]. These papers give an objective account of the Stern-Gerlach experiment in which the particles have continuous trajectories and continuous orientation of the spin vector. The concept of quantum potential is used and it is not necessary to introduce any wave packet collapse hypothesis.

Our paper is organized as follows. We first introduce the equations which govern the classical dynamics of the system, namely, Newton's equations and the Larmor equations for the precession. We show (section 2) that the same equations can be obtained from the Heisenberg formalism [10,11], that is, the quantum dynamical equations of motion are independent of the Planck's constant $\hbar$. Section 3 is devoted to the introduction of the spinorial notation [12] in order to describe the Larmor precession. Within section 4 we obtain a Pauli-Schrödinger type equation, from the Liouville equation, using a new approximate method [6] which is inspired in the Wigner original work [3]. However, since the method is entirely classical, Planck's constant does not appear in the Pauli-Schrödinger type equation. Section 5 is devoted to the application of our method to the analysis of the Stern-Gerlach type experiments [13]. Finally, our conclusions are summarized in section 6.

II. CLASSICAL EQUATIONS OF MOTION ACCORDING TO THE HEISENBERG FORMULATION

We shall denote the magnetic moment of the neutral particle (a silver atom for instance) by the vector $\vec{\mu}$. The spin vector is denoted by $\vec{S}$ and these quantities will be related by

$$\vec{\mu} = -\frac{eg}{2mc}\vec{S} \quad , \tag{1}$$

where g is the gyromagnetic factor, $-e$ is the electron charge, m is the electron mass and c is the velocity of light. The magnitude $S = |\vec{S}|$ is supposed known but its value is arbitrary. We shall also assume that the particle (rest mass M) is moving with velocity

$$\dot{\vec{r}} = \frac{\vec{p}}{M} \quad , \tag{2}$$

($|\dot{\vec{r}}| \ll c$) in a non uniform magnetic field $\vec{B}$. Therefore, the rate of variation of $\vec{p}$ is

$$\dot{\vec{p}} = \vec{\nabla}(\vec{\mu}.\vec{B}) \equiv \vec{F} \quad . \tag{3}$$

The orientation of the spin vector $\vec{S}$ also varies with time and is governed by the Larmor equation $\dot{\vec{S}} = \vec{\mu} \times \vec{B}$ or

$$\dot{\vec{\mu}} = \vec{\omega}_L \times \vec{\mu} \quad , \qquad \vec{\omega}_L \equiv \frac{eg\vec{B}}{2mc} \quad . \tag{4}$$

The above equations (2), (3) and (4) are the well known classical dynamical equations. We shall show in the following that these equations are the physical basis for our proposed classical interpretation of the Pauli-Schrödinger equation. In order to give a more clear explanation of our proposition, we shall present first the corresponding Heisenberg equations of motion for the spinning particle.

According to the Quantum Mechanics, the vectors $\vec{\mu}$ and $\vec{S}$ are operators related by the equation (1), and the components of $\vec{S}$ satisfy the commutation relation

$$i\hbar\, S_1 = [S_2, S_3] \quad . \tag{5}$$

The dynamical evolution of the system is derived from the Hamiltonian operator H

$$H = -\frac{\hbar^2}{2M}\nabla^2 - \vec{\mu}\cdot\vec{B} \quad , \tag{6}$$

and according to the Heisenberg formulation the rate of variation of $\vec{r}$ is given by

$$\dot{\vec{r}} = \frac{i}{\hbar}[H, \vec{r}] = \frac{\vec{p}}{M} \quad , \tag{7}$$

whereas the rate of variation of the momentum operator is

$$\dot{\vec{p}} = \frac{i}{\hbar}[H, \vec{p}] = \vec{\nabla}(\vec{\mu}\cdot\vec{B}) \quad . \tag{8}$$

Moreover, it is also possible to show that

$$\dot{\vec{\mu}} = \frac{i}{\hbar}[H, \vec{\mu}] = \vec{\omega}_L \times \vec{\mu} \quad , \tag{9}$$

where $\vec{\omega}_L$ was defined in (4).

The equations (7), (8) and (9) are independent of $\hbar$ and are formally identical to the corresponding classical equations (2), (3) and (4). Therefore, the physical content of both descriptions naturally allows the construction of a unified (classical and quantum) interpretation of the experiments. Moreover, the recent [14] recognition of the similarity of both approaches, for the description of the Stern-Gerlach experiment, will help us to understand better the physical content of the Pauli-Schrödinger equation and the corresponding spinorial notation.

III. THE LARMOR PRECESSION IN SPINORIAL NOTATION

Let us consider first the simple case of a uniform magnetic field $\vec{B} = (0, 0, B_0)$. The more general case will be discussed afterwards. We shall also assume that the magnetic particle is precessing at rest in the laboratory frame. The orientation of the vector $\vec{\mu}$ is such that (see Fig. 1)

$$\vec{\mu} = \mu(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \quad , \tag{10}$$

where θ is the angle between $\vec{B}$ and $\vec{\mu}$, and $\mu = |\vec{\mu}|$. The azimutal angle ϕ is a linear function of the time and is given by

$$\phi(t) = \frac{\mu|\vec{B}|}{S}\, t + \phi_0 \quad , \tag{11}$$

in accordance with the equation (4). The angle ϕ_0 is an arbitrary phase. These angles vary continuously within the range $0 \leq \theta \leq \pi$ and $0 \leq \phi_0 \leq 2\pi$.

The classical equation (4) can be cast in a spinorial notation as was shown by many authors in the past (see refs. [8,9,12]). We shall give below an explanation based on the paper by Ralph Schiller [12]. It should be remarked that in ref. [12] θ is the angle between $\vec{S}$ and $\vec{B}$ whereas here θ is the angle between $\vec{\mu}$ and $\vec{B}$ (see Fig. 1).

Let us introduce the spinor $\chi(\theta, \phi)$ defined by

$$\chi(\theta,\phi) \equiv \chi_u + \chi_d \equiv \cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}\begin{pmatrix}1\\0\end{pmatrix} + \sin\frac{\theta}{2}e^{i\frac{\phi}{2}}\begin{pmatrix}0\\1\end{pmatrix} \quad , \tag{12}$$

and also the Pauli [15] matrices

$$\sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad . \tag{13}$$

These definitions are very convenient because one can write any component of the vector $\vec{\mu}$ as $(j = 1, 2, 3)$

$$\mu_j = \mu\ \chi^\dagger(\theta,\phi)\,\sigma_j\,\chi(\theta,\phi) \quad . \tag{14}$$

If the magnetic field $\vec{B}$ varies in space, the magnetic force $\vec{F}$ (see (3) or (8)) is such that

$$F_j = \frac{\partial}{\partial x_j}\left[\mu\ \chi^\dagger(\theta,\phi)\,\vec{\sigma}\,\chi(\theta,\phi).\vec{B}\right] = \mu\cos\theta\frac{\partial B}{\partial x_j} \quad , \tag{15}$$

where $B = |\vec{B}|$ is the magnitude of the non uniform magnetic field (see Fig. 2).

Notice that, according to the spinorial notation, $\vec{F} \equiv \vec{F}_u + \vec{F}_d$ because

$$\vec{F}_u \equiv \mu\left(\chi_u^\dagger\sigma_3\chi_u\right)\vec{\nabla}B = +\mu\cos^2\frac{\theta}{2}\vec{\nabla}B \quad , \tag{16}$$

and

$$\vec{F}_d \equiv \mu\left(\chi_d^\dagger\sigma_3\chi_d\right)\vec{\nabla}B = -\mu\sin^2\frac{\theta}{2}\vec{\nabla}B \quad . \tag{17}$$

It should be remarked that $\vec{F}_u$ (and also $\vec{F}_d$) varies continuously ($0 \leq |\vec{F}_u| \leq \mu|\vec{\nabla}B|$) because $0 \leq \theta \leq \pi$. Another observation is that $\vec{F}_u$ and $\vec{F}_d$ are always anti-parallel. The factors $\cos^2\theta/2$ and $\sin^2\theta/2$ are interpreted [15], respectively, as the "orientation probabilities", *up* and *down*, of the vector $\vec{\mu}$ with respect to the vector $\vec{B}$ (see Fig. 1 and Fig. 2). Here, however, such interpretation is not required.

The classical precession (see eq.(4)) can be written as

$$\dot{\mu}_j = \frac{d}{dt}\left[\chi^\dagger(\theta,\phi)\,\mu\sigma_j\,\chi(\theta,\phi)\right] = \left[\vec{\omega}_L \times \left(\chi^\dagger\,\mu\vec{\sigma}\,\chi\right)\right]_j \quad , \tag{18}$$

or equivalently

$$i\frac{\partial}{\partial t}\chi(\theta,\phi)=\frac{\mu\vec{B}}{2S}\cdot\vec{\sigma}\,\chi(\theta,\phi) \quad , \tag{19}$$

where we have used our equation (11), and the definitions (12) and (13). Notice that $\vec{\omega_L}\equiv\mu\vec{B}/S$.

This classical equation is very interesting. It can be cast in a form which is identical to the Pauli-Schrödinger equation for a magnetic dipole precessing at rest in a magnetic field $\vec{B}$. Multiplying both sides of (19) by $\hbar$ and using equation (1) we get

$$\begin{aligned} i\hbar\frac{\partial\chi(\theta,\phi)}{\partial t} &= \frac{egS}{2mc}\vec{\sigma}.\vec{B}\,\chi(\theta,\phi)\left(\frac{\hbar}{2S}\right) \\ &= \frac{eg}{2mc}\left(\frac{\hbar}{2}\,\vec{\sigma}\right).\vec{B}\,\chi(\theta,\phi) \quad . \end{aligned} \tag{20}$$

It is remarkable that this occurs for an *arbitrary* magnitude of the spin vector $\vec{S}$. Moreover, (20) is clearly seen to be independent of $\hbar$. It is also possible to show that (20) is valid if $\vec{B}$ is time dependent (see section II of ref. [12]).

IV. DERIVATION OF THE PAULI-SCHRÖDINGER TYPE EQUATION FROM THE LIOUVILLE EQUATION

According to the classical dynamical equations, the phase space evolution of an ensemble of particles is described by the instantaneous phase space distribution which will be denoted by

$$W=W(\vec{r},\vec{p},t) \quad . \tag{21}$$

This function is associated with a particle with momentum $\vec{p}=(p_1,p_2,p_3)$, located at the point $\vec{r}=(x_1,x_2,x_3)$ and with a given magnetic dipole moment orientation θ_0 with respect to the local magnetic field $\vec{B}$ (see Fig. 2). The instantaneous variation of $\vec{r}$ and $\vec{p}$ is governed by the equations (2) and (3) or (7) and (8).

The associated Liouville equation will be written as

$$\left[\frac{\partial}{\partial t}+\dot{\vec{r}}\cdot\frac{\partial}{\partial\vec{r}}+\dot{\vec{p}}\cdot\frac{\partial}{\partial\vec{p}}\right]W=0 \quad , \tag{22}$$

and its evolution can be obtained from the solutions of equations (2) and (3) for $\dot{\vec{r}}$ and $\dot{\vec{p}}$. The equation (22), however, does not describe the precession (see (4) or (9)). Nevertheless, we shall see that eq. (22) is sufficiently accurate for our purposes.

We shall present a method for studying the mathematical problem of finding some solutions of (22), which was proposed recently by Dechoum and França [6] for the case of spinless particles. This method is based on an appropriate modification of the original proposal introduced by Wigner [3] in 1932.

Let us define a Fourier transform $Q(\vec{r},\vec{y},t)$, which is associated with $W(\vec{r},\vec{p},t)$, by

$$Q(\vec{r},\vec{y},t)\equiv\int d^3p\,W(\vec{r},\vec{p},t)\,e^{-2i\vec{p}.\vec{y}/\hbar'} \quad . \tag{23}$$

Here $\vec{y}$ is another point in configuration space, and $\hbar'$ is a free parameter with dimension of action. It is assumed that $\hbar'$ is very small ($\hbar'\ll\hbar$ for instance), and the limit $\hbar'\to 0$ will be taken in the end of the calculation. Therefore, one can conclude that $Q(\vec{r},\vec{y},t)\neq 0$ only for very small values of $|\vec{y}|$. It is important to remark that the initial orientation angles θ_0 and ϕ_0 (see Fig. 1 and Fig. 2) are being considered as *independent* variables.

The evolution equation for $Q(\vec{r},\vec{y},t)$ can be obtained easily. After substituting (23) into (22), we get

$$\left\{-i\hbar'\frac{\partial}{\partial t}+\frac{(\hbar')^2}{2M}\frac{\partial^2}{\partial\vec{y}\cdot\partial\vec{r}}+2\vec{y}\cdot\frac{\partial}{\partial\vec{r}}\left[\vec{\mu}\cdot\vec{B}(\vec{r})\right]\right\}Q=0 \quad , \tag{24}$$

where we have used (2) and (3).

Since $Q(\vec{r},\vec{y},t)\neq 0$ only for $|\vec{y}|$ small, it is possible to write

$$2\vec{y}\cdot\frac{\partial}{\partial\vec{r}}[B_j(\vec{r})]=B_j(\vec{r}+\vec{y})-B_j(\vec{r}-\vec{y}) \quad . \tag{25}$$

Therefore,

$$2\vec{y}\cdot\frac{\partial}{\partial\vec{r}}\left[\vec{\mu}\cdot\vec{B}(\vec{r})\right]=\mu\left[\chi^{\dagger}(\theta_0,\phi_0)\,\vec{\sigma}\,\chi(\theta_0,\phi_0)\right]\cdot\left[\vec{B}(\vec{r}+\vec{y})-\vec{B}(\vec{r}-\vec{y})\right]\quad, \tag{26}$$

when $|\vec{y}|\to 0$. In the last equality we have used our previous equation (14).

We shall study only those Fourier transforms $Q(\vec{r},\vec{y},t)$ which can be written as

$$Q(\vec{r},\vec{y},t)=\Psi^{\dagger}(\vec{r}+\vec{y},t|\theta_0,\phi_0)\Psi(\vec{r}-\vec{y},t|\theta_0,\phi_0)\quad, \tag{27}$$

where (see (12))

$$\Psi(\vec{r},t|\theta_0,\phi_0)\equiv\chi(\theta_0,\phi_0)\Phi(\vec{r},t)\equiv\Psi_u+\Psi_d\quad, \tag{28}$$

and $\Phi(\vec{r},t)$ is a scalar function.

A more general expression for $Q(\vec{r},\vec{y},t)$ is

$$Q(\vec{r},\vec{y},t)=\sum_k\sum_l A_{kl}(t)G_{kl}(\vec{r},\vec{y})\quad, \tag{29}$$

where $\{G_{kl}\}$ is a complete set of orthogonal functions (or states) indicated by the indices k and l. A differential equation for the coefficients A_{kl} can be obtained from (22). Therefore, there is no loss of generality in using the hypothesis (27), provided the complete set of (Fourier transformed) "phase space" states $\{G_{kl}\}$ is introduced in a later stage of the calculation (see ref. [16] for a similar procedure applied to the case of spinless particles).

Using (26), (27), (28) and the fact that $\chi^{\dagger}(\theta,\phi)\chi(\theta,\phi)\equiv 1$, it is straightforward to show that (24) leads to

$$\left[i\hbar'\frac{\partial}{\partial t}+\frac{(\hbar')^2}{2M}\nabla^2-\mu\vec{\sigma}\cdot\vec{B}(\vec{r})\right]\Psi(\vec{r},t|\theta_0,\phi_0)=0\quad. \tag{30}$$

It is interesting to notice that there is a direct correspondence between each term of (22) and (30). For instance, the Schrödinger type operator $[(\hbar')^2/2M]\nabla^2$ has its origin in the convective operator $\dot{\vec{r}}\cdot\dfrac{\partial}{\partial\vec{r}}$ of the classical Liouville equation. For $\hbar'=\hbar$, the

above equation is known as the Pauli-Schrödinger equation. It is clear from the above derivation that the equation (30) gives an *approximate* description of the classical motion of the spinning particle.

The statistical interpretation of the function $\Psi(\vec{r},t|\theta_0,\phi_0)$ is also obtained from the phase space distribution $W(\vec{r},\vec{p},t)$ and the normalization condition

$$\int d^3r\int d^3p\,W(\vec{r},\vec{p},t)=\int d^3r|\Psi(\vec{r},t|\theta_0,\phi_0)|^2=$$
$$=\int d^3r\left(|\Psi_u|^2+|\Psi_d|^2\right)=1\quad, \tag{31}$$

as it is easy to verify (see also the original paper by Pauli [15]).

A Schrödinger type equation, similar to (30), but for a spinless charged particle, bounded by a harmonic force (frequency ω_0), was obtained by Dechoum and França [6] within the realm of SED [1]. The zero-point electric field associated with the vacuum fluctuations was included in their approach. Therefore, it was possible to show that, in the limit $\hbar'\to 0$, the oscillator has an average energy of $\hbar\omega_0/2$. The presence of the Planck's constant in this result is due to the effects of the zero-point background radiation field. The mathematical interpretation of the harmonic oscillator excited states (solutions of the time independent Schrödinger type equation) was also discussed by Dechoum and França [6] and by França and Marshall [16].

The equation (30) is valid for a general $\vec{B}(\vec{r})$, provided that in the end of the calculation the limit $\hbar'\to 0$ is considered. Notice that the thermal and zero-point electromagnetic fields are *not* included in (30).

V. CLASSICAL DESCRIPTION OF A STERN-GERLACH TYPE EXPERIMENT

We shall obtain here an approximate solution of the classical (Pauli-Schrödinger type) equation (30) in the particular case in which the magnetic field $\vec{B}$ is such that (see Fig. 2)

$$\vec{B} = (-\beta x, 0, B_0 + \beta z) \tag{32}$$

for $0 \le y \le l$. The parameters β and B_0 are constants, characteristic of each experiment. The field is assumed to be zero for $l < y \le D$, where D is the distance from the magnet to the screen (or detector) with $l << 2D$.

This non uniform magnetic field gives an approximate description of the experimental situation encountered in the Stern-Gerlach type devices [17–19]. Moreover, it is easy to see from (3) that the non uniform field (32) generates different forces on the particles of the beam, depending on their position at the entrance of the Stern-Gerlach magnet, and the orientation of the vector $\vec{\mu}$ (see Fig. 2).

According to (15) and (32) the acceleration at the entrance of the magnet is such that

$$M\ddot{z} = \mu \cos\theta_0 \frac{\partial B}{\partial z} = \beta\ \mu_z(0)\ \frac{B_z^2}{B^2} + \beta\ \mu_x(0)\frac{B_x B_z}{B^2} \quad , \tag{33}$$

and

$$M\ddot{x} = \mu \cos\theta_0 \frac{\partial B}{\partial x} = -\beta\ \mu_x(0)\ \frac{B_x^2}{B^2} - \beta\ \mu_z(0)\frac{B_x B_z}{B^2} \quad , \tag{34}$$

where $\mu_x(0) \equiv \mu \sin\theta' \cos\phi'$, and $\mu_z(0) = \mu\cos\theta'$ are the components of the vector $\vec{\mu}$ at the entrance of the magnet (time $t = 0$). The angles θ' and ϕ' are the polar and the azimutal angles relative to the x, y, z coordinate system (see Fig. 2 and ref. [20]). Notice that, due to the last term in (33), $\ddot{z} \neq 0$ if $\mu_z(0) = 0$. In this case $\ddot{z}$ can take positive or negative values depending on the sign of $\mu_x(0) = \mu\cos\phi'$.

The solutions of the above non-linear equations, with the appropriate initial conditions characterizing the beam [17–19], will not be discussed here. We shall see in the following that an approximate solution of the Pauli-Schrödinger type equation (30) can be more easily constructed (in comparison with (22)). One reason for this fact is that (30) depends on the *energy* $(-\vec{\mu}\cdot\vec{B})$, whereas the original Liouville equation (22) depends on the *non-linear force* with components given by (33) and (34). Another reason is that the phase space distribution W has more dynamical variables than the auxiliary function Ψ.

A. Motion inside the magnet ($\Psi = \Psi_1$)

We shall assume that the magnetic particle is heavy (a cesium or a silver atom for instance), and spend a short time l/v_0 inside the magnet (v_0 is the velocity of the particles in the y direction, that is, $y = v_0 t$). Therefore, one can neglect the transversal convective contribution in (30), that is, we shall take (see also ref [10])

$$\frac{(\hbar')^2}{2M}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\ \Psi_1 \simeq 0 \quad . \tag{35}$$

According to (30) one can write

$$i\hbar'\frac{\partial \Psi_1(t)}{\partial t} = \mu\sigma_3 B \Psi_1(t) \quad , \tag{36}$$

where $\Psi_1(t)$ is a short notation for $\Psi_1(x, z, t)$, $B = B(x, z) = |\vec{B}|$ and σ_3 is the Pauli matrix (13). Considering the definition (28), and assuming that $\Psi_1(0) = \chi(\theta_0, \phi_0)\Phi(x, z)$, the equation (36) can be easily integrated giving

$$\Psi_1(t) = \Phi(x, z)\left\{\cos\frac{\theta_0}{2}\exp\left[-\frac{i}{2}\left(\phi_0 + \frac{2\mu B t}{\hbar'}\right)\right]\begin{pmatrix}1\\0\end{pmatrix} + \right.$$
$$\left. + \sin\frac{\theta_0}{2}\exp\left[\ \frac{i}{2}\left(\phi_0 + \frac{2\mu B t}{\hbar'}\right)\right]\begin{pmatrix}0\\1\end{pmatrix}\right\} \quad . \tag{37}$$

The function $\Phi(x, z)$ is related to the cross section of the beam of spinning atoms, conveniently prepared by the experimentalist [13,17,18], and the angles θ_0 and ϕ_0 define the polarization of the particles at the entrance of the magnet. Here $0 \le t \le t_1 = l/v_0$. Notice that

$$\phi_0 + \frac{2\mu B}{\hbar'}t \neq \phi(t) \quad , \tag{38}$$

defined in (11), so that the classical precession is not described by equation (37). For an unpolarized beam of particles the angles θ_0 and ϕ_0 are *random* variables such that $0 \le \theta_0 \le \pi$ and $0 \le \phi_0 \le 2\pi$.

B. Free motion from the magnet to the screen ($\Psi = \Psi_2$)

The screen is situated far enough from the Stern-Gerlach type magnet in order to allow the physical splitting of the beam. Notice that, in practice, a small splitting already occurs inside the magnet, and it is due to two factors: 1) the initial beam is such that $\langle p_z \rangle = 0$ but there are particles, in the ensemble, with positive and negative values of the momentum p_z; 2) the space variation of $\vec{B}(\vec{r})$ generates different forces on the particles of the beam, depending on its position and orientation inside the magnet (see (33), (34) and also Fig. 2).

As before, the motion in the y direction is simply $y = v_0 t$ and the equation for Ψ_2 is such that (see ref. [4])

$$i\hbar' \frac{\partial \Psi_2(t)}{\partial t} = -\frac{(\hbar')^2}{2M} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) \Psi_2(t) \quad , \tag{39}$$

where $\Psi_2(t)$ is a simplified notation for $\Psi_2(x, z, t)$.

We shall take $\Psi_2(0) = \Psi_1(t_1)$ where $\Psi_1(t)$ is given by (37). We shall assume that $B_0 \gg \beta|z|$ and $B_0 \gg \beta|x|$, that is, $B \simeq B_0 + \beta z$, and we shall take

$$\Phi(x, z) \simeq (4\pi^2 \alpha_1 \alpha_2)^{-1/4} \exp\left(-\frac{z^2}{4\alpha_1^2}\right) \exp\left(-\frac{x^2}{4\alpha_2^2}\right) \tag{40}$$

in expression (37). The parameters α_i are related to the width of the beam (usually $\alpha_1 \ll \alpha_2$). Typical values are described in references [13], [17] and [18].

Under the above hypotheses, the integration of (39) is lengthy but straightforward. The result can be written as $\Psi_2(t) = \cos(\theta_0/2) f_+(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin(\theta_0/2) f_-(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with

$$f_\pm(t) \simeq \frac{(4\pi^2 \sigma_x^2 \sigma_z^2)^{-1/4}}{2} \exp\left[-\frac{(z \mp z_c(t))^2}{4\sigma_z^2} \mp i \left(\frac{\mu B_0 t_1}{\hbar'} + \frac{\phi_0}{2} \right) \right] \times$$

$$\times \exp\left(\frac{-x^2}{4\,\sigma_x^2} \right) \quad , \tag{41}$$

where $\sigma_x \equiv \alpha_2 \left(1 + \frac{\hbar'^2 t^2}{4M^2 \alpha_2^2} \right) \simeq \alpha_2 \quad , \quad \sigma_z \equiv \alpha_1 \left(1 + \frac{\hbar'^2 t^2}{4M^2 \alpha_1^2} \right) \simeq \alpha_1$, and

$$z_c(t) \equiv \frac{\mu \beta t_1 t}{M} \quad . \tag{42}$$

For the sake of simplicity we have dropped from equation (41) some factors that tend to 1 in the limit $\hbar' \to 0$.

Taking $t = t_2 = D/v_0 > t_1 = l/v_0$ in equation (42) we obtain $z_c = \frac{\mu \beta l D}{M v_0^2}$, to be compared with $\Delta z = z_c\, (1 + \frac{l}{2D})$ given in equation (10-5) of ref [13]. The additional term $\frac{l}{2D}$ is the effect of the acceleration inside the magnet which has been neglected by us because we have assumed $\frac{l}{2D} << 1$. In the case of the experiment with unpolarized beam of cesium atoms by Zacharias (see [13]) $\frac{l}{2D} = 0.125$.

One can calculate the normalized distribution of particles on the screen, generated by a beam of unpolarized atoms. In the limit $\hbar' \to 0$ it is given by

$$I(x, z) = \frac{1}{4\pi} \int_0^{2\pi} d\phi_0 \int_0^{\pi} \sin\theta_0 \, d\theta_0 \lim_{\hbar' \to 0} \left\{ |\Psi_2(t_2)|^2 \right\} =$$

$$= \frac{1}{8\pi \alpha_1 \alpha_2} \left\{ \exp\left[-\frac{(z - z_c)^2}{2\alpha_1^2} \right] + \exp\left[-\frac{(z + z_c)^2}{2\alpha_1^2} \right] \right\} \exp\left(-\frac{x^2}{2\alpha_2^2} \right) \quad , \tag{43}$$

corresponding to two peaks separated by the distance $2z_c$.

As expected on physical grounds, our result is in good agreement with the experiment described by J.R. Zacharias (see [13] and Fig. 3 A), despite the various approximations used to obtain it. For comparison see also the experimental results of Esterman et al. [19] and Fig. 3 B.

An interesting observation is that $I(x, z)$ does not depend on the Planck's constant $\hbar$ used in the derivation of the Heisenberg equations presented within section 2. Moreover we can conclude that the results of the Stern-Gerlach type experiments do not allow us to infer the "directional quantization" in a magnetic field [17]. It should be stressed that the results of the sections 3, 4 and 5 are valid for an *arbitrary* magnitude of the spin vector $\vec{S}$.

VI. DISCUSSION

We have shown that it is possible to give a physical interpretation to the Pauli-Schrödinger equation for a neutral spinning particle based on classical trajectories and continuous orientation angles of the spin. This classical interpretation is valid for any magnitude of the spin $|\vec{S}|$ and magnetic dipole $|\vec{\mu}|$. The Pauli-Schrödinger type equation obtained in section 4 does not depend on the Planck's constant $\hbar$, and was derived from the classical Liouville equation in phase space. The spin vector $\vec{S}$ and the magnetic dipole $\vec{\mu}$ are not quantized, and exhibit orientation angles θ and ϕ which vary continuously (see section 3). An adequate physical description of a Stern-Gerlach type experiment was provided by the equations (30) and (43) derived from the classical Liouville equation (22). It should be mentioned that the concept of classical trajectory has also been recently used to describe the Stern-Gerlach effect for electron beams [21].

The classical equation (4), which has the spinorial form (20), is independent of $\hbar$, both being valid for a time dependent magnetic field also. Let us consider that this magnetic field is given by

$$\vec{B} = (B_1 \cos\omega t,\ B_1 \sin\omega t,\ B_0) \quad , \tag{44}$$

where ω, B_1 and B_0 are constants. Therefore, using (4) or (20) and defining $\gamma \equiv eg/2mc$, and $\omega_0 \equiv \gamma B_0$, it is possible to show that the angle $\theta(t)$ between $\vec{\mu}$ and z axis is given by

$$\frac{\mu_z(t)}{\mu} = \cos\theta(t) = 1 - \left[\frac{2(\gamma B_1)^2}{(\omega-\omega_0)^2+\gamma^2 B_1^2}\right] \sin^2\left(\frac{t}{2}\sqrt{(\omega-\omega_0)^2+\gamma^2 B_1^2}\right) \quad . \tag{45}$$

This result was firstly derived by Rabi [22], and Schwinger [23] using the Pauli equation (20). It was soon recognized, by Rabi, Ramsey and Schwinger [24], that these Rabi resonant oscillations are equally obtained from the classical or the quantum mechanical approaches. A good explanation of this equivalence is also provided by Bloembergen

[25]. The conclusion is that the magnetic dipole vector (and also the spin vector) have orientation angles which vary *continuously* with respect to the applied magnetic field. The beautiful experiments by Stern (Nobel prize 1943 for the discovery of the proton magnetic moment (see Fig. 3 B)), by Rabi (Nobel prize 1944 for the discovery of the resonance method to record the magnetic properties of the atomic nuclei), and collaborators, are the most striking confirmation of our statement (see refs. [17–19] and [22–25]).

The small forces generated by the radiation reaction, and the zero-point (and thermal) fluctuations of the eletromagnetic field, were neglected in our paper. Their effects only appear in the equilibrium (stationary) regimen. This was shown previously by Boyer [26] and by Barranco et al. [27]. According to these authors the equation of motion (4) is modified to

$$\dot{\vec{S}} = \vec{\mu} \times \vec{B}_0 + \vec{\mu} \times \vec{B}_{VF}(t) - \frac{2}{3c^3}\vec{\mu} \times \dddot{\vec{\mu}} \quad , \tag{46}$$

where $\vec{B}_0$ is a constant magnetic field in the z direction, and $\vec{B}_{VF}$ is the random magnetic field characteristic of SED [1]. The last term in (46) is the self reaction torque. Equation (46) is known as the stochastic Bhabha equation. For the sake of simplicity in references [26,27] the authors have considered only the case of $\vec{\mu}$ and $\vec{S}$ parallel.

The random magnetic field $\vec{B}_{VF}$ has an average value such that $\langle \vec{B}_{VF} \rangle = 0$ and

$$\frac{1}{4\pi}\langle \vec{B}_{VF}(t) \cdot \vec{B}_{VF}(0) \rangle = \int_0^\infty d\omega \frac{\hbar\omega^3}{2\pi^2 c^3} \coth\left(\frac{\hbar\omega}{2kT}\right) \cos(\omega t) \quad , \tag{47}$$

where $\hbar$ is the Planck's constant and T is the temperature.

According to Boyer [26], and Barranco et al. [27], the orientation angles θ and ϕ of the vector $\vec{\mu}(t)$ vary *continuously*. Therefore, the paramagnetic behaviour of the particle can be calculated according to the classical SED approach. The average value of $\mu_z(t)$ is

$$\begin{aligned} \langle \mu_z \rangle &= \frac{egS}{2mc}\int_0^\pi d\theta \cos\theta P(\theta) = \\ &= \frac{ge\hbar}{2mc}\left\{\frac{S}{\hbar}\coth\left(\frac{2S/\hbar}{\coth\left(\frac{\hbar\omega_0}{2kT}\right)}\right) - \frac{1}{2}\coth\left(\frac{\hbar\omega_0}{2kT}\right)\right\} \quad , \end{aligned} \tag{48}$$

where $\omega_0 = \mu B_0/S$, and $P(\theta)$ is the orientation probability calculated by Boyer [26] based on a Fokker-Planck equation involving the variable θ. Notice that this result depends crucially on the Planck's constant $\hbar$. Its origin can be traced back to the thermal and zero-point electromagnetic noise, whose spectral distribution is given by (47).

The approach of Boyer [26], and Barranco et al. [27], shows that the quantization of $\vec{S}$ is not necessary to give a very good account of the experimentally observed paramagnetic behaviour of the magnetic particles (see figure 2 in ref. [27]). Therefore, the SED proposal, gives a simple picture of the spinning particle, where the classical and the quantum approaches merge into very similar equations and results (see [1] and [28]). We think, however, that in certain special cases the classical approach, based on SED, presents some technical advantages when compared with the quantum approach. One example is the predicted (not observed yet) "anomalous" paramagnetic behaviour, which is generated when the paramagnetic sample is influenced by the zero-point current fluctuations of a simple RLC circuit [29]. In this particular case, the *continuous* variation of the orientation angle of the magnetic dipole is an essential calculation tool that leads to a Fokker-Planck equation [26,27,29]. Another interesting example is the detailed SED description of the Casimir interaction between the inductor of the RLC circuit and a polarizable molecule that is close to its coils [30]

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FIGURES

FIG. 1. The orientation angles of the vector $\vec{\mu}(t)$ which precesses around the magnetic field $\vec{B}$.

FIG. 2. Schematic picture of the precession in a non uniform magnetic field $\vec{B}$. The spinorial notation for the force components acting on the neutral particle are presented in equations (15), (16), (17), and the classical Larmor precession is described by equation (20). The particles in the beam (shaded area) move with velocity v_0 in the y direction.

FIG. 3. A: Beam profiles obtained by J. R. Zacharias [13]. Curve (a) shows the spreading of the beam with a low magnetic field. The gradient field β is not great enough to cause separation of the beam. Curve (b) shows separation in the high field gradient. See our expression (43) for comparison. Notice the continuous variation from curve (a) to curve (b) as a function of $\beta = \partial B_z/\partial z$. B: Beam profiles (intensity in arbitrary units) showing the magnetic field deflection of a beam of HD molecules. The experimental data was used to measure the magnetic moment of the proton (see ref.[19]).

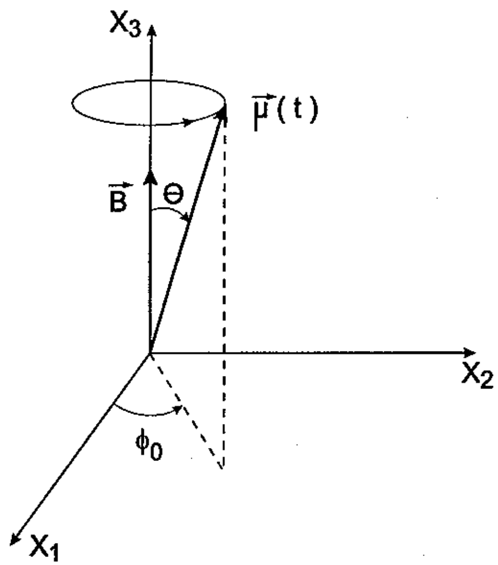

FIG. 1

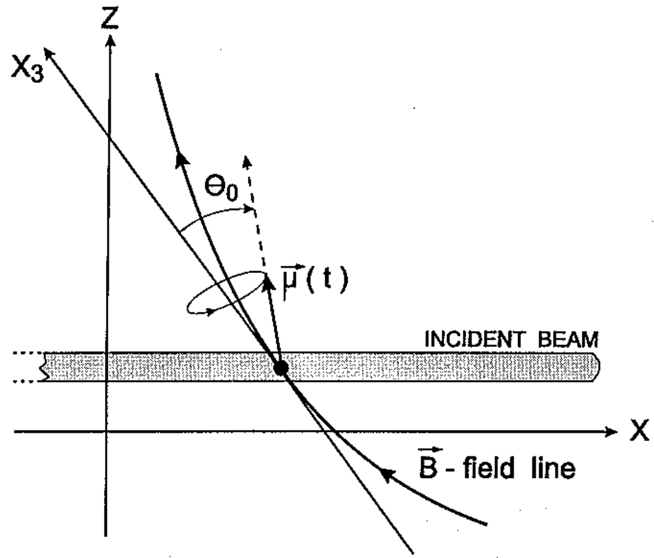

FIG. 2

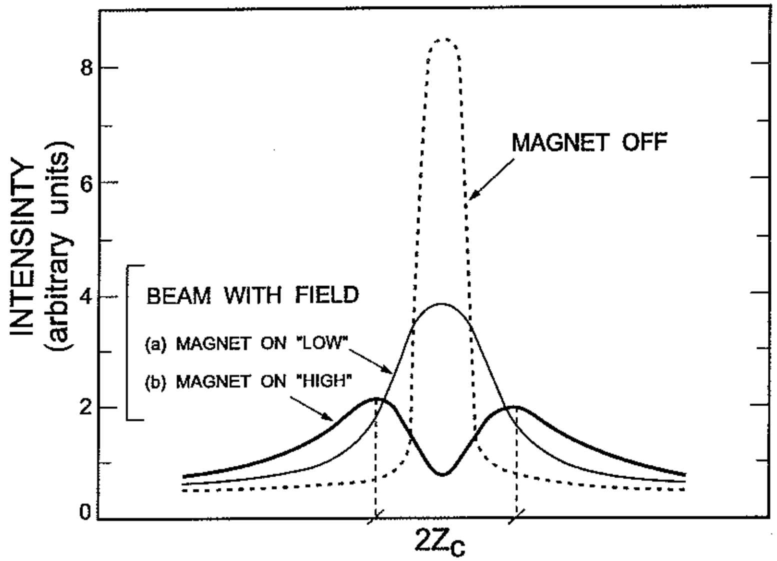
INTENSINTY
(arbitrary units)
8
6
4
2
0
MAGNET OFF
BEAM WITH FIELD
(a) MAGNET ON "LOW"
(b) MAGNET ON "HIGH"
2Z_C

FIG. 3A

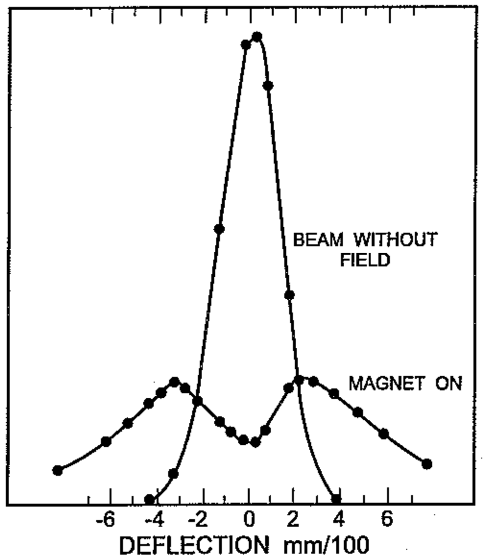
BEAM WITHOUT
FIELD
MAGNET ON
-6 -4 -2 0 2 4 6
DEFLECTION mm/100

FIG. 3B