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INITIAL-VALUE PROBLEM IN QUANTUM FIELD THEORY: AN APPLICATION TO THE RELATIVISTIC SCALAR PLASMA

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ABSTRACT

A framework to describe the real-time evolution of interacting fermion-scalar field models is set up. On the basis of the general dynamics of the fields, we derive formal equations of kinetic-type to the set of one-body dynamical variables. A time-dependent projection technique is used then to generate a nonperturbative mean-field expansion leading to a set of self-consistent equations of motion for these observable, where the lowest order corresponds to the Gaussian approximation. As an application, we consider an uniform system of relativistic spin-1/2 fermion field coupled, through a Yukawa term, to a scalar field in 3+1 dimensions, known as quantum scalar plasma. The renormalizability for the Gaussian mean-field equations, both static and dynamical, are examined and initial conditions discussed. We also investigate solutions for the gap equation and show that the energy density has a single minimum.

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I. Introduction

Over the last two decades, the interest in the real-time evolution for relativistic quantum field models stems mainly from two different areas of physics. On the one hand, inflationary cosmological models provide very appealing scenario to describe the early stage of our universe. The essential ingredient is to study the time evolution of a scalar field, which undergoes a second-order phase transition from a high-temperature symmetric phase to a low-temperature broken phase [1]. On the other hand, properties of hadronic matter manifest themselves through transient phenomena in globally off-equilibrium situations [2]. Notably, in the ultra-relativistc heavy-ion experiments there is large concentration of energy in the collision region. During the relaxation process, may hapen that the chiral condensate misaligns with the vacuum state [3]. This phenomenon might soon be probed in Brookhaven Relativistic Heavy Ion Collider (RHIC) or the CERN large Hadron Collider (LHC). In either of two context nonperturbative methods must be employed, and any microscopic models will involve a set of mutually interacting quantum fields, which can be thought of as interacting subsystem. At the other end of energy scale, interest in condensate matter [4], quantum optics [5] and theorists working on the fundamentals of quantum measurement process fall into this category as well [6].

For this class of problems, one typically tries to obtain and solve equations describing the kinetic behavior of a particular, "relevant" subsystem or of a restricted set of "relevant" observables of a more comprehensive autonomous system. Such is indeed the case, e.g., of the scalar driving field in the inflationary scenario [7] and of one-body densities [8] and certain correlation functions in heavy-ion collisions [9]. The problem thus can be stated very generally by considering several subsystems and asking the effective dynamics of a particular subsystem. The quantum state of each subsystem can be described in terms of a reduced density operator which will in general evolve *non-unitarily* on the account of correlations and decoherence effects involving different subsystems [10]. The non-unitary effects will manifest themselves specifically through the dynamical evolution of the eigenvalues of the subsystem reduced density matrices. As consequence, each individual subsystem behaves in general in a *nonisoentropic* fashion. Such complex picture is considerably reduced whenever one is able to find physical grounds to motivate a mean-field like approximation, which assume isoentropic evolution governed by effective, time-dependent Hamiltonian operador for each subsystem [11]. This method, with different denominations, has a long history in such diverse areas as atomic physics (Born-Oppenheimer), nuclear physics (Hartree-Fock-Bogoliubov), condensed matter (BCS), statistical physics (Landau-Ginzburg) and quantum optics (coherent or squeezed states). Because no higher than second moments of fluctuaction are incorporated in its implementation, the mean-field approximation is related to a Gaussian-like ansatz for the wavefunction in the variational calculations [12].

In the field-theoretical context, this has been implemented through the use of a Gaussian for the subsystem density functional in the framework of a time-dependent variational principle supplying the appropriate dynamical information, notably for bosonic fields [13]. Actually, the Gaussian *Ansatz*, having the form of a exponential of a quadratic form in the field operators, implies that many-point correlation functions can be factored in terms of two-point functions. This is well known in the context of the derivation of the Hartree-Fock approximation to the nonrelativistic many-body problem [14]. This factorization has been used by Chang to implement the Gaussian approximation for the $\lambda\phi^4$ theory [15] . The dynamics of the reduced two-point density then itself becomes isoentropic, since irreducible higher-order correlation effects are neglected.

An alternative method to improve the Gaussian approximation within the context of the self-interacting $\lambda\phi^4$ theory was proposed recently [16]. The approach basically follows a time-dependent projection technique discussed some time ago by Willis and Picard [17] in the context of master equation for coupled systems and extended later by Nemes and de Toledo Piza to study nonrelativistic many-fermion dynamics [18]. The method consists essentially in writting the correlation information of the *full density* in terms of a memory kernel acting on the uncorrelated density with the help of a time-dependent projector. At this point, an systematic mean-field expansion for two-point correlations can be perfomed. The lowest order recovers the results of the usual Gaussian mean-field approximation. The higher orders describe the dynamical correlation effects between the subsystems and are expressed through suitable memorial integrals added to the mean-field dynamical equations. Thus, the resulting equations acquire the structure of kinetic equations, with the memory integrals performing as *collisional* dynamics terms which eliminate the isoentropic constraint. Numerically, few lowest orders of these equations are treatable and the results have been shown that the collisional terms are able to correct partially both, quantitavely and qualitatively, the failures presented in the mean-field approximation [16, 19]. In a recent paper, Natti and de Toledo Piza have extended the method to study relativistic fermion field theories [20]. In particular, they have considered chiral Gross-Neveu model in $1+1$-dimensions [21], obtaining a set effective dynamical equations in Gaussian approximation. The calculation beyond of the Gaussian approximation to this case is currently in progress.

With sucess of the applications of the method in the previous contexts it is natural to

try to extend the discussion in a system involving both scalar and fermion fields. As a first step towards this end, Takano Natti and de Toledo Piza [22] have obtained relevant dynamics for the Jaynes-Cummings Hamiltonian, a well known model in quantum optics [23]. Main motivation stems from the fact that it is a soluble model and thus provides a clear understanding of the physical phenomena involved. The exact numerical result are then useful in controlling various approximations necessary for the treatment of more realistic case. Furthermore, the model can be seen as $0+1$ dimensional quantum field theory known as relativistic scalar plasma [24] in zero spatial dimensions. In [22] one has verified that the inclusion of the present approximation of the collision effects in the dynamical equations is not only fundamental to generate the qualitative behavior associated with the decorrelation process related to the initial damping of the Rabi oscillations, but also successfully describes such effects quantitatively over a time span covering at least several Rabbi periods.

In continuation of previous works, we report in this paper an aplication of the technique to describe the real-time evolution for relativistic scalar plasma model in $3+1$ dimensions. This is non-trivial, renormalizable model for which many results are available in the literature so that it offers suitable testing ground for the proposed approach. It is of course difficult to give, within this model, a definite physical meaning and compare to experimental data. However, it corresponds to one of the simplest quantum-field theoretical models used to discuss the relativistic dense matter in the contexts of heavy-ion collisions and the high-density astrophysical system [24, 25, 26]. We will, therefore, consider this model as a first simple application within the context of interacting boson-fermion system. An outline of the paper is as follows. On the basis of the general dynamics of the fields, we derive in Sec.II a set of formal equations of motion for one-body observables of an interacting boson-fermion system, which are the groundwork for approximation schemes. We shall show in Sec.III that the time-dependent projection technique discussed in previous works can also be applied in this case. Sec. IV will illustre this general scheme in the simplest context of scalar plasma system in Gaussian mean-field (isoentropic) approximation. We discuss also the self-consistent renormalization for the resulting equations in the equilibrium and dynamical situations. In the former case we discuss the solutions for these variational equations and the nonequilibrium initial conditions are discussed in the later context.

II. Gaussian Variables and Their Effective Dynamics

In this section, we shall describe a formal treatment of the kinetics of the bosonic $\phi(x)$ and fermionic $\psi(x)$ interacting quantum fields. The basic idea of our approach is to focus on the time evolution of a restricted set of simple variables. We argue that a large number of relevant physical observables are related to one-body operators. Consequently, we consider as variables of interest the expectation values of linear, $\phi(x)$, and bilinear field operators such as $\phi(x)\phi(x)$, $\bar{\psi}(x)\psi(x)$, $\psi(x)\psi(x)$ and etc. The dynamics of these quantities are kept under direct control when one works variationally using a Gaussian functional *Ansatz* [12], and will, therefore, be referred to as Gaussian observables. In order to keep as close as possible to the formulation appropriate for the many-body problem, we work in fact with expressions which are bilinear in the creation and annihilation parts of the fields in Heisenberg picture. (See sections II of Ref.[16] and Ref.[20] for more details).

II–a. Gaussian Variables

Let us begin by expanding the boson field operators $\phi(x)$ and $\Pi(x)$ as

$$\begin{aligned} \phi(\mathbf{x},t) &= \sum_{\mathbf{p}} \frac{1}{(2Vp_0)^{1/2}} \left[b_{\mathbf{p}}(t)e^{i\mathbf{p}.\mathbf{x}} + b^\dagger_{\mathbf{p}}(t)e^{-i\mathbf{p}.\mathbf{x}}\right] \\ \Pi(\mathbf{x},t) &= i\sum_{\mathbf{p}} \left(\frac{Vp_0}{2}\right)^{1/2} \left[b^\dagger_{\mathbf{p}}(t)e^{-i\mathbf{p}.\mathbf{x}} - b_{\mathbf{p}}(t)e^{i\mathbf{p}.\mathbf{x}}\right] \ , \end{aligned} \qquad (1)$$

where $b^\dagger_{\mathbf{p}}(t)$ and $b_{\mathbf{p}}(t)$ are the usual boson creation and annihilation operators satisfying the standard commutation relations at equal times

$$\begin{aligned} &[b_{\mathbf{p}}(t), b^\dagger_{\mathbf{p}'}(t')]_{t=t'} = \delta_{\mathbf{p},\mathbf{p}'} \\ &[b^\dagger_{\mathbf{p}}(t), b^\dagger_{\mathbf{p}'}(t')]_{t=t'} = [b_{\mathbf{p}}(t), b_{\mathbf{p}'}(t')]_{t=t'} = 0 \ . \end{aligned} \qquad (2)$$

In Eq.(1) V is the volume of the periodic box,

$$(p_0)^2 = \mathbf{p}^2 + \Omega^2 \quad \text{and} \quad px = p_0t - \mathbf{p}.\mathbf{x} \ ,$$

where Ω is the expansion mass parameter which will be fixed later.

We also consider Dirac spin-1/2 fields $\psi(x)$ and $\bar{\psi}(x)$ and are expanded as follows:

$$\begin{aligned} \psi(\mathbf{x},t) &= \sum_{\mathbf{k},s} \left(\frac{M}{k_0}\right)^{1/2} \frac{1}{\sqrt{V}} \left[u_1(\mathbf{k},s)a^{(1)}_{\mathbf{k},s}(t)e^{i\mathbf{k}.\mathbf{x}} + u_2(\mathbf{k},s)a^{(2)\dagger}_{\mathbf{k},s}(t)e^{-i\mathbf{k}.\mathbf{x}}\right] \\ \bar{\psi}(\mathbf{x},t) &= \sum_{\mathbf{k},s} \left(\frac{M}{k_0}\right)^{1/2} \frac{1}{\sqrt{V}} \left[\bar{u}_1(\mathbf{k},s)a^{(1)\dagger}_{\mathbf{k},s}(t)e^{-i\mathbf{k}.\mathbf{x}} + \bar{u}_2(\mathbf{k},s)a^{(2)}_{\mathbf{k},s}(t)e^{i\mathbf{k}.\mathbf{x}}\right] \ , \end{aligned} \qquad (3)$$

where $a^{(1)\dagger}_{\mathbf{k},s}(t)$ and $a^{(1)}_{\mathbf{k},s}(t)$ [$a^{(2)\dagger}_{\mathbf{k},s}(t)$ and $a^{(2)}_{\mathbf{k},s}(t)$] are fermion creation and annihilation operators associated with positive [negative]-energy solutions $u_1(\mathbf{k},s)$ [$u_2(\mathbf{k},s)$] of Dirac's equation. Canonical quantization demands that the creation and annihilation operators satisfy the standard anticommutation relations at equal times

$$\begin{aligned} \{a^{(\lambda)\dagger}_{\mathbf{k},s}(t), a^{(\lambda')}_{\mathbf{k}',s'}(t')\}_{t=t'} &= \delta_{\mathbf{k},\mathbf{k}'}\delta_{s,s'}\delta_{\lambda,\lambda'} \quad \text{for} \quad \lambda,\lambda' = 1,2 \\ \{a^{(\lambda)\dagger}_{\mathbf{k},s}(t), a^{(\lambda')\dagger}_{\mathbf{k}',s'}(t')\}_{t=t'} &= \{a^{(\lambda)}_{\mathbf{k},s}(t), a^{(\lambda')}_{\mathbf{k}',s'}(t')\}_{t=t'} = 0 \ . \end{aligned} \qquad (4)$$

In Eq.(3) we have used the notations

$$(k_0)^2 = \mathbf{k}^2 + M^2 \quad \text{and} \quad kx = k_0t - \mathbf{k}.\mathbf{x}$$

being M the mass parameter for the fermions.

The next step is to identify the Gaussian observables. In general, the state of the system is given in terms of a many-body density operator $\mathcal{F}$ in the Heisenberg picture, a time-independent, non-negative, Hermitian operator with unit trace. Although the procedure is quite general, we will illustre our approach in the simplest context of an unifom system. The extension to inhomogenoeous field configurations is discussed in Ref. [27].

The first variable of interest is the nonvanishing mean values of the boson field

$$\langle\phi(\mathbf{x},t)\rangle = \sum_{\mathbf{p}} \frac{1}{(2Vp_0)^{1/2}} \left[B_{\mathbf{p}}(t)e^{i\mathbf{p}.\mathbf{x}} + B^*_{\mathbf{p}}(t)e^{-i\mathbf{p}.\mathbf{x}}\right] \qquad (5)$$

being

$$B_{\mathbf{p}}(t) \equiv \langle b_{\mathbf{p}}(t)\rangle = Tr_{\text{BF}}\left[b_{\mathbf{p}}(t)\mathcal{F}\right] \ , \quad B^*_{\mathbf{p}}(t) \equiv \langle b^\dagger_{\mathbf{p}}(t)\rangle = Tr_{\text{BF}}\left[b^\dagger_{\mathbf{p}}(t)\mathcal{F}\right] \ . \qquad (6)$$

Here and in what follows the symbol $Tr_{\rm BF}$ denotes a trace over both bosonic and fermionic variables. Partial traces over bosonic or fermionic variables will be written as $Tr_{\rm B}$ and $Tr_{\rm F}$ respectively. In the case of uniform system the mean values of the fields $\phi(\mathbf{x},t)$ are x-independent. This requires

$$B_{\mathbf{p}}(t) \stackrel{\text{uniform}}{\longrightarrow} B_{\mathbf{p}}(t)\delta_{\mathbf{p},0} = Tr_{\rm BF} b_0(t)\mathcal{F} = B(t) \ . \tag{7}$$

Thus, the mean-values of ϕ and Π become

$$\begin{aligned} \langle\phi(t)\rangle &= \frac{1}{\sqrt{2V\Omega}}\left[B(t)+B^*(t)\right] \\ \langle\Pi(t)\rangle &= i\sqrt{\frac{\Omega}{2V}}\left[B^*(t)-B(t)\right] \ . \end{aligned} \tag{8}$$

With the help of (6) and (7) we define the shifted boson operators $d_{\mathbf{p}}(t)$ and $d^\dagger_{\mathbf{p}}(t)$ as

$$d_{\mathbf{p}}(t) \equiv b_{\mathbf{p}}(t) - B(t)\delta_{\mathbf{p},0} \quad \text{with} \quad [d_{\mathbf{p}}(t), d^\dagger_{\mathbf{p}'}(t')]_{t=t'} = \delta_{\mathbf{p},\mathbf{p}'} \ , \tag{9}$$

which by construction have vanishing $\mathcal{F}$-expectation values.

The other Gaussian variables are the mean values of the bilinear forms of boson and fermion operators. As usual [28], we can construct the extended one-boson density matrix $\mathcal{R}_{\rm p}$ for an uniform system as

$$\begin{aligned} \mathcal{R}_{\rm p}(t) &= \begin{bmatrix} \Lambda_{\rm p}(t) & \Xi_{\rm p}(t) \\ \Xi_{-\rm p}(t) & 1+\Lambda_{-\rm p}(t) \end{bmatrix} \\ &= \begin{bmatrix} \langle d^\dagger_{\mathbf{p}}(t) d_{\mathbf{p}}(t)\rangle & \langle d_{\mathbf{p}}(t) d_{-\mathbf{p}}(t)\rangle \\ \langle d^\dagger_{\mathbf{p}}(t) d^\dagger_{-\mathbf{p}}(t)\rangle & \langle d_{\mathbf{p}}(t) d^\dagger_{\mathbf{p}}(t)\rangle \end{bmatrix} \ , \end{aligned} \tag{10}$$

where the one-boson density matrix $\Lambda_{\rm p}$ is hermitian and the pairing density matrix $\Xi_{\rm p}$ is symmetric. In (10) translation invariance has been used so that the densities defined there are diagonal in momentum space, which is no longer true in the case of inhomogeneous systems (see Cap.7 of [28] for details).

For the fermion field we have the following nonvanishing mean values of bilinear for of field operators:

$$\begin{aligned} R_{\mathbf{k},s,\lambda';\mathbf{k},s,\lambda}(t) &= Tr\left\{\left[a^{(\lambda')\dagger}_{\mathbf{k},s}(t)a^{(\lambda)}_{\mathbf{k},s}(t)\right]\mathcal{F}\right\} \quad \text{for} \quad \lambda,\lambda'=1,2 \\ \Pi_{\mathbf{k},s,\lambda';\mathbf{k},s,\lambda}(t) &= Tr\left\{\left[a^{(\lambda')}_{-\mathbf{k},s}(t)a^{(\lambda)}_{\mathbf{k},s}(t)\right]\mathcal{F}\right\} \quad \text{for} \quad \lambda,\lambda'=1,2 \ . \end{aligned}$$

Using these objects we can construct the extended one-fermion density [28] as

$$\begin{aligned} \mathcal{R}_{\mathbf{k},s} &= \begin{bmatrix} R_{\mathbf{k},s}(t) & \Pi_{\mathbf{k},s}(t) \\ -\Pi^*_{\mathbf{k},s}(t) & I_2 - R^*_{\mathbf{k},s}(t) \end{bmatrix} \\ &= \begin{bmatrix} \langle a^{(\lambda')\dagger}_{\mathbf{k},s}(t)a^{(\lambda)}_{\mathbf{k},s}(t)\rangle & \langle a^{(\lambda')}_{-\mathbf{k},s}(t)a^{(\lambda)}_{\mathbf{k},s}(t)\rangle \\ \langle a^{(\lambda')\dagger}_{-\mathbf{k},s}(t)a^{(\lambda)\dagger}_{\mathbf{k},s}(t)\rangle & \langle a^{(\lambda)}_{\mathbf{k},s}(t)a^{(\lambda)\dagger}_{\mathbf{k},s}(t)\rangle \end{bmatrix} \ , \end{aligned}$$

where the hermitian matrix $R_{\mathbf{k},s}$ and the antisymmetric matrix $\Pi_{\mathbf{k},s}$ are the one-fermi density and pairing density, respectively. These together with the quantities defined in E (8) and (10) constitute the selected set of observables and provide an adequate starting poi for our kinetic treatment.

II–b. Quasi-Particle Operators and Bogoliubov Transformation

The procedure to deal with the pairing density consists in reducing the extended o body density defined in (10) and (12) to a diagonal form [29]. This can be achieved by canonical transformation of the Bogoliubov-type. Thus, let us first define the quasi-bos operators as

$$\begin{aligned} \beta_{\mathbf{p}}(t) &= x^*_{\mathbf{p}}(t)\left[b_{\mathbf{p}}(t) - B(t)\delta_{\mathbf{p},0}\right] + y^*_{\mathbf{p}}(t)\left[b^\dagger_{\mathbf{p}}(t) - B(t)\delta_{\mathbf{p},0}\right] \\ \beta^\dagger_{\mathbf{p}}(t) &= x_{\mathbf{p}}(t)\left[b^\dagger_{\mathbf{p}}(t) - B(t)\delta_{\mathbf{p},0}\right] + y_{\mathbf{p}}(t)\left[b_{\mathbf{p}}(t) - B(t)\delta_{\mathbf{p},0}\right] \ . \end{aligned}$$

The coefficient of the transformation can be find by requiring

$$\langle \beta_{-\mathbf{p}}(t)\beta_{\mathbf{p}}(t)\rangle = \langle \beta^{\dagger}_{-\mathbf{p}}(t)\beta^{\dagger}_{\mathbf{p}}(t)\rangle = 0 \ .$$

Notice that from the reflection symmetry one must have $x_{\mathbf{p}}(t) = x_{\mathrm{p}}(t)$, $y_{\mathbf{p}}(t) = y_{\mathrm{p}}(t)$ where p denotes modulus of $\mathbf{p}$. A systematic way to determine $x_{\mathrm{p}}(t)$ and $y_{\mathrm{p}}(t)$ is to solve the following secular problem:

$$G\mathcal{R}_{\mathrm{p}}\mathcal{X}_{\mathrm{p}} = \mathcal{X}_{\mathrm{p}}GN_{\mathrm{p}} \ , \tag{14}$$

where

$$\mathbf{G} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \mathcal{X}_{\mathrm{p}}(t) = \begin{bmatrix} x_{\mathrm{p}}(t) & y^*_{\mathrm{p}}(t) \\ y_{\mathrm{p}}(t) & x^*_{\mathrm{p}}(t) \end{bmatrix} \quad N_{\mathrm{p}}(t) = \begin{bmatrix} \nu_{\mathrm{p}}(t) & 0 \\ 0 & 1+\nu_{\mathrm{p}}(t) \end{bmatrix} , \tag{15}$$

and the eigenvalues $\nu_{\mathrm{p}}(t) = \langle \beta^{\dagger}_{\mathbf{p}}\beta_{\mathbf{p}}\rangle$ are the quasi-boson occupation number. Since the Bogoliubov transformation is canonical one can verify that $\mathcal{X}_{\mathrm{p}}$ satisfies the orthogonality and completeness relation

$$\mathcal{X}^{\dagger}_{\mathrm{p}}G\mathcal{X}_{\mathrm{p}} = \mathcal{X}_{\mathrm{p}}G\mathcal{X}^{\dagger}_{\mathrm{p}} = G \Rightarrow x^2_{\mathrm{p}} - y^2_{\mathrm{p}} = 1 \ . \tag{16}$$

Analogous consideration can be made to the fermionic operators with the eigenvalue problem defined now by

$$\mathcal{X}^{\dagger}_{\mathrm{k},s}\mathcal{R}_{\mathrm{k},s}\mathcal{X}_{\mathrm{k},s} = N_{\mathrm{k},s} \ . \tag{17}$$

In this equation the unitary matrix $\mathcal{X}_{\mathrm{k},s}(t)$ has the following structure:

$$\mathcal{X}_{\mathrm{k},s} = \begin{bmatrix} U^*_{\mathrm{k},s} & V^*_{\mathrm{k},s} \\ V_{\mathrm{k},s} & U_{\mathrm{k},s} \end{bmatrix} , \ U_{\mathrm{k},s} = \begin{bmatrix} U_{11} & 0 \\ 0 & U_{22} \end{bmatrix} , \ V_{\mathrm{k},s} = \begin{bmatrix} 0 & V_{12} \\ V_{21} & 0 \end{bmatrix} . \tag{18}$$

As before k will denote the modulus of $\mathbf{k}$ because of reflection symmetry. The matrices $N_{\mathrm{k},s}(t)$ are diagonal

$$\begin{aligned} N_{\mathrm{k},s} &= \begin{bmatrix} \nu_{\mathrm{k},s} & 0 \\ 0 & I_2 - \nu_{\mathrm{k},s} \end{bmatrix} \\ &= \begin{bmatrix} \langle \alpha^{(\lambda)\dagger}_{\mathbf{k},s}(t)\alpha^{(\lambda)}_{\mathbf{k},s}(t)\rangle & \langle \alpha^{(\lambda)}_{-\mathbf{k},s}(t)\alpha^{(\lambda)}_{\mathbf{k},s}(t)\rangle \\ \langle {\alpha^{(\lambda)}_{-\mathbf{k},s}}^{\dagger}(t){\alpha^{(\lambda)}_{\mathbf{k},s}}^{\dagger}(t)\rangle & \langle \alpha^{(\lambda)}_{\mathbf{k},s}(t){\alpha^{(\lambda)}_{\mathbf{k},s}}^{\dagger}(t)\rangle \end{bmatrix} , \end{aligned} \tag{19}$$

where $\alpha^{(\lambda)}_{\mathbf{k},s}$ are the quasi-fermion operators defined by the Bogoliubov transformation

$$\begin{bmatrix} \alpha^{(1)}_{\mathbf{k},s}(t) \\ \alpha^{(2)}_{\mathbf{k},s}(t) \\ {\alpha^{(1)}_{-\mathbf{k},s}}^{\dagger}(t) \\ {\alpha^{(2)}_{-\mathbf{k},s}}^{\dagger}(t) \end{bmatrix} = \begin{bmatrix} U^*_{\mathrm{k},s} & V^*_{\mathrm{k},s} \\ V_{\mathrm{k},s} & U_{\mathrm{k},s} \end{bmatrix} \begin{bmatrix} a^{(1)}_{\mathbf{k},s}(t) \\ a^{(2)}_{\mathbf{k},s}(t) \\ {a^{(1)}_{-\mathbf{k},s}}^{\dagger}(t) \\ {a^{(2)}_{-\mathbf{k},s}}^{\dagger}(t) \end{bmatrix} . \tag{20}$$

Thus, the elements of the diagonal submatrix $\nu_{\mathrm{k},s}(t)$ can be interpreted as quasi-fermion occupation numbers. Finally, the unitary conditions for $\mathcal{X}_{\mathrm{k},s}(t)$, i.e.,

$$\mathcal{X}^{\dagger}_{\mathrm{k},s}\mathcal{X}_{\mathrm{k},s} = I_4 \quad \text{and} \quad \mathcal{X}_{\mathrm{k},s}\mathcal{X}^{\dagger}_{\mathrm{k},s} = I_4 \ , \tag{21}$$

imply the following constraint equations for $U_{\mathrm{k},s}$ and $V_{\mathrm{k},s}$

$$\begin{aligned} &V_{\mathrm{k},s}V^{\dagger}_{\mathrm{k},s} + U_{\mathrm{k},s}U^{\dagger}_{\mathrm{k},s} = I_2 \quad , \quad V_{\mathrm{k},s}U^{T}_{\mathrm{k},s} + U_{\mathrm{k},s}V^{T}_{\mathrm{k},s} = 0_2 \ , \\ &V^{\dagger}_{\mathrm{k},s}V_{\mathrm{k},s} + U^{T}_{\mathrm{k},s}U^*_{\mathrm{k},s} = I_2 \quad , \quad V^{T}_{\mathrm{k},s}U^*_{\mathrm{k},s} + U^{\dagger}_{\mathrm{k},s}V_{\mathrm{k},s} = 0_2 \ . \end{aligned} \tag{22}$$

In summary, we have obtained in this subsection the transformation rules to the quasi-particle representation, which is more appropriate to treat the two-particle correlations. In fact, with the help of Eqs.(13) and (20) (actually of this inverse) it is then an easy task to express $\phi(x)$, $\psi(x)$ and $\bar{\psi}(x)$, Eqs.(1) and (3), in term of $\beta^{\dagger}_{\mathbf{p}}(t)$, $\beta_{\mathbf{p}}(t)$, ${\alpha^{(\lambda)}_{\mathbf{k},s}}^{\dagger}(t)$ and $\alpha^{(\lambda)}_{\mathbf{k},s}(t)$. In doing so, one finds that the plane waves of $\phi(x)$, $\psi(x)$ and $\bar{\psi}(x)$ are modified by a complex, momentum-dependent redefinition of Ω and M involving the Bogoliubov parameters. The complex character of these parameters is actually crucial in dynamical situations, where the imaginary parts will allow for the description of time-odd (i.e., velocity-like) properties.

II–c. Formal Equation of Motion

What we have achieved so far amounts to obtaining an expansion of the fields $\phi(x)$, $\psi(x)$ and $\bar{\psi}(x)$ such that the mean values in $\mathcal{F}$ of Gaussian observables are parametrized in terms of the Bogoliubov coefficients, $x_{\mathrm{p}}(t)$ and $y_{\mathrm{p}}(t)$, in terms of the elements of $U_{\mathrm{k},s}(t)$ and $V_{\mathrm{k},s}(t)$ matrices and of the occupation numbers $\nu_{\mathrm{p}}(t) = Tr\left(\beta_{\mathrm{p}}^\dagger\beta_{\mathrm{p}}\mathcal{F}\right)$ and $\nu_{\mathrm{k},s}^{(\lambda)}(t) = Tr\left({\alpha_{\mathrm{k},s}^{(\lambda)}}^\dagger \alpha_{\mathrm{k},s}^{(\lambda)}\mathcal{F}\right)$ for $\lambda = 1, 2$. In general, all these quantities are time dependent under the Heisenberg dynamics of the field operators, and we now proceed to obtain the corresponding equations of motion.

We begin with the boson subsystem. The dynamical equation for the mean value of the boson field $\langle\phi(t)\rangle$ results directly of the Heisenberg equation

$$i\langle\dot{\phi}(t)\rangle = Tr_{\mathrm{BF}}[\phi(t), H]\mathcal{F} \ , \tag{23}$$

where H is the Hamiltonian of the system and $\langle\phi(t)\rangle$ is given by (8). Equivalently, we might write (23) in terms of $B(t)$, since its real and imaginary part are proportional to $\langle\phi(t)\rangle$ and $\langle\Pi(t)\rangle$ respectively. In doing so, one has

$$i\dot{B}_{\mathbf{p}}\delta_{\mathbf{p},0} = x_{\mathrm{p}} Tr_{\mathrm{BF}}[\beta_{\mathrm{p}}\delta_{\mathbf{p},0}, H]\mathcal{F} - y_{\mathrm{p}}^* Tr_{\mathrm{BF}}[\beta_{-\mathbf{p}}^\dagger\delta_{-\mathbf{p},0}, H]\mathcal{F} \ , \tag{24}$$

where we have used (13).

For the remaining bosonic Gaussian quantities, we first rewrite the eigenvalue equation (14), using (16), as

$$\mathcal{X}_{\mathrm{p}}^\dagger\mathcal{R}_{\mathrm{p}}\mathcal{X}_{\mathrm{p}} = N_{\mathrm{p}} \ . \tag{25}$$

Taking the time derivative we have

$$\begin{aligned} \mathcal{X}_{\mathrm{p}}^\dagger\dot{\mathcal{R}}_{\mathrm{p}}\mathcal{X}_{\mathrm{p}} &= \dot{N}_{\mathrm{p}} - \dot{\mathcal{X}}_{\mathrm{p}}^\dagger\mathcal{R}_{\mathrm{p}}\mathcal{X}_{\mathrm{p}} - \mathcal{X}_{\mathrm{p}}^\dagger\mathcal{R}_{\mathrm{p}}\dot{\mathcal{X}}_{\mathrm{p}} \\ &= \dot{N}_{\mathrm{p}} - \dot{\mathcal{X}}_{\mathrm{p}}^\dagger G\mathcal{X}_{\mathrm{p}} G N_{\mathrm{p}} - N_{\mathrm{p}} G\mathcal{X}_{\mathrm{p}}^\dagger G\dot{\mathcal{X}}_{\mathrm{p}} \ . \end{aligned} \tag{26}$$

The left hand side of the Eq.(26) can be evaluated using the Heisenberg equation of moti

$$i\mathcal{X}_{\mathrm{p}}^\dagger\dot{\mathcal{R}}_{\mathrm{p}}\mathcal{X}_{\mathrm{p}} = \begin{pmatrix} Tr_{\mathrm{BF}}[\beta_{\mathrm{p}}^\dagger\beta_{\mathrm{p}}, H]\mathcal{F} & Tr_{\mathrm{BF}}[\beta_{\mathrm{p}}\beta_{-\mathrm{p}}, H]\mathcal{F} \\ Tr_{\mathrm{BF}}[\beta_{\mathrm{p}}^\dagger\beta_{-\mathrm{p}}^\dagger, H]\mathcal{F} & Tr_{\mathrm{BF}}[\beta_{\mathrm{p}}\beta_{\mathrm{p}}^\dagger, H]\mathcal{F} \end{pmatrix} \ . \qquad (2$$

The right hand side of the Eq.(26), on the other hand, can be evaluated explicitly using (1 and (15). Equating the result to (27) yields

$$\begin{aligned} \dot{\nu}_{\mathrm{p}} &= Tr_{\mathrm{BF}}[\beta_{\mathrm{p}}^\dagger\beta_{\mathrm{p}}, H]\mathcal{F} & (2 \\ i(1+2\nu_{\mathrm{p}})(\dot{x}_{\mathrm{p}}y_{\mathrm{p}} - x_{\mathrm{p}}\dot{y}_{\mathrm{p}}) &= Tr_{\mathrm{BF}}[\beta_{\mathrm{p}}^\dagger\beta_{-\mathrm{p}}^\dagger, H]\mathcal{F} \ , & (2 \end{aligned}$$

which describe the time evolution of the bosonic variables.

The procedure can, of course, apply to the fermionic variables. The analogous equatio to (26) and (27) in this case read respectively as

$$\mathcal{X}_{\mathrm{k},s}^\dagger\dot{\mathcal{R}}_{\mathrm{k},s}\mathcal{X}_{\mathrm{k},s} = \dot{N}_{\mathrm{k},s} - \dot{\mathcal{X}}_{\mathrm{k},s}^\dagger\mathcal{X}_{\mathrm{k},s}N_{\mathrm{k},s} - N_{\mathrm{k},s}\mathcal{X}_{\mathrm{k},s}^\dagger\dot{\mathcal{X}}_{\mathrm{k},s} \qquad (3$$

$$i\mathcal{X}_{\mathrm{k},s}^\dagger\dot{\mathcal{R}}_{\mathrm{k},s}\mathcal{X}_{\mathrm{k},s} = \begin{bmatrix} Tr\left([{\alpha_{\mathbf{k},s}^{(\lambda')}}^\dagger\alpha_{\mathbf{k},s}^{(\lambda)}, H]\mathcal{F}\right) & Tr\left([\alpha_{-\mathbf{k},s}^{(\lambda')}\alpha_{\mathbf{k},s}^{(\lambda)}, H]\mathcal{F}\right) \\ Tr\left([{\alpha_{-\mathbf{k},s}^{(\lambda')}}^\dagger{\alpha_{\mathbf{k},s}^{(\lambda)}}^\dagger, H]\mathcal{F}\right) & Tr\left([\alpha_{\mathbf{k},s}^{(\lambda')}{\alpha_{\mathbf{k},s}^{(\lambda)}}^\dagger, H]\mathcal{F}\right) \end{bmatrix} . \qquad (3$$

The right-hand side of the Eq.(30) can also be evaluated explicitly using (18-22), namely,

$$i\mathcal{X}_{\mathrm{k},s}^\dagger\dot{\mathcal{R}}_{\mathrm{k},s}\mathcal{X}_{\mathrm{k},s} = \begin{bmatrix} i\dot{\nu}_{\mathrm{k},s} + [\nu_{\mathrm{k},s}, h_{\mathrm{k},s}^*]_- & -g_{\mathrm{k},s}^* + \{\nu_{\mathrm{k},s}, g_{\mathrm{k},s}^*\}_+ \\ -g_{\mathrm{k},s} + \{\nu_{\mathrm{k},s}, g_{\mathrm{k},s}\}_+ & -i\dot{\nu}_{\mathrm{k},s} + [\nu_{\mathrm{k},s}, h_{\mathrm{k},s}]_- \end{bmatrix} \ , \qquad (3$$

where the matrices $h_{\mathrm{k},s}$ and $g_{\mathrm{k},s}$ are given in terms of $U_{\mathrm{k},s}$ and $V_{\mathrm{k},s}$,

$$\begin{aligned} h_{\mathrm{k},s} &= -i(\dot{V}_{\mathrm{k},s}^T V_{\mathrm{k},s}^* + \dot{U}_{\mathrm{k},s}^\dagger U_{\mathrm{k},s}) & (3 \\ g_{\mathrm{k},s} &= -i(\dot{V}_{\mathrm{k},s}^T U_{\mathrm{k},s}^* + \dot{U}_{\mathrm{k},s}^\dagger V_{\mathrm{k},s}) \ . & (3 \end{aligned}$$

From (31) and (32) we obtain dynamical equations which describe the time evolution of tl fermionic Gaussian variables. They read

$$i\dot{\nu}_{\mathbf{k},s} + [\nu_{\mathbf{k},s}, h^*_{\mathbf{k},s}]_- = Tr\left([\alpha^{(\lambda')\dagger}_{\mathbf{k},s}\alpha^{(\lambda)}_{\mathbf{k},s}, H]\mathcal{F}\right) \tag{35}$$

$$-\dot{g}^*_{\mathbf{k},s} + \{\nu_{\mathbf{k},s}, g^*_{\mathbf{k},s}\}_+ = Tr\left([\alpha^{(\lambda')}_{-\mathbf{k},s}\alpha^{(\lambda)}_{\mathbf{k},s}, H]\mathcal{F}\right) . \tag{36}$$

Equations of motion (24), (28), (29), (35) and (36), together with the unitarity conditions (16) and (21), determine the time rate of change of the bosonic and fermionic Gaussian observables in terms of expectation values of appropriate commutators. They are, however, clearly not closed equations when the Hamiltonian H involves interacting fields. In this case, in fact, the time derivatives of the Gaussian observables are given in terms of traces which are not expressible in terms of the Gaussian observables themselves, since they will involve also many-boson (-fermion) densities. This situation can be dealt with in terms of the projection technique reviewed in the next section.

III. Projection Technique and Approximation Scheme

This section discusses the time-dependent projection technique in the context of interacting boson-fermion field models [17]. We shall show that it permits one to obtain closed approximations to the equations of motion obtained in previous section. The framework was developed earlier in the context of nonrelativistic nuclear many-fermion dynamics [18] and was recently applied to the $\lambda\phi^4$ field theory [16]. The formulation leads to a nonperturbative mean-field like expansion for the dynamics of the two-point correlation function and from which one recovers the results of the Gaussian mean-field approximations in lowest order. When the calculation is carried to higher orders one adds dynamical correlation of higher orders to the simplest mean-field approximation.

We begin by decomposing the full density $\mathcal{F}$ as

$$\begin{aligned}\mathcal{F} &= \mathcal{F}_0(t) + \mathcal{F}'(t) \\ &\equiv \mathcal{F}_0^{\mathrm{B}}\mathcal{F}_0^{\mathrm{F}} + \mathcal{F}'(t) ,\end{aligned} \tag{37}$$

where $\mathcal{F}_0(t)$ is a Gaussian *Ansatz* which achieves a Hartree-Fock factorization of traces involving more than two field operators. The factorized form of the $\mathcal{F}_0(t)$ embodies what we refer to as the double mean field approximation. The subsystem densities $\mathcal{F}_0^{\mathrm{B}}$ and $\mathcal{F}_0^{\mathrm{F}}$ are in

fact unit trace gaussian densities, written in the form of an exponential of a bilinear, Hermitian expression in the creation and annihilation parts of the bosonic and of the fermionic fields respectively [14]. In the momentum basis, they reads

)

$$\begin{aligned}\mathcal{F}_0^{\mathrm{B}} &= \frac{\exp\left[\sum_{(a,b)} A_{a,b}b_a^\dagger b_b + B_{a,b}b_a^\dagger b_b^\dagger + C_{a,b}b_a b_b\right]}{Tr\left\{\exp\left[\sum_{(a,b)} A_{a,b}b_a^\dagger b_b + B_{a,b}b_a^\dagger b_b^\dagger + C_{a,b}b_a b_b\right]\right\}} , \\ \mathcal{F}_0^{\mathrm{F}} &= \frac{\exp\left[\sum_{(c,d)} D_{c,d}a_c^\dagger a_d + E_{c,d}a_c^\dagger a_d^\dagger + F_{c,d}a_c a_d\right]}{Tr\left\{\exp\left[\sum_{(c,d)} D_{c,d}a_c^\dagger a_d + E_{c,d}a_c^\dagger a_d^\dagger + F_{c,d}a_c a_d\right]\right\}} .\end{aligned} \tag{38}$$

The parameters in Eq.(38) are fixed by requiring that mean values in $\mathcal{F}_0$ of expressions that are linear or bilinear in the fields reproduce the corresponding $\mathcal{F}$ averages [see Eqs.(41) below]. The densities $\mathcal{F}_0^{\mathrm{B}}$ and $\mathcal{F}_0^{\mathrm{F}}$ are time-dependent object, which acquire a particularly simple form when expressed in terms of the Bogoliubov quasiboson and quasifermion operators,

$$\mathcal{F}_0^{\mathrm{B}} = \prod_{\mathbf{p}} \frac{1}{1+\nu_{\mathbf{p}}}\left(\frac{\nu_{\mathbf{p}}}{1+\nu_{\mathbf{p}}}\right)^{\beta_{\mathbf{p}}^\dagger\beta_{\mathbf{p}}} , \tag{39}$$

$$\mathcal{F}_0^{\mathrm{F}} = \prod_{\mathbf{k},s,\lambda} [\nu^{(\lambda)}_{\mathbf{k},s}\alpha^{(\lambda)\dagger}_{\mathbf{k},s}\alpha^{(\lambda)}_{\mathbf{k},s} + (1-\nu^{(\lambda)}_{\mathbf{k},s})\alpha^{(\lambda)}_{\mathbf{k},s}\alpha^{(\lambda)\dagger}_{\mathbf{k},s}] . \tag{40}$$

They have a unit trace and by construction satisfies

$$\begin{aligned}
&Tr_{\mathrm{BF}}(\beta_a\mathcal{F}_0) = Tr_{\mathrm{BF}}(\beta_a\mathcal{F}) = Tr_{\mathrm{BF}}\left(\beta_a^\dagger\mathcal{F}_0\right) = Tr_{\mathrm{BF}}\left(\beta_a^\dagger\mathcal{F}\right) = 0 \\
&Tr_{\mathrm{BF}}(\beta_a\beta_b\mathcal{F}_0) = Tr_{\mathrm{BF}}(\beta_a\beta_b\mathcal{F}) = 0 \\
&Tr_{\mathrm{BF}}\left(\beta_a^\dagger\beta_b^\dagger\mathcal{F}_0\right) = Tr_{\mathrm{BF}}\left(\beta_a^\dagger\beta_b^\dagger\mathcal{F}\right) = 0 \\
&Tr_{\mathrm{BF}}\left(\beta_a^\dagger\beta_b\mathcal{F}_0\right) = Tr_{\mathrm{BF}}\left(\beta_a^\dagger\beta_b\mathcal{F}\right) = \nu_a\delta_{a,b} \\
&Tr_{\mathrm{BF}}\left(\beta_a\beta_b^\dagger\mathcal{F}_0\right) = Tr_{\mathrm{BF}}\left(\beta_a\beta_b^\dagger\mathcal{F}\right) = (1+\nu_a)\delta_{a,b} \\
&Tr_{\mathrm{BF}}(\alpha_a\mathcal{F}_0) = Tr_{\mathrm{BF}}(\alpha_a\mathcal{F}) = Tr_{\mathrm{BF}}\left(\alpha_a^\dagger\mathcal{F}_0\right) = Tr_{\mathrm{BF}}\left(\alpha_a^\dagger\mathcal{F}\right) = 0 \\
&Tr_{\mathrm{BF}}(\alpha_a\alpha_b\mathcal{F}_0) = Tr_{\mathrm{BF}}(\alpha_a\alpha_b\mathcal{F}) = 0
\end{aligned} \tag{41}$$

$$Tr_{\text{BF}}\left(\alpha_a^\dagger\alpha_b^\dagger\mathcal{F}_0\right) = Tr_{\text{BF}}\left(\alpha_a^\dagger\alpha_b^\dagger\mathcal{F}\right) = 0$$
$$Tr_{\text{BF}}\left(\alpha_a^\dagger\alpha_b\mathcal{F}_0\right) = Tr_{\text{BF}}\left(\alpha_a^\dagger\alpha_b\mathcal{F}\right) = \nu_a\delta_{ab}$$
$$Tr_{\text{BF}}\left(\alpha_a\alpha_b^\dagger\mathcal{F}_0\right) = Tr_{\text{BF}}\left(\alpha_a\alpha_b^\dagger\mathcal{F}\right) = (1-\nu_a)\delta_{ab} \ .$$

The remainder density $\mathcal{F}'(t)$, defined by the Eq.(37), is a traceless, pure correlation part of the full density. In view of the special form used for $\mathcal{F}_0(t)$ it will in general contain correlations of two types : *inter-subsystem* (boson-fermion) correlations and *intra-subsystem* (boson-boson and fermion-fermion) correlations (see Ref.[22] for an application in quantum optics).

The crucial step is to observe that $\mathcal{F}_0(t)$ can be written as a time-dependent projection of $\mathcal{F}$, i.e.,

$$\mathcal{F}_0(t) = \mathcal{P}(t)\mathcal{F} \ , \ \ \mathcal{P}(t) = \mathcal{P}(t)\mathcal{P}(t) \ . \tag{42}$$

In order to obtain $\mathcal{P}(t)$ we require that, in addition, to Eqs.(42), it satifies (see Appendix A of Ref.[16] for construction of $\mathcal{P}$)

$$i\dot{\mathcal{F}}_0(t) = [\mathcal{P}(t),\mathcal{L}]\,\mathcal{F} = [\mathcal{F}_0(t),H] + \mathcal{P}(t)\,[H,\mathcal{F}] \ , \tag{43}$$

where $\mathcal{L}$ is the Liouvillian time-displacement generator defined as

$$\mathcal{L}\cdot = [H,\ \cdot\,] \ . \tag{44}$$

The relation (43) is just the Heisenberg picture counterpart of the condition $\dot{\mathcal{P}}(t)\mathcal{F} = 0$ which has been used to define $\mathcal{P}(t)$ in the Schrödinger picture [17].

The remaining formal steps towards closing the equations of motion for the Gaussian variables are now straightforward. To do so, we note that the existence of the projector $\mathcal{P}(t)$ allows one to find an equation relating the correlation part $\mathcal{F}'(t)$ to the Gaussian part $\mathcal{F}_0(t)$ of the full density. This can be immediately obtained from the Eqs.(37) and (43) and reads

$$(i\partial_t + \mathcal{P}(t)\mathcal{L})\,\mathcal{F}'(t) = (\mathcal{I} - \mathcal{P}(t))\,\mathcal{L}\mathcal{F}_0(t) \ . \tag{45}$$

This equation has the formal solution

$$\mathcal{F}'(t) = \mathcal{G}(t,0)\mathcal{F}'(0) - i\int_0^t dt'\ \mathcal{G}(t,t')\,(\mathcal{I}-\mathcal{P}(t'))\,\mathcal{L}\mathcal{F}_0(t') \ , \tag{46}$$

where the first term accounts for initial correlations possibly contained in $\mathcal{F}$ and $\mathcal{G}(t,t'$
stand for the time-ordered Green's function

$$\mathcal{G}(t,t') = T\left(\exp\left[i\int_{t'}^t d\tau\mathcal{P}(\tau)\mathcal{L}\right]\right) \ . \tag{47}$$

It is worthwhile to notice that $\mathcal{F}'(t)$, and therefore also $\mathcal{F}$, can be *formally* expressec in terms of $\mathcal{F}_0(t')$ (for $t' \leq t$) and of initial correlations $\mathcal{F}'(0)$. This allows us, therefor to express also the dynamical equations discussed in Sec II-c as functionals of $\mathcal{F}_0(t')$ anc of the initial correlations. Since, on the other hand, the reduced density $\mathcal{F}_0(t')$ is given b the one-body densities alone, we see that the resulting equations are now essentially closec in terms of Gaussian observables. Note, however, that the complicated time dependence o the Heisenberg field operators is explicitly probed through the memory effects present ir the expression (46) for $\mathcal{F}'(t)$. Workable approximations scheme are therefore needed for the actual evaluation for the equations of motion.

An attempt to deal with the collision terms within the general situation of quantum dissipative systems has proposed in Ref.[30] and the method has been used in field-theoretical context [16, 22]. It consists in replacing the full Heisenberg time-evolution of operators by a simpler mean-field type evolution governed by

$$H_0 = \mathcal{P}^\dagger(t)H \ .$$

(see Refs.[16, 22] for explit form of $\mathcal{P}^\dagger(t)$ and H_0). One finds from this that the field operators at different times are related by a phase factor written in terms of Gaussian variables. Consistent with this picture a systematic expansion for $\mathcal{G}(t,t')$, and therefore the correlation density $\mathcal{F}'(t)$, can be derived. In doing so, the equations of motion discussed in the previous section can be evaluated leading to a set of self-consistent equations for the Gaussian variables. An important feature of this scheme (which holds also for higher orders of the expansion [30]) is that the mean energy is conserved, namely

$$\frac{\partial}{\partial t}\langle H\rangle = 0$$

where

$$\langle H\rangle = Tr\ H\mathcal{F}_0(t) + Tr\ H\mathcal{F}'(t) \ .$$

Few lowest orders of this expansion are treatable both analytically and numerically. The zero order, which corresponds to $\mathcal{F}'(t) = 0$, recovers the usual Gaussian approximation. The higher orders describe the correlation effects and are added to the mean field results in form of collision integrals.

In summary, the adopted approximation scheme can be interpreted as follows. The dynamical evolution is split into: i) a pure mean-field part, related to the contributions to the dynamical equations involving the projected density $\mathcal{F}_0(t)$, Eqs.(39) and (40); ii) correlation effects, approximated by the contributions with adopted form for $\mathcal{F}'(t)$, Eq.(46). The later evolution is nonunitary in the sense that they change the coherence properties of $\mathcal{F}_0(t)$ through the time evolution of the occupation number $\nu_{\rm p}(t)$ and $\nu_{{\rm k},s}(t)$ [see Eqs.(28) and (35)]. In fact, one can see this by replacing $\mathcal{F}$ by $\mathcal{F}_0(t)$ in these equations and obtains $\dot{\nu}_{\rm p}(t) = 0$ and $\dot{\nu}_{{\rm k},s}(t) = 0$. As consequence, the entropy function associated with $\mathcal{F}_0(t)$, i.e., $S(t) = -Tr_{\rm BF}\mathcal{F}_0(t)\log\mathcal{F}_0(t)$ will change in time due to the correlation contributions, which perform as collisions terms from the point of view of the one-body densities.

Although the procedure is quite general and can be, readily, utilized for field models involving scalar and spin-1/2 particles, the following sections will adopt, as first application, the simplest context of relativistic scalar plasma model. We shall consider in the present paper the lowest (mean-field) approximation, corresponding to $\mathcal{F}'(t) = 0$. Collisional dynamics will be reported elsewhere.

IV. Relativistic Quantum Plasma

Section II obtained a framework to describe the effective one-body dynamics in the context of scalar-fermion field theory. The resulting equations of motion are formally exact and we have discussed in Section III an approximation scheme to get closed dynamics in terms of Gaussian variables. As first simple application, we shall consider in this section a system of Dirac fermion field coupled, through a Yukawa term, to a non-self-interaction scalar field in 3+1 dimensions, known as relativistic scalar plasma model [24]. Hence, in Subsection IV-a we obtain the Gaussian mean-field equations of motion for this model. The renormalization procedure is scrutinized in detail in IV-b. Thus, we shall discuss the solutions for the resulting gap equation and show that the energy density allows always a single minimum. Finally, in subsection IV-c, we shall investigate the renormalizability for the Gaussian equations of motion using the energy conservation as the key. Nonequilibrium initial states are then discussed in order to have well defined dynamics.

IV–a. Effective Dynamics of Relativistic Scalar Plasma

Let us begin by writting the bare Hamiltonian as [24]

$$
\begin{aligned}
H &= \int_{\mathbf{x}} \mathcal{H}\,, \\
\mathcal{H} &= -\bar{\psi}(i\vec{\gamma}.\vec{\partial} - m)\psi - g\bar{\psi}\phi\psi + \frac{1}{8\pi}\left[(4\pi)^2\Pi^2 + |\partial\phi|^2 + \mu^2\phi^2\right]\,,
\end{aligned}
\tag{48}
$$

[We use the notation: $\int_{\mathbf{x}} = \int d^3x$] where m and μ are, respectively, the mass of the spin-1/2 and scalar particles and g is the coupling constant. We shall study here the off-equilibrium situation of the system characterized by the time-evolution of Gaussian observables whose dynamics are governed by Eqs.(24), (28), (29), (35) and (36). Looking at these equations one sees that its evaluation consist in taking the traces of appropriate commutators in the Fock space.

To do so, one has to expand $\mathcal{H}$ in the quasi-particle basis. At this point, let us make some comments on the technical details. First all, one needs to have a convenient representation for γ-matrices and the Dirac spinors $u_\lambda(\mathbf{k}, s)$ in order to Fourier expand the fermion field. We choose (See Appendix A of Ref. [20] for representation of γ-matrices)

$$
\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} . \tag{49}
$$

In this representation the Dirac spinors $u_1(\mathbf{k}, s)$ and $u_2(\mathbf{k}, s)$ read as

$$
\begin{aligned}
u_1(\mathbf{k}, s) &= \left(\frac{k_0 + M}{2M}\right)^{1/2} \begin{pmatrix} \chi_s \\ \dfrac{\sigma\cdot\mathbf{k}\chi_s}{(k_0 + M)} \end{pmatrix}, \\
u_2(\mathbf{k}, s) &= \left(\frac{k_0 - M}{2M}\right)^{1/2} \begin{pmatrix} \chi_s \\ \dfrac{\sigma\cdot\mathbf{k}\chi_s}{(k_0 - M)} \end{pmatrix},
\end{aligned}
\tag{50}
$$

where M is the expansion mass parameter given in (3).

Next, we can use the constraint equations (16) and (21) to parametrize the coefficients of the Bogoliubov transformations of bosons and fermions respectively as

$$x_{\rm p} = \cosh\kappa_{\rm p} + i\frac{\eta_{\rm p}}{2}$$
$$y_{\rm p} = \sinh\kappa_{\rm p} + i\frac{\eta_{\rm p}}{2} \tag{51}$$

$$U_{11} = U_{22} = \cos\varphi_{{\rm k},s} \ \text{ and } \ U_{12} = U_{21} = 0$$
$$V_{12} = -V_{21} = \sin\varphi_{{\rm k},s} e^{i\gamma_{{\rm k},s}} \ \text{ and } \ V_{11} = V_{22} \tag{52}$$

with $\kappa_{\rm p}$, $\eta_{\rm p}$, $\varphi_{\rm k}$ and $\gamma_{\rm k}$ real. Thus, the Hamiltonian (48) can be now written in the quasi-particle basis and we have all the ingredients to compute the equations of motion for the Gaussian variables. The rest of this subsection will summarize the main results.

The mean value of linear form of the scalar field gives two equation motion corresponding to the evolution for the homogeneous condensate and its canonical moment,

$$\langle\dot{\phi}\rangle = 4\pi\langle\Pi\rangle \tag{53}$$

$$\langle\dot{\Pi}\rangle = -\frac{\mu^2}{4\pi}\langle\phi\rangle - g\sum_s\int_{\rm k}\frac{M}{k_0}\left[\cos 2\varphi_{{\rm k},s} + \frac{|{\bf k}|}{M}\sin 2\varphi_{{\rm k},s}\cos\gamma_{{\rm k},s}\right](1-\nu^{(1)}_{{\rm k},s}-\nu^{(2)}_{{\rm k},s}) \tag{54}$$

[$\int_{\rm k}$ denotes $(2\pi)^{-3}\int d^3{\rm k}$]. The results for the equations of motion (28) and (29) read as

$$\dot{\nu}_{\rm p} = 0 \tag{55}$$

$$\dot{\kappa}_{\rm p} = -4\pi({\bf p}^2+\Omega^2)^{1/2}\eta_{\rm p}e^{\kappa_{\rm p}} \tag{56}$$

$$\dot{\eta}_{\rm p}e^{-\kappa_{\rm p}} = ({\bf p}^2+\Omega^2)^{1/2}\left[4\pi e^{2\kappa_{\rm p}} - \frac{1}{4\pi}\frac{({\bf p}^2+\mu^2)}{({\bf p}^2+\Omega^2)}e^{-2\kappa_{\rm p}}\right] . \tag{57}$$

The equation (55) shows that the occupancy $\nu_{\rm p}$ are constant in the mean-field approximation as we have mentioned before. The results (56) and (57) correspond, respectively, to the imaginary and real parts of (29), and are the counterpart of the two-point dynamics in the usual functional variational calculation [32, 33].

For the fermionic sector of the system, the resulting pairing dynamics written in terms of real parameters read as

$$\dot{\nu}^{(1)}_{{\rm k},s} = 0 \ \text{ and } \ \dot{\nu}^{(2)}_{{\rm k},s} = 0 \ , \tag{58}$$

$$\dot{\varphi}_{{\rm k},s} = \frac{|{\bf k}|}{k_0}[M-(m-g\langle\phi\rangle)]\sin\gamma_{{\rm k},s} \tag{59}$$

$$\dot{\gamma}_{{\rm k},s} = 2\frac{[{\bf k}^2+(m-g\langle\phi\rangle)M]}{k_0} + 2\frac{[M-(m-g\langle\phi\rangle)]}{k_0}|{\bf k}|\cot 2\varphi_{{\rm k},s}\cos\gamma_{{\rm k},s} \ . \tag{60}$$

Observe that there is no spin-dependence in the above equations. Therefore, we might simplify our notation by writting

$$\varphi_{{\rm k},s} = \varphi_{\rm k} \ , \ \gamma_{{\rm k},s} = \gamma_{\rm k} \ \text{ and } \ \nu^{(\lambda)}_{{\rm k},s} = \nu^{(\lambda)}_{\rm k} \ . \tag{61}$$

Furtermore, the sum $\sum_s$ is just a constant factor $= 2$.

Another physical quantity of interest is the energy of the system, which can be evaluated easily in this approximation

$$\frac{\langle H\rangle}{V} = \frac{1}{V}TrH\mathcal{F}_0$$
$$= -2\int_{\rm k}\left[\frac{({\bf k}^2+mM)}{k_0}\cos 2\varphi_{\rm k} + \frac{(m-M)}{k_0}|{\bf k}|\sin 2\varphi_{\rm k}\cos\gamma_{\rm k}\right](1-\nu^{(1)}_{\rm k}-\nu^{(2)}_{\rm k})$$
$$+\frac{1}{4\pi}\left(\frac{\mu^2}{2}\langle\phi\rangle^2 + \frac{\langle\Pi\rangle^2}{2}\right)$$

$$+\frac{1}{16\pi}\int_{\mathrm{p}}\left[e^{-2\kappa_{\mathrm{p}}}\frac{(\mathbf{p}^2+\mu^2)}{(\mathbf{p}^2+\Omega^2)^{1/2}}+(4\pi)^2(\mathbf{p}^2+\Omega^2)^{1/2}(e^{2\kappa_{\mathrm{p}}}+\eta_{\mathrm{p}}^2)\right](1+2\nu_{\mathrm{p}})$$

$$+g\langle\phi\rangle 2\int_{\mathrm{k}}\frac{M}{k_0}\left[\cos 2\varphi_{\mathrm{k}}+\frac{|\mathbf{k}|}{M}\sin 2\varphi_{\mathrm{k}}\cos\gamma_{\mathrm{k}}\right](1-\nu_{\mathrm{k}}^{(1)}-\nu_{\mathrm{k}}^{(2)}) \tag{62}$$

[$\int_{\mathrm{p}}$ denotes $(2\pi)^{-3}\int d^3\mathrm{p}$]. As we have mention in section III, an important feature of this scheme is that the mean energy is conserved in all orders [30], which can be verified explicitly using the equations of motion (53)-(60). Notice also that the results above contain divergent integrals, therefore renormalization is required in order to have a well defined dynamics. We shall focus on this point in the next subsections.

IV–b. Static Equations and Renormalization

This subsection will discuss (53)-(60) in the equilibrium situation. We investigate the solution of these equations and study renormalization conditions. Hence, we set

$$\dot{\gamma}_{\mathrm{k}}=\dot{\varphi}_{\mathrm{k}}=\dot{\kappa}_{\mathrm{p}}=\dot{\eta}_{\mathrm{p}}=\langle\dot{\phi}\rangle=\langle\dot{\Pi}\rangle=0 \tag{63}$$

and consider zero occupancy, $\nu_{\mathrm{p}}=\nu_{\mathrm{k}}^{(\lambda)}=0$. Thus, Eqs.(53)-(60) become

$$[M-(m-g\langle\phi\rangle|_{\mathrm{eq}})]\sin\gamma_{\mathrm{k}}|_{\mathrm{eq}}=0 \tag{64}$$

$$\cot 2\varphi_{\mathrm{k}}|_{\mathrm{eq}}=-\frac{[\mathbf{k}^2+(m-g\langle\phi\rangle|_{\mathrm{eq}})M]}{[M-(m-g\langle\phi\rangle|_{\mathrm{eq}})]|\mathbf{k}|\cos\gamma_{\mathrm{k}}|_{\mathrm{eq}}} \tag{65}$$

$$\eta_{\mathrm{p}}|_{\mathrm{eq}}=0 \tag{66}$$

$$4\pi(\mathbf{p}^2+\Omega^2)e^{2\kappa_{\mathrm{p}}|_{\mathrm{eq}}}-\frac{1}{4\pi}(\mathbf{p}^2+\mu^2)e^{-2\kappa_{\mathrm{p}}|_{\mathrm{eq}}}=0 \tag{67}$$

$$\langle\Pi\rangle|_{\mathrm{eq}}=0 \tag{68}$$

$$\langle\phi\rangle|_{\mathrm{eq}}=-\frac{4\pi g}{V\mu^2}2\int_{\mathrm{k}}\frac{M}{k_0}\left(\cos 2\varphi_{\mathrm{k}}|_{\mathrm{eq}}+\frac{|\mathbf{k}|}{M}\sin 2\varphi_{\mathrm{k}}|_{\mathrm{eq}}\cos\gamma_{\mathrm{k}}|_{\mathrm{eq}}\right). \tag{69}$$

Any solution of this set of equations corresponds to a extremum of the energy density, c vacuum of the theory.

b-1. Equilibrium conditions and mass parameters

Looking at the equations (64)-(65) one notes that there are two situation to be considered

i) $M=m-g\langle\phi\rangle|_{\mathrm{eq}}$
In this case we have

$$\tan\varphi_{\mathrm{k}}|_{\mathrm{eq}}=0 \ \text{ or } \ \cos\varphi_{\mathrm{k}}|_{\mathrm{eq}}=1 \quad \sin\varphi_{\mathrm{k}}|_{\mathrm{eq}}=0 \tag{70}$$

and γ_{k} can be any value. The equilibium condition require, therefore, that Bogoliubo transformation, given in (20) or (52), to be an identity matrix. On the other hand i) is variational condition which fix the mass parameter M introduced in the field expansion.

ii) $M\neq m-g\langle\phi\rangle|_{\mathrm{eq}}$
Here the pairing angles are written as functions of $\langle\phi\rangle|_{\mathrm{eq}}$,

$$\sin\gamma_{\mathrm{k}}|_{\mathrm{eq}} = 0 \tag{71}$$

$$\begin{aligned}\cos 2\varphi_{\mathrm{k}}|_{\mathrm{eq}} &= -\frac{(\mathbf{k}^2+\bar{m}M)}{[(\mathbf{k}^2+\bar{m}M)^2+(\bar{m}-M)^2\mathbf{k}^2]^{1/2}}\\ \sin 2\varphi_{\mathrm{k}}|_{\mathrm{eq}} &= \frac{(\bar{m}-M)|\mathbf{k}|}{[(\mathbf{k}^2+\bar{m}M)^2+(\bar{m}-M)^2\mathbf{k}^2]^{1/2}}\,,\end{aligned} \tag{72}$$

where $\bar{m}$ is the effective fermion mass,

$$\bar{m}\equiv m-g\langle\phi\rangle|_{\mathrm{eq}}\ . \tag{73}$$

The above discussion suggest that the system might present two distinct equilibrium conditions for the fermionic pairing dynamics. The former means that M given in i), say M_1, is actually the optimal mass parameter in the expansian of the fermion field [33]. As consequence, the Bogoliubov coefficients form an identity matrix. In the later case, an arbitrary M is used as mass parameter and modified further to incorporating the pairing effects, say M_2. However, the minimal conditions given by (71)-(72) will in fact switch M_2

to M_1 found in the first case. To see this explicitly, we substitute (71)-(72) into (69) and obtain the same gap equation below in either of the two cases,

$$\mu^2\langle\phi\rangle|_{eq} = 16\pi g\bar{m}G(\bar{m}) \ , \tag{74}$$

where

$$G(\bar{m}) = \int_{\mathbf{k}} \frac{1}{2(\mathbf{k}^2+\bar{m}^2)^{1/2}} = \frac{1}{8\pi^2}\left(\Lambda_{\mathbf{k}}^2 + \bar{m}^2\log\frac{2\Lambda_{\mathbf{k}}}{\sqrt{e}\bar{m}}\right) \ , \tag{75}$$

being $\Lambda_{\mathbf{k}}$ the cutoff of the integral. In the earlier calculation we have considered the general mass parameter M. Nevertheless, (74) shows that the mass gap equations turns out to be independent of M (See also (85) below for the Gaussian mean-field energy). Henceforth, we shall fix $M = m$ in rest of discussion of this paper.

b-2. Gap equation and countertems

Notice in (74) and (75) that the coefficients of logarithmic divergence is cubic in $\bar{m}$. Furthermore, from (73) one sees that the divergent terms are polinomial of third degree in $\langle\phi\rangle|_{eq}$. As usual [25], one might take care of the above infinities by adding to the original Hamiltonian counterterms of the form

$$4\pi H_C = \frac{A}{1!}\phi + \frac{\delta\mu^2}{2!}\phi^2 + \frac{C}{3!}\phi^3 + \frac{D}{4!}\phi^4 \ . \tag{76}$$

At this point it is worthwhile to remember that ϕ is a non-self-interacting scalar field in this model. Therefore, to insure that there is no meson-meson scattering [26], the bosonic quantum fluctuation will not be considered in this system. We shall then set the pairing dynamics to zero, i.e.,

$$\kappa_{\mathrm{p}}|_{eq} = 0 \ \text{ and } \ \eta_{\mathrm{p}}|_{eq} = 0 \ . \tag{77}$$

With these assumptions and using the counterterms introduced in (76) the mass gap equation becomes

$$\begin{aligned}[A + 2\pi CG(\Omega)] \ &+ \ \left[\mu^2 + \delta\mu^2 + 2\pi DG(\Omega)\right]\langle\phi\rangle|_{eq} \\ &+ \ \frac{C}{2}\langle\phi\rangle|_{eq}^2 + \frac{D}{6}\langle\phi\rangle|_{eq}^3 = 16\pi g\bar{m}G(\bar{m}) \ . \end{aligned} \tag{78}$$

The arbitrary parameters A, $\delta\mu$, C and D can be found now easily by adjusting the coeffients of $\langle\phi\rangle|_{eq}$ in both of sides of (78). Thus, the *self-consistency condition* will require

$$\begin{aligned} D &= 48\pi g^4 L(m) && (79)\\ \delta\mu^2 &= -96\pi^2 g^4 L(m)G(\mu) - 16\pi g^2 G(0) + 24\pi m^2 g^2 L(m) && (80)\\ C &= -48\pi m g^3 L(m) && (81)\\ A &= 96\pi^2 m g^3 L(m)G(\mu) + 16\pi m g G(0) - 8\pi m^3 g L(m) \ , && (82)\end{aligned}$$

where

$$L(m) \equiv \int_{\mathbf{k}} \frac{1}{2\mathbf{k}^2(\mathbf{k}^2+m^2)^{1/2}} \tag{83}$$

and the function G is defined by (75). We have also chosen μ and m as the mass scales for boson and fermion fields respectively.

Substituting these counterterms in (78) we have appropriate cancelations by construction. Besides, there is also a combination of type

$$L(m)[G(\mu) - G(\Omega)]$$

coming from the first and second term. Since Ω is an arbitrary expansion mass parameter, one can, therefore, remove this divergence by taking $\Omega = \mu$. With these ingredients we find the following renormalized gap equation

$$\frac{\pi}{2}\mu^2\langle\phi\rangle|_{eq} - g\bar{m}^3\left[\ln\left(\frac{\bar{m}}{m}\right) + \frac{1}{2}\right] = 0 \ . \tag{84}$$

The above equation together with (64)-(65) determine the stationary points or vacuum of the theory in the Gaussian mean-field approximation.

b-3. Mean energy and stationary points

Let us examine next the energy density when it is stationary with respect to the fermion variables, i.e.,

$$\begin{aligned}\frac{\langle H\rangle}{V}(\varphi_{\mathbf{k}}|_{eq}, \gamma_{\mathbf{k}}|_{eq}, \langle\phi\rangle) \ &= \ \left(\frac{A}{4\pi} + \frac{C}{2}G(\mu)\right)\langle\phi\rangle + \left(\frac{\mu^2}{8\pi} + \frac{\delta\mu^2}{8\pi} + \frac{D}{4}G(\mu)\right)\langle\phi\rangle^2 \\ &+ \ \frac{C}{24\pi}\langle\phi\rangle^3 + \frac{D}{96\pi}\langle\phi\rangle^4 + \int_{\mathbf{k}} G(\bar{m}) \ , \end{aligned} \tag{85}$$

where we have added the counteterms contribution to (62) and disregarded unimportant additive constants. Substituting (79)- (82) into this result one finds, after some algebra, the renormalized version of mean-field energy as function of mean-field value $\langle\phi\rangle$ or $\bar{m}$ is

$$\frac{\langle H\rangle}{V}=\frac{1}{8\pi^2}\left[\frac{\pi\mu^2}{g^2}(m-\bar{m})^2+\bar{m}^4\ln\left(\frac{\bar{m}}{m}\right)+\frac{(\bar{m}^4-m^4)}{4}\right] , \tag{86}$$

where we have added appropriate constant in order to have $E(\langle\phi\rangle=0)=0$. One can now discuss the possible solution of (84) by analysing the minima of (86).

Let us define $x\equiv g\langle\phi\rangle|_{\rm eq}/m$ and $E(x)\equiv(8\pi^2/m^4)\langle H\rangle/V$. Thus (84) and (86) can be rewritten, respectively, as

$$\frac{\pi\mu^2}{2g^2m^2}x-(1-x)^3\left[\ln(1-x)+\frac{1}{2}\right]=0 \tag{87}$$

$$E(x)=\frac{\pi\mu^2}{g^2m^2}x^2+(1-x)^4\left[\ln(1-x)+\frac{1}{4}\right]-\frac{1}{4} . \tag{88}$$

The combination $\frac{4g^2m^2}{\pi\mu^2}$ has been used by Kalman as an effective coupling constant [24].

The different behavior of $E(x)$ are shown in Figs.(1)-(4) for several combination of μ/m and g^2. Notice first that this function has the domain at $0\le x\le 1$, which is the physical range for the fermion mass. The point $x=0$ corresponds to $\langle\phi\rangle=0$ or $\bar{m}=m$ and $x=1$ is the case when the effective fermion mass $\bar{m}=0$. Qualitatively, the results indicate that the system always presents a single *minimum*. The Figs.(1)-(2) show $E(x)$ for several values of g^2 with μ/m fixed (see figure captions for numerical values of the parameters). The positions of x_{min}, which indicate the vacua of the system, approach to $x=0$ when we decrease g^2. In the limit of $g^2\to 0$, one gets $x_{min}\to 0$, as it can be verified in the Fig.(3). In this case, $m\approx\bar{m}$ is the optimal fermion mass, as it must be in the free field theory. On the other hand, the Figs.(3)-(4) plot the function $E(x)$ keeping same value of Yukawa coupling g^2, but with different ratios of μ/m. Comparing these two curves one sees that $x_{min}\to 0$ when $\mu/m\to\infty$. In other words, when the meson mass is large, the force range is small, as usual in the Yukawa theory. In the limit infinity μ, the fermion particles of the system cannot interact. It can be seen also from the Kalman's formula, where m/μ plays roles of effective coupling constant. The above discussions suggest that the field has always a stable vacuum. This means that there is a finite range around the minimum where the dynamics of the system is well defined. In the next subsection, we shall therefore discuss renomalization conditions for the time-dependent equations.

IV–c. Renormalizability and Initial Conditions for Time-Dependent Equations

The last section has discussed the problem of renormalization for the vacuum sector of the relativistic scalar plasma model in the Gaussian mean-field approximation. We have shown that the physical quantities can be made finite with the counterterms introduced in (76) and (79)-(82). Here we shall consider the system in an off-equilibrium situation and study renormalizability for the equations of motion. (See, e.g., Refs.[12, 32] for the issue of renormalization of time-dependent equations in ϕ^4 field theory)

Let us begin by rewriting (54) with the counterterm contribution

$$\begin{aligned}\langle\dot{\Pi}\rangle &= -\frac{1}{4\pi}\left(A+\frac{C}{4}G(\mu)\right)-\frac{1}{4\pi}\left(\mu^2+\delta\mu^2+\frac{D}{4}G(\mu)\right)\langle\phi\rangle\\ &- \frac{C}{8\pi}\langle\phi\rangle^2-\frac{D}{24\pi}\langle\phi\rangle^3-\frac{\mu^2}{4\pi}\langle\phi\rangle-2g\left[I_1(m)+I_2(m)\right] ,\end{aligned} \tag{89}$$

where the divergent integrals $I_1(m)$ e $I_2(m)$ are given by

$$I_1(m) = \int_{\mathbf{k}}\frac{m}{(\mathbf{k}^2+m^2)^{1/2}}\cos 2\varphi_{\mathbf{k}} \tag{90}$$

$$I_2(m) = \int_{\mathbf{k}}\frac{|\mathbf{k}|}{(\mathbf{k}^2+m^2)^{1/2}}\sin 2\varphi_{\mathbf{k}}\cos\gamma_{\mathbf{k}} . \tag{91}$$

The other equations of motion are not modified by the counterterms. These integals are divergent if, for instance, $\lim_{k\to\infty}\varphi(\mathbf{k},t)$ =const, which must be kept under control with the counterterms and appropriate choice of initial states. However, it is hard to know the large momentum behavior for the dynamical variables as functions of time, since, in principle, one needs to solve the equations of motion, which is not easy for nonlinear problems. On the other hand, we have seen in Section III that the energy is conserved in this approximation scheme. This property suggests us to choose correct sets of initial conditions, in the sense that the total energy is finite, and therefore energy conservation will keep the dynamical variables limited at all time.

Following this key, we recalculate the mean-value of energy

$$\frac{\langle H\rangle}{V}=\frac{1}{V}TrH\mathcal{F}_0$$

$$= -2\int_{\mathbf{k}} k_0 \cos 2\varphi_{\mathbf{k}} + 2g\langle\phi\rangle\,[I_1(m)+I_2(m)]$$

$$+\frac{1}{4\pi}\left(\frac{\mu^2}{2}\langle\phi\rangle^2+\frac{\langle\Pi\rangle^2}{2}\right)+\left(\frac{A}{4\pi}+\frac{C}{2}G(\mu)\right)\langle\phi\rangle$$

$$+\left(\frac{\mu^2}{8\pi}+\frac{\delta\mu^2}{8\pi}+\frac{D}{4}G(\mu)\right)\langle\phi\rangle^2+\frac{C}{24\pi}\langle\phi\rangle^3+\frac{D}{96\pi}\langle\phi\rangle^4 \tag{92}$$

vith the assumptions discussed in the previous sections. Thus, this expression differs from 62) by the counterterm and bosonic pairing contribution. Using the equations of motion 53), (59), (60) and (89) one can verify easily the energy conservation. Therefore, if the ystem starts with a finite amount of energy, by the conservation law only well defined lynamics are allowed.

We shall thus find a criterion for initial conditions in order to get finite energy density. 'o do so, we define a new set of variables as follows:

$$\begin{aligned}\cos 2\varphi_{\mathbf{k}}(t) &= \cos 2\varphi_{\mathbf{k}}|_{eq} + R(\mathbf{k},t) &(93)\\ \sin 2\varphi_{\mathbf{k}}(t) &= \sin 2\varphi_{\mathbf{k}}|_{eq} + S(\mathbf{k},t) &(94)\\ \cos \gamma_{\mathbf{k}}(t) &= \cos \gamma_{\mathbf{k}}|_{eq} + W(\mathbf{k},t)\ , &(95)\end{aligned}$$

'here $\cos 2\varphi_{\mathbf{k}}|_{eq}$, $\sin 2\varphi_{\mathbf{k}}|_{eq}$ and $\cos \gamma_{\mathbf{k}}|_{eq}$ are solutions of the time-independent equations given y (71) and (72). In terms of these new variables the energy density (92) becomes

$$\begin{aligned}\frac{\langle H\rangle}{V} &= -\int_{\mathbf{k}}\frac{(\mathbf{k}^2+m\bar{m})}{k_0}R(\mathbf{k})+\int_{\mathbf{k}}\frac{(m-\bar{m})^2}{k_0^2K_0}\mathbf{k}^2W(\mathbf{k})\\ &- \int_{\mathbf{k}}\frac{(m-\bar{m})|\mathbf{k}|}{k_0}S(\mathbf{k})[1+W(\mathbf{k})]+E_0 &(96)\end{aligned}$$

here $K_0=\sqrt{\mathbf{k}^2+\bar{m}^2}$ and

$$E_0=\frac{1}{8\pi^2}\left[\frac{\pi\mu^2}{g^2}(m-\bar{m})^2+\bar{m}^4\ln\left(\frac{\bar{m}}{m}\right)+\frac{(\bar{m}^4-m^4)}{4}\right]\ . \tag{97}$$

From (96) one sees that the initial conditions of R, S, W can no longer be arbitrary, they ust vanish as fast as $|\mathbf{k}|^{-n}$ $(n>0)$ in a such way that $E<\infty$. With this assumption one ıds the following simple relation valid for large $\mathbf{k}$,

$$R(\mathbf{k},0)=\frac{|\mathbf{k}|(m-\bar{m})}{\mathbf{k}^2+m\bar{m}}S(\mathbf{k},0) \qquad (\mathbf{k}\to\infty)\ . \tag{98}$$

Using this relation, Eq.(96) can be rewritten as

$$\frac{\langle H\rangle}{V}=-\int_{\mathbf{k}}\left[\frac{(m-\bar{m})^2\mathbf{k}^2}{k_0^2K_0}W(\mathbf{k})+\frac{(m-\bar{m})|\mathbf{k}|}{k_0}S(\mathbf{k})W(\mathbf{k})\right]+E_0\ , \tag{99}$$

Analysing this relation, we conclude that the initial conditions with

$$S(\mathbf{k},0)\underset{|\mathbf{k}|\to\infty}{\longrightarrow}|\mathbf{k}|^{-l} \tag{100}$$

and

$$W(\mathbf{k},0)\underset{|\mathbf{k}|\to\infty}{\longrightarrow}|\mathbf{k}|^{-l-1}, \tag{101}$$

being $l>2$, will give finite results for these integrals. Hence, the energy conservation will enforce the dynamical variables limited at all time.

In summary, we have presented a framework to study the initial-value problem for interacting fermion-scalar field models. The method allows one to get a set of self-consistent equations for expectation values of linear and bilinear field operators. The lowest order of this approximation scheme corresponds to the results of variational mean-field-like calculation, and the collision terms represent the correlation effects involving different subsystems from the viewpoint of one-body density. The technique is quite general and model independent. In particular, we have implemented a zero-order calculation within the simplest context of relativistic scalar plasma system. We have shown in detail that the standard form for renormalization also applies to these nonperturbative calculation and we have obtained finite expression for energy density. A simple numerical calculation suggests that the system has always a single stable minimum, although further investigation will be necessary for other oscillation modes. The standard procedure to this question is throught the RPA analysis [12], where the stability is indicated by its the eigenvalues. It is interesting to mention that the excitation modes described by the RPA equations are the quantum particles of the field. In fact, the physics of one meson and two spin-1/2 fermion can be investigated from this equation [34]. We have also discussed the renormalization for the time-dependent equations. Using energy conservation as the key, we found that there are a finite range around the vacuum where the dynamics of the system is well defined.

Finally, we comment that the projection technique discussed in the Sec.III can be readly applied [16, 22] to include dynamical correlation corrections to the mean-field approximation. In this case, the occupation number are no longer constant and will affect the effective dynamics of the Gaussian observables. The framework presented here serves also as groundwork to finite density and finite temperature discussions [35]. In particular, a finite-matter density calculation beyond the mean-field approximation allows one to study collisional observables such as transport coefficients [36]. The extension of this procedure to explore nonuniform systems is straighforward but lengthy. In this case, the spatial dependence of the field are expanded in natural orbitals of extented one-body density. These orbitals can be given in terms of a momentum expansion through the use of a more general Bogoliubov transformation [27, 28].

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Figure Captions

FIGURE 1. The behavior of the ground-state mean-field energy density $E(x)$ of the uniform scalar plasma system as a function of fermionic effective mass $x = g\langle\phi\rangle/m = 1-\bar{m}/m$ for any values of the coupling constant g and mass scale $\mu/m = 0.1$ fixed.

FIGURE 2. The behavior of the ground-state mean-field energy density $E(x)$ of the uniform scalar plasma system as a function of fermionic effective mass $x = g\langle\phi\rangle/m = 1-\bar{m}/m$ for any values of the coupling constant g and mass scale $\mu/m = 2$ fixed.

FIGURE 3. The behavior of the ground-state mean-field energy density $E(x)$ of the uniform scalar plasma system as a function of fermionic effective mass $x = g\langle\phi\rangle/m = 1-\bar{m}/m$ for coupling constant $g^2 = \pi/100$ and mass scale $mu/m = 2$.

FIGURE 4. The behavior of the ground-state mean-field energy density $E(x)$ of the uniform scalar plasma system as a function of fermionic effective mass $x = g\langle\phi\rangle/m = 1-\bar{m}/m$ for coupling constant $g^2 = \pi/100$ and mass scale $mu/m = 0.1$.

Figure 1

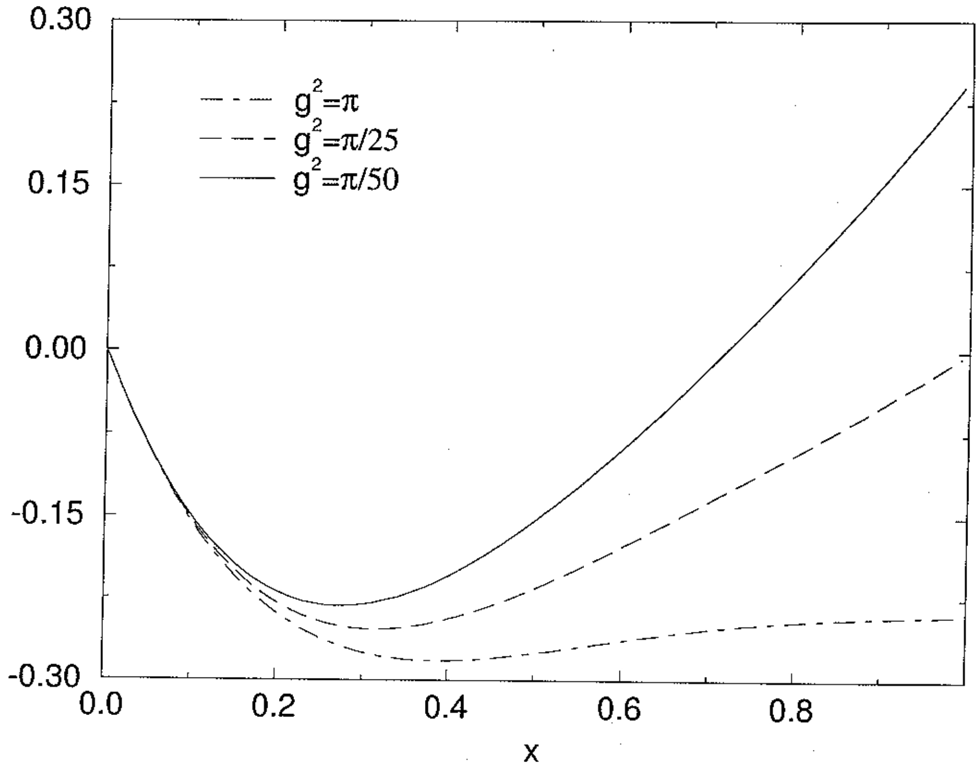

Figure 2

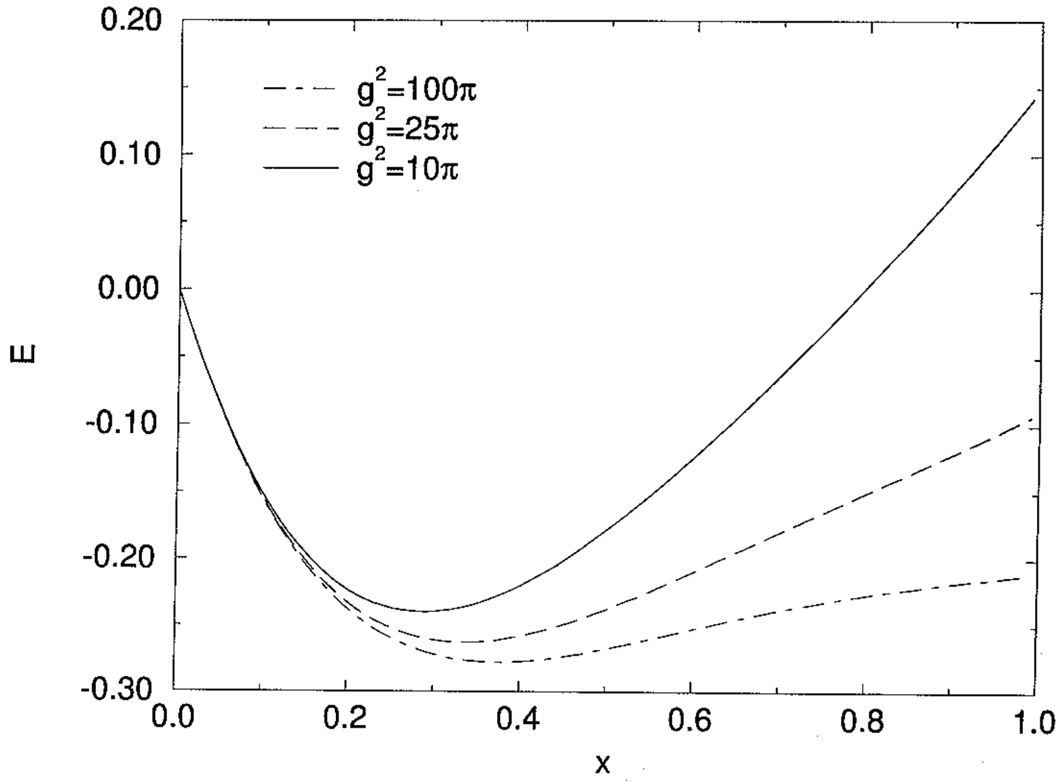

Figure 3

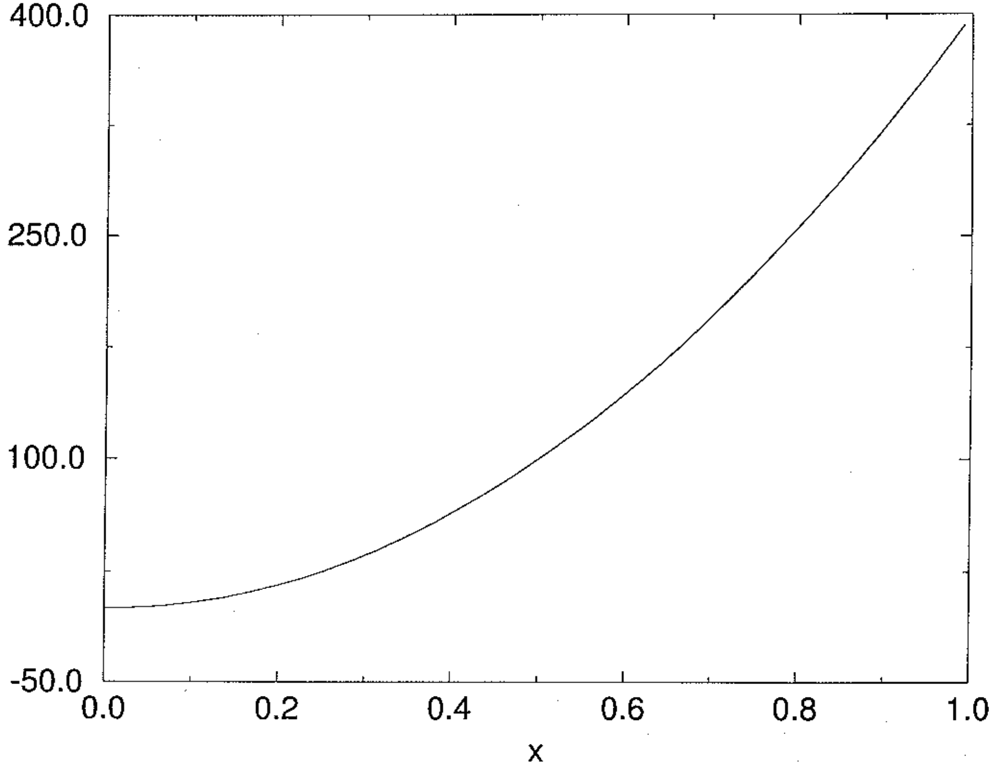

Figure 4