

**UNIVERSIDADE DE SÃO PAULO**

**PUBLICAÇÕES**

**INSTITUTO DE FÍSICA  
CAIXA POSTAL 66318  
05315-970 SÃO PAULO – SP  
BRASIL**

**IFUSP/P-1316**

**A MODIFICATION OF THE 10d SUPERPARTICLE  
ACTION INSPIRED BY THE GUPTA-BLEULER  
QUANTIZATION SCHEME**

**S. Bellucci**

INFN – Laboratori Nazionali di Frascati  
P.O. Box 13, 00044 Frascati, Italy

**A. Galajinsky**

Instituto de Física, Universidade de São Paulo

Agosto/1998

# A modification of the 10d superparticle action inspired by the Gupta-Bleuler quantization scheme

S. Bellucci\*

INFN-Laboratori Nazionali di Frascati,  
P.O. Box 13, 00044 Frascati, Italy

and

A. Galajinsky†

Instituto de Física, Universidade de São Paulo,  
C. Postal 66318, 05315-970, São Paulo, Brasil

## Abstract

We reconsider the issue of the existence of a complex structure in the Gupta-Bleuler quantization scheme. We prove an existence theorem for the complex structure associated with the  $d = 10$  Casalbuoni-Brink-Schwarz superparticle, based on an explicitly constructed Lagrangian that allows a holomorphic-antiholomorphic splitting of the fermionic constraints consistent with the vanishing of all first class constraints on the physical states.

As it is well known, the puzzle of the covariant quantization of the superparticle, superstring models can be viewed as the problem of mixed first and second class fermionic constraints in the Hamiltonian formalism [1–3]. One of the interesting approaches to treat the second class constraints is the Gupta-Bleuler-type quantization scheme [4–7] which, for the case at hand, reduces to the construction of a specific complex structure  $J$  on a phase space of the models<sup>1</sup>. The latter provides a holomorphic-antiholomorphic splitting of the mixed constraints which proved to yield a successful covariant quantization of the  $4d$  superparticle [5].

A recipe how to construct such a  $J$  in an arbitrary space-time dimensions has been proposed in the recent work [8]. The strategy adopted was to decompose the tensor  $J$  into irreducible representations of the Lorentz

\*e-mail: bellucci@lnf.infn.it

†e-mail: galajin@fma.if.usp.br

<sup>1</sup>For simplicity, in what follows we shall discuss the superparticle case only.

group and then reduce equations defining  $J$  to those for irreps. The explicit solution in  $d = 10$  has been found

$$J_{ab} = \frac{1}{\alpha}(A_a B_b - A_b B_a), \quad (1.a)$$

$$\alpha = \pm \sqrt{A^2 B^2 - (AB)^2}, \quad (1.b)$$

$$(Ap) = 0, \quad (Bp) = 0, \quad (1.c)$$

requiring the extension of the original phase space  $(x^n, p_n), (\theta^\alpha, p_{\theta\alpha})$  by two new vector variables  $A, B$ . Generally such an extension can easily be realized by introducing two pairs of canonically conjugate variables  $(A^n, p_{A_n}), (B^n, p_{B_n})$  subject to the first class constraints.

$$p_{A_n} = 0, \quad p_{B_n} = 0, \quad (2)$$

and treating the constraints (1.c) as gauge fixing conditions to some of the constraints (2). However, as the first class constraints remaining in Eq.(2) do not commute with the complex structure, in passing to a quantum description the vanishing of the constraints on physical states would be incompatible with the vanishing of the holomorphic constraints on those states.

In this brief note we suggest a way to cure this inconsistency. The idea is to completely fix the gauge freedom in the sector  $(A, p_A), (B, p_B)$  by introducing further auxiliary variables. If the first class constraints from the sector of the new variables turn out to commute with  $A$  and  $B$ , the complete description is self-consistent.

The action to be examined reads

$$S = \int d\tau \frac{1}{2e} (\dot{x}^n - i\theta\Gamma^n\dot{\theta} - \omega_1 A^n - \omega_2 B^n - \mu_i \Lambda^n{}_i)^2 - \rho_1 (A^2 - 1) - \rho_2 (B^2 - 1) - \nu_{1i} (A\Lambda_i) - \nu_{2i} (B\Lambda_i) - \Phi_{ij} (\Lambda_i \Lambda_j + \Delta_{ij}) - \sum_{i=1}^8 \mu_i \quad (3)$$

where

$$\Delta_{ij} \equiv \begin{cases} 0, & i = j \\ 1, & i \neq j \end{cases} \quad i, j = 1, \dots, 8$$

Here the summation over repeated indices is understood. As compared to the Casalbuoni-Brink-Schwarz model [9] one finds a set of auxiliary variables  $(A^n, B^n, \Lambda^n{}_i, \mu_i, \nu_{1i}, \nu_{2i}, \Phi_{ij}, \omega_1, \omega_2, \rho_1, \rho_2)$ , with  $\Phi_{ij}$  being symmetric.

Consider the model (3) in the Hamiltonian formalism. Introducing momenta  $(p_e, p^n, p_{\theta\alpha}, p_A^n, p_B^n, p_{\Lambda ni}, p_{\mu_i}, p_{\nu_{1i}}, p_{\nu_{2i}}, p_{\Phi_{ij}}, p_{\omega_1}, p_{\omega_2}, p_{\rho_1}, p_{\rho_2})$  canonically conjugate to the configuration space variables one has a set of primary constraints

$$p_e = 0, \quad p_\theta + i\theta\Gamma^n p_n = 0, \quad (4a)$$

$$p_A^n = 0, \quad p_B^n = 0, \quad p_{\Lambda ni} = 0, \quad (4b)$$

$$p_{\mu_i} = 0, \quad p_{\nu_{1i}} = 0, \quad p_{\nu_{2i}} = 0, \quad (4c)$$

$$p_{\omega_1} = 0, \quad p_{\omega_2} = 0, \quad p_{\rho_1} = 0, \quad (4d)$$

$$p_{\rho_2} = 0, \quad p_{\Phi_{ij}} = 0, \quad (4e)$$

and the relation to eliminate  $\dot{x}^n$

$$\dot{x}_n = ep_n + i\theta\Gamma_n \dot{\theta} + \omega_1 A_n + \omega_2 B_n + \mu_i \Lambda_{ni}. \quad (5)$$

The canonical Hamiltonian is

$$\begin{aligned} H = & (p_\theta + i\theta\Gamma^n p_n)\lambda_\theta + p_e \lambda_e + p_A \lambda_A + p_B \lambda_B + p_{\Lambda_i} \lambda_{\Lambda_i} + p_{\mu_i} \lambda_{\mu_i} + \\ & + p_{\nu_{1i}} \lambda_{\nu_{1i}} + p_{\nu_{2i}} \lambda_{\nu_{2i}} + p_{\Phi_{ij}} \lambda_{\Phi_{ij}} + p_{\omega_1} \lambda_{\omega_1} + p_{\omega_2} \lambda_{\omega_2} + p_{\rho_1} \lambda_{\rho_1} + p_{\rho_2} \lambda_{\rho_2} + \\ & + e \frac{p^2}{2} + \omega_1 (pA) + \omega_2 (pB) + \rho_1 (A^2 - 1) + \rho_2 (B^2 - 1) + \nu_{1i} (A\Lambda_i) + \\ & + \nu_{2i} (B\Lambda_i) + \Phi_{ij} (\Lambda_i \Lambda_j + \Delta_{ij}) + \mu_1 ((p\Lambda_1) + 1) + \mu_2 ((p\Lambda_2) + 1) + \dots \\ & + \mu_8 ((p\Lambda_8) + 1), \end{aligned} \quad (6)$$

where the  $\lambda$ 's denote Lagrange multipliers corresponding to the primary constraints.

The consistency conditions for the primary constraints imply the secondary ones <sup>2</sup>

$$p^2 = 0, \quad p\Lambda_i + 1 = 0, \quad \Lambda_i \Lambda_j + \Delta_{ij} = 0, \quad (7a)$$

$$pA = 0, \quad A^2 - 1 = 0, \quad A\Lambda_i = 0, \quad (7b)$$

$$pB = 0, \quad B^2 - 1 = 0, \quad B\Lambda_i = 0, \quad (7c)$$

$$\omega_1 p^n + 2\rho_1 A^n + \nu_{1i} \Lambda^n_i = 0, \quad (7d)$$

$$\omega_2 p^n + 2\rho_2 B^n + \nu_{2i} \Lambda^n_i = 0, \quad (7e)$$

$$\nu_{1i} A^n + \nu_{2i} B^n + \mu_i p^n + 2\Phi_{ij} \Lambda^n_j = 0, \quad (7f)$$

<sup>2</sup>We define the Poisson brackets of the variables  $(\Lambda, p_\Lambda), (\Phi, p_\Phi)$  in the form  $\{\Lambda^n_i, p_{\Lambda m j}\} = \delta^n_m \delta_{ij}$ ,  $\{\Phi_{ij}, p_{\Phi k s}\} = \frac{1}{2}(\delta_{ik} \delta_{js} + \delta_{is} \delta_{jk})$ .

and determines half of the  $\lambda_\theta$

$$\Gamma^n p_n \lambda_\theta = 0. \quad (8)$$

Consider now Eq. (7d). Multiplying it by  $A^n$  and taking into account Eq. (7b) one gets

$$\rho_1 = 0. \quad (9)$$

Subsequent multiplication of the remaining equation  $\omega_1 p^n + \nu_{1i} \Lambda^n_i = 0$  by  $p^n, \Lambda^n_i$  reduces it to a system of linear homogeneous equations which has the trivial solution

$$\omega_1 = 0, \quad \nu_{1i} = 0, \quad (10)$$

since the matrix

$$\begin{pmatrix} p^2 & p\Lambda_i \\ p\Lambda_j & \Lambda_i \Lambda_j \end{pmatrix},$$

is nondegenerate on the constraint surface (7a)-(7f). In the same spirit Eqs. (31e),(31f) simplify to

$$\omega_2 = 0, \quad \rho_2 = 0, \quad \nu_{2i} = 0, \quad \mu_i = 0, \quad \Phi_{ij} = 0. \quad (11)$$

The preservation in time of the secondary constraints (7a)-(7c),(9),(10),(11) determine some of the Lagrange multipliers

$$p\lambda_A = 0, \quad A\lambda_A = 0, \quad \Lambda_i \lambda_A + A\lambda_{\Lambda_i} = 0 \quad (12a)$$

$$p\lambda_B = 0, \quad B\lambda_B = 0, \quad \Lambda_i \lambda_B + B\lambda_{\Lambda_i} = 0 \quad (12b)$$

$$p\lambda_{\Lambda_i} = 0, \quad \Lambda_i \lambda_{\Lambda_j} + \Lambda_j \lambda_{\Lambda_i} = 0, \quad (12c)$$

$$\lambda_{\rho_1} = 0, \quad \lambda_{\omega_1} = 0, \quad \lambda_{\nu_{1i}} = 0, \quad (12d)$$

$$\lambda_{\rho_2} = 0, \quad \lambda_{\omega_2} = 0, \quad \lambda_{\nu_{2i}} = 0, \quad (12e)$$

$$\lambda_{\mu_i} = 0, \quad \lambda_{\Phi_{ij}} = 0, \quad (12f)$$

and no tertiary constraints appear.

Taking into account Eqs. (4),(9),(10) and (11) one concludes that the variables  $(\rho_1, p_{\rho_1}), (\rho_2, p_{\rho_2}), (\mu_i, p_{\mu_i}), (\nu_{1i}, p_{\nu_{1i}}), (\nu_{2i}, p_{\nu_{2i}}), (\Phi_{ij}, p_{\Phi_{ij}}), (\omega_1, p_{\omega_1}), (\omega_2, p_{\omega_2})$  are unphysical and can be omitted after introducing the associated Dirac bracket. Thus, the only nontrivial constraints to be analyzed are those from Eqs. (7a)-(7c), together with the corresponding momenta (4b).

Let us now return to Eqs. (7). The constraints (7b) together with the corresponding momentum  $p_{A^n} = 0$  are second class. In a full agreement

be solved explicitly. Actually, since the vectors  $p^n, A^n, \Lambda^n_i$  satisfying Eq. (7) are linearly independent for any fixed value compatible with Eq. (7), the matrix

$$\begin{pmatrix} p_0 & \dots & p_9 \\ A_0 & \dots & A_9 \\ \Lambda_{0i} & \dots & \Lambda_{9i} \end{pmatrix}, \quad (13)$$

is invertible on the constraint surface. The latter fact implies that the system of linear inhomogeneous equations (12a) has a unique solution for any fixed value of  $p^n, A^n, \Lambda^n_i$ . Analogously, the constraints (7c) and  $p_{Bn} = 0$  are second class and Eq. (12b) uniquely determines  $\lambda_B$ .

Thus, it remains to discuss the constraints (7a) and the corresponding momentum  $p_{\Lambda ni} = 0$ . In order to extract the first class constraints contained in  $p_{\Lambda ni} = 0$ , let us construct operators projecting onto subspaces orthogonal to  $(A^n, B^n)$  and  $(p^n, \Lambda_{ni})$  respectively. The explicit form of the projectors is

$$\begin{aligned} \Pi_{(A,B)}^m{}_n &= \delta^m{}_n + \frac{(AB)}{1-(AB)^2} A^m B_n + \frac{(AB)}{1-(AB)^2} B^m A_n - \\ &- \frac{1}{1-(AB)^2} A^m A_n - \frac{1}{1-(AB)^2} B^m B_n, \end{aligned} \quad (14)$$

$$\begin{aligned} \Pi_{(A,B)}^m{}_n A^n &\equiv 0, & \Pi_{(A,B)}^m{}_n B^n &\equiv 0, \\ \Pi_{(A,B)}^m{}_n p^n &= p^m, & \Pi_{(A,B)}^m{}_n \Lambda^n_i &= \Lambda^m_i \end{aligned} \quad (15)$$

$$\begin{aligned} \Pi_{(p,\Lambda)}^m{}_n &= \delta^m{}_n - \frac{(p\Lambda_j)\nabla^{ji} p^m \Lambda_{ni}}{(p\Lambda)\nabla(p\Lambda)} - \frac{(p\Lambda_j)\nabla^{ji} \Lambda^m_i p_n}{(p\Lambda)\nabla(p\Lambda)} + \\ &+ \frac{\nabla^{ji} \Lambda^m_i \Lambda_{nj}}{(p\Lambda)\nabla(p\Lambda)} - \frac{(p\Lambda_k)\nabla^{ki} \Lambda^m_i (p\Lambda_s)\nabla^{sj} \Lambda_{nj}}{(p\Lambda)\nabla(p\Lambda)} - \\ &- \frac{p^m p^n}{(p\Lambda)\nabla(p\Lambda)}, \end{aligned} \quad (16)$$

$$\begin{aligned} \Pi_{(p,\Lambda)}^m{}_n p^n &\equiv 0, & \Pi_{(p,\Lambda)}^m{}_n \Lambda^n_i &\equiv 0, \\ \Pi_{(p,\Lambda)}^m{}_n A^n &= A^m, & \Pi_{(p,\Lambda)}^m{}_n B^n &= B^m, \end{aligned} \quad (17)$$

where  $\nabla$  is the inverse matrix to  $\Delta$ ,  $\nabla_{ij}\Delta_{jk} = \delta_{ik}$ . Note also that  $(p\Lambda)\nabla(p\Lambda) = \frac{8}{7} \neq 0$ . In the presence of the projectors the first class constraints can be written in the form

$$\tilde{p}_\Lambda^m \equiv \Pi_{(p,\Lambda)}^m{}_n \Pi_{(A,B)}^n{}_k p_\Lambda^k = 0. \quad (18)$$

At the next stage, one needs to construct the Dirac bracket associated with all the second class constraints of the problem, which will look like q

$$\begin{aligned} \{M, N\}_D &= \{M, pA\} \dots \{p_{A_n}, N\} + \{M, A^2 - 1\} \dots \{p_{A_n}, N\} + \\ &+ \{M, A\Lambda_i\} \dots \{p_{A_n}, N\} + \{M, pB\} \dots \{p_{B_n}, N\} + \\ &+ \{M, B^2 - 1\} \dots \{p_{B_n}, N\} + \{M, B\Lambda_i\} \dots \{p_{B_n}, N\} + \\ &+ \{M, p\Lambda_i + 1\} \dots \{p_{\Lambda_j p}, N\} + \{M, p\Lambda_i + 1\} \dots \{p_{\Lambda_k \Lambda_s} + p_{\Lambda_s \Lambda_k}, N\} + \\ &+ \{M, \Lambda_i \Lambda_j + \Delta_{ij}\} \dots \{p_{\Lambda_k p}, N\} + \{M, \Lambda_i \Lambda_j + \Delta_{ij}\} \dots \\ &\{p_{\Lambda_k \Lambda_s} + p_{\Lambda_s \Lambda_k}, N\} + \{M, A\Lambda_i\} \dots \{p_{\Lambda_j p}, N\} + \{M, A\Lambda_i\} \dots \\ &\{p_{\Lambda_k \Lambda_s} + p_{\Lambda_s \Lambda_k}, N\} - (-1)^{\epsilon(MN)} (M \leftrightarrow N), \end{aligned} \quad (19)$$

where ... denotes some specific functions and  $\{M, N\}$  is the usual Poisson bracket. As it is seen, under this bracket  $A^n, B^m$  commute both with each other and with the first class constraints  $\tilde{p}_\Lambda^m = 0$  from the sector of additional variables. This implies that the subsequent split of the fermionic constraints  $p_\theta + i\theta\Gamma^n p_n = 0$  into holomorphic and antiholomorphic sets will be consistent with the vanishing of the first class constraints  $\tilde{p}_\Lambda^m = 0$  on physical states. This was the problem to solve.

Thus, in this letter we have reconsidered the complex structure in the Gupta-Bleuler quantization scheme, introducing a gauge fixing procedure based on the addition of a set of auxiliary variables, which makes the vanishing of the first class constraints on physical states compatible with the holomorphic-antiholomorphic splitting of the fermionic constraints. We have built explicitly the corresponding Lagrangian formulation.

Since this Lagrangian looks like a monster, we have little hope to be really able to quantize a model on the basis of this scheme. However, our understanding is that the Lagrangian above can be viewed as the existence theorem for the complex structure associated with the 10d Casalbuoni-Brink-Schwarz model.

## Acknowledgements

One of the authors (A.G) thanks A.A. Deriglazov and P.M. Lavrov for useful discussions. He was supported by INTAS-RFBR grant 95-829 and by FAPESP.

## References

- [1] L. Brink, M. Henneaux, and C. Teitelboim, Nucl. Phys. B **293** (1987) 505.
- [2] A. Dresse, J. Fisch, M. Henneaux, and C. Schomblond, Phys. Lett. B **210** (1988) 141.
- [3] A. Deriglazov, A. Galajinsky, and S. Lyakhovich, Nucl. Phys. B **473** (1996) 245.
- [4] Z. Hasiewicz, J. Kowalski-Glikman, J. Lukierski, and J.W. van Holten, J. Math. Phys. **32** (1991) 2358.
- [5] S. Aoyama, J. Kowalski-Glikman, J. Lukierski, and J.W. van Holten, Phys. Lett. B **201** (1988) 487; **216** (1989) 133.
- [6] W. Kalau, Int. J. Mod. Phys. A **8** (1993) 391.
- [7] J. Kowalski-Glikman, Ann. Phys. **232** (1994) 1.
- [8] S. Bellucci and A. Galajinsky, hep-th/9712247, to appear in Phys. Lett. B.
- [9] R. Casalbuoni, Phys. Lett. B **62** (1976) 49;  
L. Brink and J.H. Schwarz, Phys. Lett. B **100** (1981) 310.