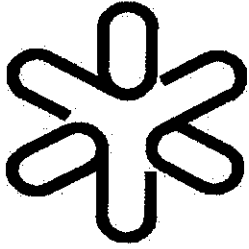


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**Fractal Structure of Random Matrices**

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# Fractal Structure of Random Matrices\*

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## Abstract

A multifractal analysis is performed on the universality classes of random matrices and the transition ones. Our results indicate that the eigenvector probability distribution is a linear sum of two  $\chi^2$ -distribution throughout the transition between the universality ensembles of random matrix theory and Poisson.

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The Anderson localization [1] is a wave phenomenon characterized by a destructive interference that gives rise to inaccessible regions in the configuration space of a physical system. It occurs in many situations [2] and, in particular, in condensed matter physics, it is the responsible for the metal-insulator phase transition (MIT) caused by an increasing of disorder in a quantum disordered system, in which, as a consequence, the material undergoes a transformation from a metallic to an insulator phase. The eigenstates of the system, in the metallic phase, extend over all the available space and, in the other hand, at the insulator side, it is localized around the impurities. This situation implies, as the transformation proceeds, in a modification of the fractal dimension of the wavefunctions. This aspect of the transition has been studied through the use of the multifractal analysis [3,4] which was introduced some years ago [5].

Random matrix ensembles are another powerful theoretical tool to study transitions from extended to localized states [6]. The phase of the extended states, i.e., the metallic phase in the MIT case, is approached by the universal ensembles of random matrix theory (RMT) [7], namely, the Gaussian Orthogonal Ensemble (GOE), if the system has time-reversal invariance, and the Gaussian Unitary Ensemble (GUE), if not. On the other hand, the insulator phase, where the states become localized, can be simulated by a Poissonian ensemble. Accordingly, the energy levels of the strong mixing metallic states are expected to follow Wigner-Dyson statistics of RMT, while the levels of the uncorrelated localized states of the insulator phase, have fluctuations that follow the Poisson statistics. The Maximum Entropy Principle (MEP) has been used to generate matrix ensembles [8] that make the connection between two universal

limiting situations, e.g., RMT and Poisson. The joint probability distribution of the matrix elements of the interpolating ensembles obtained are given by [9]

$$P(H, \beta, \alpha) = K_N \exp[-\alpha_0 \text{Tr}(H^2)] \exp\{-\beta \text{Tr}[(H - H_0)^2]\} \quad (1)$$

where  $N$  is the dimension of the matrix,  $K_N$  is the normalization constant,  $\alpha_0$  is fixed by a choice of units and  $\beta$  is the parameter that controls the transition. When  $\beta$  varies from zero to infinity, the ensemble undergoes a transition from the RMT ensemble to the ensemble of  $H_0$ . By choosing  $H_0$  to be a diagonal, we have the desired transition from RMT to Poisson. This connection between RMT and Poisson is not expected to be universal and, in fact, other possible interpolating ensembles have already been proposed, e.g., band matrices [10] and  $U(N)$  invariant ensembles [11]. However, the above formalism has the advantage that the chaoticity parameter  $\beta$  is easily expressed in terms of the coupling constant [8,12]. In this letter, we extend the multifractal analysis to the abstract space of random matrices to investigate statistical properties of the eigenstates of this interpolating ensemble.

Multifractal is a mathematical object that is not characterized by a unique dimension but by an infinite spectrum of dimensions [13]. For example, in a classical dynamical system, the probability given by the frequency with which different "cells" of a partition of the strange attractor is visited, in the time evolution of a chaotic system, is a multifractal. In the quantum case, chaoticity may be defined as the situation in which the eigenfunctions spread uniformly over all components with respect to any basis. Of course, this reflects the fact that the system does not have any other good

quantum numbers but the energy and it is the quantum mechanical equivalent of the absence of integrals of motion in the classical case. This property may be considered as an statement about the dimension of the wave function. On the other hand, when transitions towards regularity occur, one should expect that the presence of conserved quantities implies in some localization that should be followed by modifications in the respective dimensions of the eigenstates. This can be seen in Fig. 1, where it is plotted the logarithm of the components of a random eigenstate of the above ensemble as a function of its label, calculated at a critical value of the parameter  $\beta$  [9]. It is also shown in the figure, a Gaussian fit that makes clear the localization. It can be seen that only a small fraction of the components contribute to the normalization. In this sense the state does not occupy all the available space. Here the support is given by the states of the basis and the probability distribution by the way these basis states are populated by the eigenstates of the Hamiltonian.

The critical behavior of the above transition ensemble has been defined in Ref. [9] applying Shannon's concept of entropy to the eigenstates components treated as probabilities. As a start point to introduce the multifractal formalism, we observe that this entropy is a particular case of Tsallis generalized entropy [14] of a probability distribution  $p_i$ , with  $i = 1, N_p$ , which in terms of the partition function

$$\chi(q) = \sum_{i=1}^{N_p} p_i^q \quad (2)$$

is defined by

$$S_q = \frac{1 - \chi(q)}{q - 1}. \quad (3)$$

for any real  $q$ . Associated with  $S_q$ , a spectrum of dimension functions,  $D_q$ , can then be introduced as [5]

$$D_q = \lim_{l \rightarrow 0} \frac{\ln [1 - (q - 1) S_q]}{(q - 1) \ln l}, \quad (4)$$

where  $l$  is a characteristic size associated with the partition.

Some positive integer values of  $q$  have an immediate interpretation. Thus,  $D_0 = -\frac{\ln N_c}{\ln l}$ , where  $N_c$  is the number of occupied cells, i.e., those with probability different from zero, gives the fractal dimension of the support. For  $D_1$  we obtain

$$D_1 = \frac{S_1}{\ln l} \quad (5)$$

where

$$S_1 = - \sum_i p_i \ln p_i \quad (6)$$

is, by definition the Shannon's information entropy, and  $D_1$  is the information dimension.  $D_2$  is the correlation dimension.

We now assume the scaling  $p_i \sim l^{\alpha'}$  and, also, that the exponents  $\alpha'$  vary continuously, as  $i$  is varied, with a certain distribution  $\rho(\alpha')$  that scales with  $l^{f(\alpha')}$ . Thus the partition function can be transformed into an integral as

$$\chi(q) = \int d\alpha' \rho(\alpha') l^{q\alpha' - f(\alpha')} \quad (7)$$

Since  $l$  is a small quantity, this integral will be determined asymptotically by the value  $\alpha' = \alpha$  that minimizes the exponent of  $l$  in the integrand. This yields

$$D_q = \frac{1}{q - 1} [q\alpha - f(\alpha)] \quad (8)$$

from which we can deduce

$$\alpha = \frac{d}{dq} [(q-1) D_q] \quad (9)$$

These two equations give  $\alpha$  and  $f(\alpha)$  in terms of  $D_q$  or, alternatively,  $D_q$  if we know  $\alpha$  and  $f(\alpha)$ . Some universal properties follow from these definitions. Thus  $f(\alpha)$  is a convex function whose maximum is located at  $\alpha(0)$  with  $f[\alpha(0)] = D_0$  and  $\alpha$  varies in the interval  $[D_\infty, D_{-\infty}]$ .  $f(\alpha)$  gives the dimension of the support of those  $p_i$  that scale with  $\alpha$ .

As a first application of the above formalism, we consider the special case of the transition from GOE to the Poisson for matrices of dimension  $N = 2$ . In this case, the probability distribution of a given component  $C$  can be worked out analytically. It has been shown in Ref. [15] that it is given by

$$P(y) = \frac{\alpha_0}{2\pi\beta} \sqrt{1 + \frac{\beta}{\alpha_0}} \frac{1}{\alpha_0 \sqrt{y(1-y)}} \frac{1}{\frac{\alpha_0}{4\beta} + y(1-y)} \quad (10)$$

where  $0 < y = C^2 < 1$ . When  $\beta = 0$ , the GOE limit, the distribution is that of the component of a two dimensional unit vector that can point evenly in any direction. In the other limit,  $\beta \rightarrow \infty$ , the distribution goes to a sum of delta functions and the vector is completely localized,  $y$  can then have only the values 0 or 1. This means that for small values of ratio  $\beta/\alpha_0$ , the distribution is dominated by the two power law singularities located at the extremities of the segment. On the other hand, as  $\beta$  increases, the two poles  $\frac{1}{2} \left( 1 \mp \sqrt{1 + \frac{\alpha_0}{\beta}} \right)$  of the denominator, approach the interval from the left(-) and the right(+), respectively, and deform the power law behavior. In order to perform a more detailed analysis, we partition the interval  $[0, 1]$  into  $N_p$  equal

subintervals of size  $l = 1/N_p$ . We have then two kind of contributions to the partition function. Those that come from the power law singularities at  $y = 0$  and  $y = 1$  and those from the rest of the segment. To calculate the contributions of the first ones we integrate from 0 to  $l$  and from  $1 - l$  to 1 and for the others we just approximate them as  $\rho(y)l$ . The partition function is then given by

$$\chi(q) \sim \left[ \arctan \left( \sqrt{\frac{\beta l}{\alpha_0}} \right) \right]^q + l^{q-1}. \quad (11)$$

For small values of the parameter  $\beta$ , we can assume  $\beta l/\alpha_0 \ll 1$  and the arctan can then be replaced by its argument and the first term becomes  $l^{q/2}$ . We then deduce

$$D_q = \begin{cases} 1, & q < 2 \\ \frac{q}{2(q-1)}, & q > 2 \end{cases} \quad (12)$$

from which we derive a  $f(\alpha)$ -spectrum with only two points,  $\alpha = 1$  with  $f(\alpha) = 1$  and  $\alpha = \frac{1}{2}$  with  $f(\alpha) = 0$ . These values mean that the extremity points have fractal dimension zero while the others have the dimension of the support. Incidentally, we remark that we have exactly here the same distribution of that provide by the iteration of the logistic map.

In the second situation, we assume that although the  $l$  are very small  $\beta$  is so big that we can not linearize the arctan anymore. The first term scales then as  $l^0$  and we deduce

$$D_q = \begin{cases} 1, & q < 1 \\ 0, & q > 1 \end{cases} \quad (13)$$

and the two points  $\alpha = 1$  with  $f(\alpha) = 1$  and  $\alpha = 0$  with  $f(\alpha) = 0$  for the  $f(\alpha)$ -spectrum. We remind that we are here getting close to the situation in which



the distribution becomes a sum of delta functions concentrated in the limits of the interval. The components approximate then the two values 1 and 0. That is why the exponent  $\alpha$  vanishes.

We see, in both cases discussed above, that the dimension function exhibits discontinuity, in its first derivative in one case and in the function itself in the other. In the thermodynamics picture of the multifractal analysis, these discontinuity are interpreted as phase transitions [16] and what we learn from the above results is that the transition chaos-order, here obtained by the variation of the parameter  $\beta$ , is followed by a change in the qualitative behavior of the dimension function. This modification may also be considered as a phase transition with respect to the variation of the external parameter  $\beta$ . This is exhibited in the figure where we can see that the information dimension  $D_1$  plotted as a function of  $\ln \frac{\beta}{\alpha_0}$  shows a typical first order phase transition pattern. One should expect that what we are observing in this case of ensemble of  $2 \times 2$  matrices are universal features of the transition RMT- Poisson. We pass now to the discussion of ensembles of matrices of size arbitrarily large.

We start discussing the GOE limit. It is known that, in this case, the probability distribution of the components is that of the components of a unit vector in the hypersphere in the space of  $N$  dimensions. It can be proved [17] that this is given by

$$P(y) = \frac{2\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{N-1}{2}\right)} y^{-\frac{1}{2}} (1-y)^{\frac{N-3}{2}} \quad (14)$$

The distribution is again characterized by the power singularities at the extremities  $y = 0$  and  $y = 1$ . The partition function associated to a division of the interval into  $N_p$

cells of equal sizes  $l = 1/N_p$  approaches, for small  $l$ , the behavior

$$\chi(q) \sim l^{\frac{q}{2}} + l^{q-1} + l^{\frac{2}{N-1}}. \quad (15)$$

and, in the limit when  $l \rightarrow 0$ , we find the dimension function

$$D_q = \begin{cases} \frac{N-1}{2} \frac{q}{q-1}, & q < -\frac{2}{N-3} \\ 1, & -\frac{2}{N-3} < q < 2 \\ \frac{1}{2} \frac{q}{q-1}, & q > 2 \end{cases} \quad (16)$$

We have therefore a double phase transition separating three states, two of them are defined by the equation of state  $\frac{q-1}{q} D_q = \text{const.}$  which is generated by power law singularities [5] and, in the middle of them, there is the state whose equation is given by  $D_q = \text{const.}$ . For the  $f(\alpha)$ -spectrum we deduce, making use of Eqs. (8) and (9),

$$f(\alpha) = \begin{cases} 0, & \alpha = \frac{1}{2} = D_\infty \\ 1, & q = 1 \\ 0, & \alpha = \frac{N-1}{2} = D_{-\infty} \end{cases} \quad (17)$$

which means that the two singular contributions from the extremities of the interval have zero fractal dimension while the rest of the others have the dimension of the support.

The more general case, with  $\beta \neq 0$  and  $N > 2$ , has been investigated by numerical simulation of ensemble of matrices. The dimension function in the limit  $l \rightarrow 0$  was extrapolated from the dimensions obtained with two small  $l$ 's by the relation

$$D_q(0) = \frac{D_q(l_1) \ln l_1 - D_q(l_2) \ln l_2}{\ln l_1 - \ln l_2}$$

which follows from the assumption that for sufficiently small  $l$ , the partition function behaves as

$$\chi_q \simeq A(0) l^{(q-1)D_q(0)}.$$

The results obtained in this way for matrices of dimension  $N = 100$  are shown in Fig. 2. We see that the structure with two phase transitions and three states of the GOE case evolves to the picture given by Eq. (13), typical of the limiting distribution with two  $\delta$ -functions at the extrema of the interval. The correspondent  $f(\alpha)$ -spectrum is shown in the next figure, Fig. 3, for three values of the chaoticity parameter and matrices of dimension  $N = 10$ . It is seen that the singular behavior of the GOE limit, Eq. (17) has been smoothed out by the numerical simulation. Finally, in the subsequent figure, Fig. 4, the first-order phase transition exhibited by the information dimension  $D_1$ , is seen. In the same figure, it is also shown the Shannon's entropy of the eigenstates given by Eq. (6) with  $p_i = |C_i^k|^2$  averaged over the  $k$  states. In Ref. [9], the inflection point of this entropy has been taken as a definition of the critical value of the chaoticity parameter which separates the phase of localized and extend states. We can conclude from this figure that this definition is consistent with the results obtained for the information dimension.

One important point to remark here is the consequence the present analysis has on the question of what is the probability distribution of wavefunction components and, also, of matrix elements of an operator- strength function- in the intermediate regime between RMT and Poisson or, in more general terms, between chaos and or-

der. It has been proposed that a  $\chi^2$ -distribution of  $\nu$  degrees freedom would fit this distribution [18]. However, numerical simulations seem to suggest that a combination of two  $\chi^2$ -distributions is necessary in order to have a good description through all the intermediate steps of the transition [12]. The above results of the multifractal analysis seem to give a theoretical support to this empirical observation. Indeed, what we have shown is that the chaoticity parameter acts like an external thermodynamic variable that induces a first-order phase transition. Therefore, one should expect a modification of the nature of the probability distribution as the transition proceeds. In the  $N = 2$  case, we have seen that this change in the structure of the function, Eq. (10), is provided by the coming into action of two poles that lie outside of the physical domain. For large size of the matrices it is the appearance of an extra  $\chi^2$ -distribution that takes care of the modification of the distribution in the passage from chaos to order.

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**Figure Captions:**

Fig. 1 Logarithm of the components of a random eigenstate for a case intermediate between GOE and Poisson. The calculations were done with matrices of dimension  $N = 100$ . A Gaussian fit is also shown.

Fig. 2 The dimension function  $D_q$  for matrices of size  $N = 100$  for four values of the parameter  $\beta(\alpha_0 = 1)$ .

Fig. 3 The  $f(\alpha)$ -spectrum for matrices of size  $N = 10$  for four values of the parameter  $\beta(\alpha_0 = 1)$ .

Fig. 4 Information dimension  $D_1$  for matrices of size  $N = 100$ , showing a first-order phase transition as a function of the logarithm of the chaoticity parameter. Also shown is Shannon's entropy of the eigenstates.

N=100

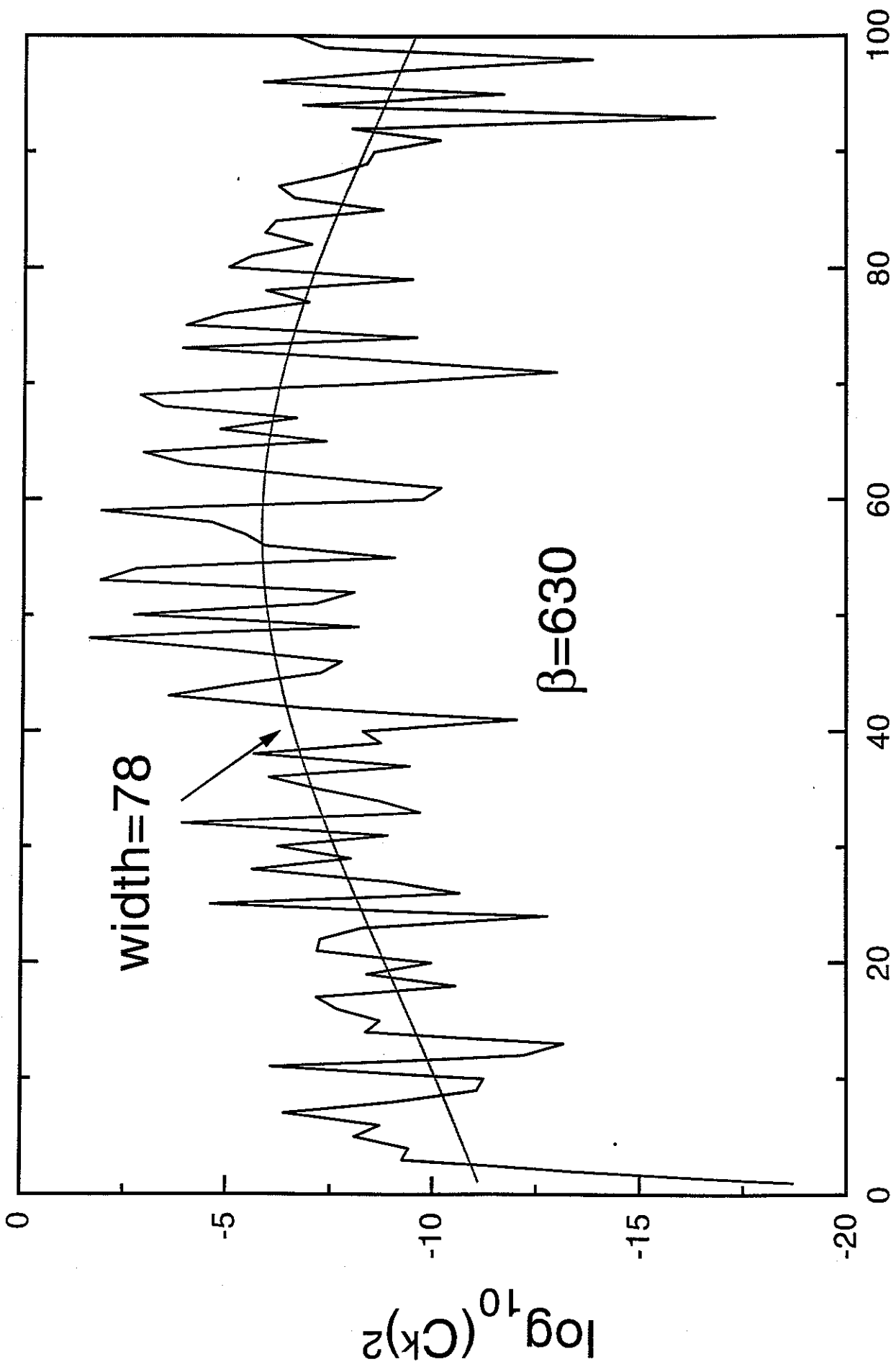


Fig. 1

$N=100$

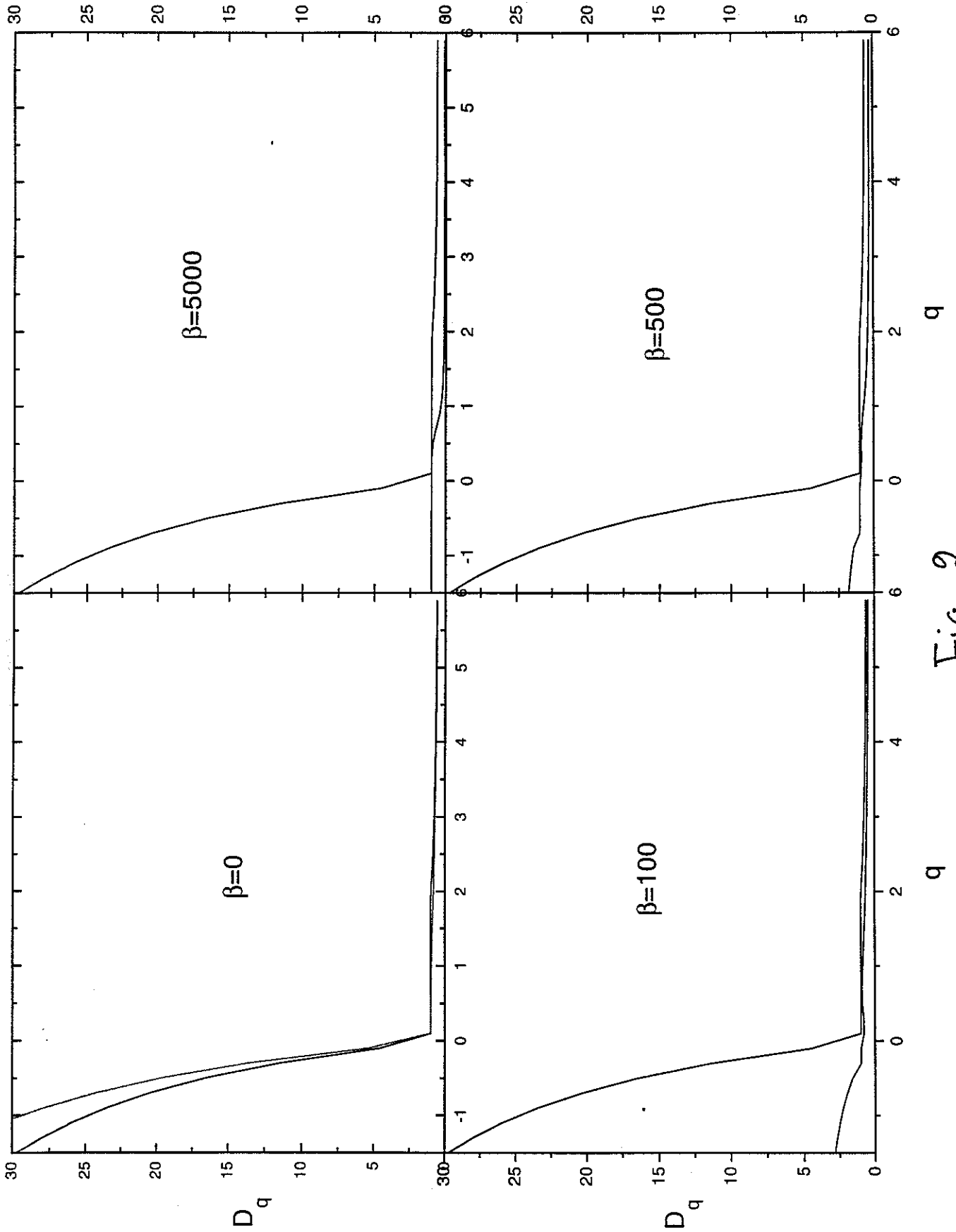


Fig. 2



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