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ON AN INFRARED-FINITE NORMAL PRODUCT FORMALISM

by

M. Gomes* & R. Köberle
Instituto de Física, Universidade de São Paulo

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ABSTRACT

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An infrared-finite normal product formalism is set up by subtracting Feynman integrands at some spacelike point $p = n$, $n^2 = -\mu^2$. We show how the usual machinery, like equations of motion, Zimmermann identities, Callan-Symanzik and renormalization group equations, work. As an application we discuss the Ward identity for a charged scalar boson.

ON AN INFRARED-FINITE NORMAL PRODUCT FORMALISM

I. INTRODUCTION

In the usual BPHZ renormalization scheme[1] formal Feynman integrals are made finite by subtracting a suitable number of terms at zero external momenta. Since at these momenta the massless theory has infrared divergencies, in order to study this limit one has to perform a finite renormalization on the Green functions and composite fields, imposing normalization conditions at some spacelike point $p^2 = \mu^2$ [2]. It seems convenient to have at one's disposal a formalism, which avoids infrared divergent intermediate steps, especially if one is interested in soft quantization of theories with spontaneously broken symmetry, Gauge theories etc.

We will set up such a scheme by subtracting our Feynman integrands at some point $p_\mu = n_\mu$, with $n^2 = -\mu^2$. The details will be spelled out in Section II in the context of the A^4 theory. In Section III we address ourselves at questions of infrared finiteness of Green functions and normal products. The Ward identity for a charged scalar field will be discussed in Section IV.

II. GREEN FUNCTIONS AND COMPOSITE FIELDS

Considering A^4 theory as an example, we may define Green functions by[3]

$$G^{(N)}(X_1,\ldots, X_N) = \langle 0|T|A(X_1)\ \ldots\ A(X_N)||0\rangle =$$

$$= \text{finite part of } {}^{(0)}\langle 0|T|A^{(0)}(X_1)\ldots\ A^{(0)}(X_N)\ \exp i \qquad \text{(II.1)}$$

$$\int :\ L_{int}(A^{(0)})(y):d^4y|0\rangle^{(0)}$$

where $A^{(0)}$ is the free field given by

$$L_0 = \frac{1}{2}\ \partial_\mu A\partial^\mu A - \frac{1}{2}\ m^2A^2 \qquad \text{(II.2)}$$

and

$$L_{int} = \frac{1}{2}\ a\ A^2 - \frac{g}{4!}\ A^4 \qquad \text{(II.3)}$$

Expression (II.1) is a sum of Feynmann integrals of the type

$$\int dk_1\ \ldots\ dk_s\ I_G(k,P,m,\varepsilon,\eta)$$

where G indicates the corresponding Feynmann diagram. The finite part prescription consists of replacing I_G by R_G :

$$R_G = \sum_{U\varepsilon F_G} \prod_{\gamma\varepsilon U} (-S^{\delta(\gamma)})\ I_G \qquad . \qquad \text{(II.4)}$$

where F is the set of all forests of G and $S^{\delta(\gamma)}$ is a generalized Taylor operator acting on a set of linearly independent ingoing momenta of the graph γ. Different sets will produce different integrands, but the same integrals. For simplicity we write down S^{δ} only when acting an one variable, the generalization to more variables being obvious:

$$S^{\delta(\gamma)}F(P) = F\left(n_o(\delta,1)\right) + \left(P-n_1(\delta,1)\right)^{\alpha} \left.\frac{\partial F}{\partial P^{\alpha}}\right|_{P=n_2(\delta,1)} +$$

$$+ \frac{1}{2}\left(P-n_1(\delta,2)\right)^{\alpha}\left(P-n_1(\delta,2)\right)^{\beta} \left.\frac{\partial^2 F}{\partial P^{\alpha}\partial P^{\beta}}\right|_{P=n_2(\delta,2)} + \dots \tag{II.5}$$

$$+ \dots + \frac{1}{\delta!}\left(P-n_1(\delta,\delta)\right)^{\alpha_1}\dots\left(P-n_1(\delta,\delta)\right)^{\alpha_\delta} \left.\frac{\partial^{\delta} F}{\partial P^{\alpha_1}\dots\partial P^{\alpha_\delta}}\right|_{P=n_2(\delta,\delta)}$$

where $n^{\mu}_{i}(\delta,j)$ are constant four-vectors. $\delta(\gamma)$ is the superficial degree of divergence given in our model by $\delta(\gamma) = 4 - N_{\gamma}$, where N_{γ} is the number of external legs of the subgraph γ. The finite part of equ. (II.1) is defined by

$$\lim_{\varepsilon\to o} \int dk_1 \dots dk_S R_G(k,P,m,\varepsilon,n) \ ,$$

which is a well defined tempered distribution (Zimmermann's theorem follows with trivial modification).

In order to eliminate an eventual directional dependence on n_{μ}, which would violate Lorentz invariance, we impose the restriction, that $p.n=0$, whenever a momentum p_μ occuring in a generalized Taylor operator is contracted with n^μ produced by the action of a derivate $\frac{\partial}{\partial p_\mu}$ evaluated at n_μ.

This amounts to a finite renormalization of Green functions and normal products.

We will not need equ. (II.5) in all its complexity, but each model will posses a particularly well suited subtraction scheme. In general one will have to make a compromise between the simplicity of the equations of motion and the simplicity of normalization conditions and the rule for pulling a derivative inside a normal product. For the purpose of our illustrative A^4 model, we will choose the following subtraction operator τ^δ:

$$\tau^{(o)}F(P) = F(n)$$

$$\tau^{(1)}F(P) = F(n) + (P-n)^\alpha \left.\frac{\partial F}{\partial P^\alpha}\right|_{P=n}$$

$$\tau^{(2)}F(P) = F(0)+(P-n)^\alpha \left.\frac{\partial F}{\partial P^\alpha}\right|_{P=n} + \frac{1}{2}(P-n)^\alpha (P-n)^\beta \left.\frac{\partial^2 F}{\partial P^\alpha \partial P^\beta}\right|_{P=n} \tag{II.6}$$

$$\vdots$$

$$\tau^{(\delta)}F(P) = F(n)+(P-n)^\alpha \left.\frac{\partial F}{\partial P^\alpha}\right|_{P=n} +\ldots+ \frac{1}{\delta!}(P-n)^{\alpha_1}(P-n)^{\alpha_\delta}\left.\frac{\partial^\delta F}{\partial P^{\alpha_1}\ldots\partial P^{\alpha_\delta}}\right|_{P=n}$$

All independent ingoing momenta are treated on equal footing. Only in $\tau^{(2)}F(P)$ the first subtraction is made at $P=0$ in order to have an a-type counterterm vanishing when $m\to 0$. At the price of a more complex subtraction scheme, one could have eliminated this counterterm altogether. The possibility of such a subtraction at p=0 has to be verified in each theory.

We remark that the last subtraction in the two-point function will not be infrared finite, but this fact will not jeopardise the purpose of this paper, which is to produce infrared finite Green functions and normal products. We do get, for example, an infrared finite b-type counterterm, in case we want to enforce the usual normalization condition on the two-point function by adding a term of the type $\frac{1}{2} b\, \partial_\mu A(x) \partial^\mu A(x)$ to the Lagrangian.

We now state the normalization conditions following from equs. (II.6), which will be needed later on. Every logarithmically divergent Feynman integral vanishes after subtraction, when the independent momenta p_i are set equal to η. For example the four-point function $\Gamma^{(4)}(P_1,P_2,P_3,-P_1-P_2-P_3)$ satisfies :

$$\Gamma^{(4)}(\eta,\eta,\eta,-3\eta) = \text{trivial contribution} \qquad \text{(II.7)}$$

For quadratically divergent expressions of say two independent momenta $F(P_1,P_2)$ we get after subtraction:

$$D_1 F(P_1,P_2) = F(0,0) = 0$$

$$D_2^{P_1 P_2} F(P_1,P_2) = \left(g^{\mu\nu} - \frac{\eta^\mu \eta^\nu}{\eta^2}\right) \frac{\partial^2 F}{\partial P_i^\mu \partial P_j^\nu}\bigg|_{P=\eta} = 0 \qquad \text{(II.8)}$$

Notice that only in the process of manufacturing finite integrals, when applying the operations (II.6) do we put $n.p=0$, that is the $n.p=0$ rule is part of our "finite part" prescription.

Our A^4 theory is thus normalized as

$$\Gamma^{(2)}(P,-P)\Big|_{P^2=m^2} = 0$$

$$\frac{\partial \Gamma^{(2)}(P,-P)}{\partial P^2}\Bigg|_{P^2=-\mu^2} = i \qquad , \; n^2=-\mu^2 \tag{II.9}$$

$$\Gamma^{(4)}\,(n,n,n,-3n) = ig$$

Up to 2nd order our two-point function for small m^2 is:

$$\Gamma^{(2)}(P,-P)=i\{P^2 - m^2 + a - \frac{1}{12}\frac{1}{(4\pi)^4}\, g^2P^2[\,\log(-\frac{P^2}{\mu^2}) - 1\,]\} \tag{II.10}$$

$$a = \frac{g^2}{12(4\pi)^4}\; m^2\; [\,\log(\frac{m^2}{\mu^2}) - 1\,]$$

Had we wanted to implement the usual normalization $\Gamma^{(2)}(n,-n)=-i(m^2+\mu^2)$condition by adding an ip^2b term the result for a and b would be:

$$a = \frac{g^2}{12(4\pi)^4} m^2 \log \frac{m^2}{\mu^2} \tag{II.11}$$

$$b = -\frac{g^2}{12(4\pi)^4} \{1 - \frac{m^2}{\mu^2} \log \frac{m^2}{\mu^2}\}, \qquad (m \to 0)$$

We see that $a \to 0$ as $m \to 0$ and b stays finite in this limit.

The infrared finiteness of these Green functions will be demonstrated in the following section.

Normal products are introduced as usual. If $\mathcal{O}$ is a formal product of the basic field and its derivativies of canonical dimension d, then the normal product of degree $\delta = d + \alpha (\alpha = 0, 1, \ldots)$ is defined by

$$<0|T\ N_\delta[\mathcal{O}](X)\ A(X_1) \ldots\ A(X_n)|\ 0> = \tag{II.12}$$

$$= \text{finite part of } {}^{(o)}<0|T:\mathcal{O}^{(o)}:(X) A^{(o)}(X_1) \ldots A^{(o)}(X_n) \exp i \int :$$

$$: L_{int}(A^{(o)}:(y) d^4 y | 0)^{(o)}$$

where the number of subtractions to be made for proper diagrams is given by

$$\delta(\gamma) = \begin{cases} \delta - N_\gamma, & \text{if the vertex associated with } \mathcal{O} \text{ belongs to } \gamma \\ 4 - N_\gamma, & \text{otherwise} \end{cases} \tag{II.13}$$

These subtractions are made by applying the Taylor operators around $p = \eta$

$$t^{(\delta)}F(P) = F(n) + (P-n)^{\alpha}\frac{\partial F}{\partial P^{\alpha}}\bigg|_{P=n} + \ldots + \frac{1}{\delta!}(P-n)^{\alpha_1}\ldots(P-n)^{\alpha_\delta}\frac{\partial^{\delta} F}{\partial P^{\alpha_1}\ldots\partial P^{\alpha_\delta}}\bigg|_{P=n}$$

(II.14)

i.e., only $t^{(2)}$ differs from $\tau^{(2)}$ as given by equ.(II.6). In order to do this, a set of linearly independent ingoing momenta has to be chosen and different choices now define different **normal** products. We will in the sequel only use normal products obtained via (II.10) by choosing as n independent momenta the ones associated with the fields $A(X_1)\ldots A(X_n)$, since these lead to equations of motion inside bilinear normal products without the $N_4(\partial_\mu A\partial^\mu A)$-term.

The prescription contained in equs. (II.12)-(II.14) produces a set of infrared finite normal products as will be shown in the next section.

One can in the usual way derive equations of motion and Zimmerman identities for these normal products. We may on the other hand, for $\delta \leq 4$ say, introduce a set of normal products, called $\hat{N}_\delta$, replacing the operators of equ.(II.14) by the ones used in the definition for Green functions, namely equ. (II.6).

The equations of motion are then

$$(\partial_X^2+m^2)\langle 0|TA(X)A(X_1)\ldots A(X_n)|0\rangle = \langle 0|T\{aA(X)-\frac{1}{3!}g\hat{N}_3(A^3)(X)\}X\rangle -$$
$$-\, i\sum_{k=1}^{N}\delta(X-X_k)\langle 0|TA(X_1)\ldots\hat{A}(X_k)\ldots A(X_N)|0\rangle$$

(II.15)

$$\langle 0|T\hat{N}_4[A\partial^2A](X)X\rangle = \langle 0|T\hat{N}_4\{-\frac{1}{3!}gA^4(X)+(a-m^2)A^2(X)\}X|0\rangle -$$
$$-\, i\sum_k\delta(X-X_k)\langle 0|\ TX|0\rangle$$

where $\hat{A}(X_k)$ means, that the field $A(X_k)$ has been deleted and $X = \prod_{i=1}^{N} A(X_i)$.

Power counting shows that the minimally subtracted $\hat{N}_4$'s are at most logarithmically divergent, whereas $\hat{N}_4(A^2)$ satisfies the Zimmermann idendity.

$$\hat{N}_2(A^2)=\hat{N}_4(uA^2)+r\hat{N}_4(\partial_\mu A\partial^\mu A)+s\hat{N}_4(A\partial^2A)+t\hat{N}_4\ (A^4) \qquad \text{(II.16)}$$

where the coefficients are given by

$$r=-\frac{1}{8}(g^{\mu\nu}-\frac{n^\mu n^\nu}{n^2})\frac{\partial}{\partial P^\mu}\frac{\partial}{\partial Q^\nu}\langle 0|T\hat{N}_2[A^2](0)\tilde{A}(P)\tilde{A}(Q\ )|0\rangle^{prop}\Big|_{P=Q=n}$$

$$s=-\frac{1}{8}(g^{\mu\nu}-\frac{n^\mu n^\nu}{n^2})\frac{\partial}{\partial P^\mu}\frac{\partial}{\partial P^\nu}\langle 0|T\hat{N}_2[A^2](0)\tilde{A}(P)\tilde{A}(P)|0\rangle^{prop}\Big|_{P=n}$$

$$t=\frac{1}{4!}\langle 0|T\hat{N}_2[A^2](0)(\tilde{A}(n))^4|0\rangle^{prop} \qquad \text{(II.17)}$$

$$u=\frac{1}{2!}\langle 0|T\hat{N}_2[A^2](0)\tilde{A}(P)\tilde{A}(P)|0\rangle^{prop}\Big|_{P=0}$$

This last coefficient is logarithmically divergent, since, although we will in the next section show that $\tilde{N}_2[A^2]$ is actually finite at non-exceptional momenta, $p=0$ does not belong to this class. Thus all the normal products occuring in equ. (II.16) are at most logarithmically divergent. In the next section we show via differential equations that in reality all $\tilde{N}_4(A\partial^2A)$, $\tilde{N}_4(\partial_\mu A\partial^\mu A)$ and $\tilde{N}_4(A^4)$ are finite, only $\tilde{N}_4(A^2)$ being logarithmically divergent. The multiplication by u produces the finite object $\tilde{N}_4(uA^2)$, which is not any more subtracted at $P=0$.

III. PROOF OF INFRARED FINITENESS

The Callan-Symanzik (CS) and Renormalization Group (RG) equations[4] can be used to prove the existence, for non-exceptional momenta, of the zero mass limit of the theory described in the previous section. As in ref.[5], the derivation is made easier by the use of Differential Vertex Operations (DVO) introduced by

$$\Delta_{\delta,\mathcal{O}} G^{(N)}(X_1,\ldots X_n) = \text{Finite part of } \int dX^{(o)} \langle |T{:}\mathcal{O}^{(o)}{:}(X)A^o(X_1\ldots A^o(X_N)\,.$$

$$.\exp(i\int dy\ :L_I^{(o)}:(y)|\rangle^{(o)} \qquad \text{(III.1)}$$

where the formal X integration means that the graphs to be subtracted by the scheme of equation (II.14) with degree function $\delta(\gamma)$ given by equation (II.13) have zero momentum entering at the special vertex (note that with this definition a DVO differs from an integrated normal product because of the subtraction terms). The DVOs we will need are the following:

$$\begin{aligned}
&\Delta_o \quad \text{corresponding to } \mathcal{O} = \frac{i}{2} A^2 \quad \text{and} \quad \delta=2 \\
&\Delta_1 \quad \text{corresponding to } \mathcal{O} = \frac{i}{2} A^2 \quad \text{and} \quad \delta=4 \\
&\Delta_2 \quad \text{corresponding to } \mathcal{O} = \frac{i}{2} \partial_\mu A \partial^\mu A \quad \text{and} \quad \delta=4 \\
&\Delta_3 \quad \text{corresponding to } \mathcal{O} = \frac{i}{4!} A^4 \quad \text{and} \quad \delta=4
\end{aligned} \tag{III.2}$$

Counting identities and differentiation formulas with respect to any of the parameters of the theory can be worked out in a standard way by manipulation of the DVOs above. Hereby one should pay attention to the fact that a DVO depends on η through the subtractions. Thus in calculating $\frac{\partial \Gamma^{(N)}}{\partial \mu^2}$ using

$$\Gamma^{(N)} = \sum_{\lambda_1, \lambda_3 = 0}^{\infty} \frac{a^{\lambda_1} (-g)^{\lambda_3}}{\lambda_1! \, \lambda_3!} \Delta_1^{\lambda_1} \Delta_2^{\lambda_3} \Gamma_o^{(N)} \tag{III.3}$$

one gets, in addition to the usual term $\frac{\partial a}{\partial \mu^2} \Delta_1$ also contributions coming from the differentiation of subtractions for proper graphs with two and four legs, i.e.

$$\frac{\partial \Gamma^{(N)}}{\partial \mu^2} = \frac{\partial a}{\partial \mu^2} \Delta_1 \Gamma^{(N)} + \alpha_2 \Delta_2 \Gamma^{(N)} + \alpha_3 \Delta_3 \Gamma^{(N)} \qquad \text{(III.4)}$$

where the coefficients α_2 and α_3 can be obtained directly from the forest formula or, most easily, from the normalization conditions for DVOs. We obtain

$$\alpha_2 = - \frac{i}{6} \left(g_{\mu\nu} - \frac{n_\mu n_\nu}{n^2}\right) \partial^\mu_P \partial^\nu_P \frac{\partial \Gamma^{(2)}}{\partial \mu^2} (P,-P)\Big|_{P=n}$$

$$\alpha_3 = - i \frac{\partial \Gamma^{(4)}}{\partial \mu^2} (P_1, P_2, P_3, - \sum_1^3 P_i) \qquad \text{(III.5)}$$

Analogously, one derives a Zimmermann identity

$$\Delta_o \Gamma^{(N)} = r_1 \Delta_1 \Gamma^{(N)} + r_2 \Delta_2 \Gamma^{(N)} + r_3 \Delta_3 \Gamma^{(N)} \qquad \text{(III.6)}$$

with $r_1 = -i \Delta_o \Gamma^{(2)} (0,0)$;

$$r_2 = - \frac{i}{6} \left(g_{\mu\nu} - \frac{n_\mu n_\nu}{n^2}\right) \partial^\mu_P \partial^\nu_P \Delta_o \Gamma^{(2)} (P,-P)\Big|_{P=n}$$

$$r_3 = -i \Delta_o \Gamma^{(4)} (n,n,n;-3n)$$

The other formulas have the usual form:

$$\frac{\partial}{\partial \mu^2} \Gamma^{(N)} = \left(\frac{\partial a}{\partial m^2} - 1\right) \Delta_1 \Gamma^{(N)} \tag{III.7}$$

$$\frac{\partial}{\partial g} \Gamma^{(N)} = \frac{\partial a}{\partial g} \Delta_1 \Gamma^{(N)} - \Delta_3 \Gamma^{(N)} \tag{III.8}$$

$$N\Gamma^{(N)} = 2(a-m^2)\, \Delta_1 \Gamma^{(N)} + 2\Delta_2 \Gamma^{(N)} - 4g\Delta_3 \Gamma^{(N)} \tag{III.9}$$

Following the usual argument we establish now the CS and RG equations

$$\left[m^2 \frac{\partial}{\partial m^2} + \mu^2 \frac{\partial}{\partial \mu^2} + \beta \frac{\partial}{\partial g} - N\gamma \right] \Gamma^{(N)} = \alpha m^2 \Delta_o \Gamma^{(N)} \tag{III.10}$$

$$\left[\mu^2 \frac{\partial}{\partial \mu^2} + \sigma \frac{\partial}{\partial g} - N\tau \right] \Gamma^{(N)} = 0 \tag{III.11}$$

Where the coefficients $\alpha, \beta, \gamma, \sigma$ and τ can be determined using equations (III.4), (III.6), (III.7) to (III.9) into (III.10) and (III.11) and equating to zero the coefficient of each DVO. Subtracting (III.11) from (III.10) we obtain

$$\left[m^2 \frac{\partial}{\partial m^2} + \lambda_1 \frac{\partial}{\partial g} - N\lambda_2 \right] \Gamma^{(N)} = \alpha m^2 \Delta_o \Gamma^{(N)} \tag{III.12}$$

with $$\lambda_2 = \frac{i}{12} \alpha m^2 \left(g^{\mu\nu} - \frac{n^\mu n^\nu}{n^2}\right) \partial^P_\mu \partial^P_\nu \Delta_o \Gamma^{(2)}(P, -P) \Big|_{P=n} \tag{III.13}$$

$$\lambda_1 = 4g\lambda_2 + i\alpha m^2 \Delta_o \Gamma^{(4)}(n, n, n, -3n)$$

obtained by application of the normalization conditions at η. With some modifications, similar equations can be derived for proper functions containing one normal product vertex. Thus, for example $\Gamma^{(N)}_{2,A^2}$, which indicates the proper vertex functions containing one N_2A^2 normal product vertex, satisfies equations (III.7) and (III.8) and also

$$\frac{\partial}{\partial\mu^2}\Gamma^{(N)}_{2,A^2} = \frac{\partial a}{\partial\mu^2}\Delta_1\Gamma^{(N)}_{2,A^2} + \alpha_2\Delta_2\Gamma^{(N)}_{2,A^2} + \alpha_3\Delta_3\Gamma^{(N)}_{2,A^2} +$$

$$+ \alpha_4\Gamma^{(N)}_{2,A^2} \qquad \text{(III.14)}$$

$$\alpha_4 = \frac{1}{2}\frac{\partial}{\partial\mu^2}\Gamma^{(2)}_{2,A^2}(-2\eta;\eta,\eta)$$

$$\Delta_o\Gamma^{(N)}_{2,A^2} = r_1\Delta_1\Gamma^{(N)}_{2,A^2} + r_2\Delta_2\Gamma^{(N)}_{2,A^2} + r_3\Delta_3\Gamma^{(N)}_{2,A^2} + r_4\Gamma^{(N)}_{2,A^2}$$

$$r_4 = \frac{1}{2}\Delta_o\Gamma^{(N)}_{2,A^2}(-2\eta;\eta,\eta) \qquad \text{(III.15)}$$

$$(N-2)\,\Gamma^{(N)}_{2,A^2} = 2(a-m^2)\,\Delta_1\Gamma^{(N)}_{2,A^2} + 2\Delta_2\Gamma^{(N)}_{2,A^2} - 4g\Delta_3\Gamma^{(N)}_{2,A^2} \qquad \text{(III.16)}$$

The therm $\alpha_4\Gamma^{(N)}_{2,A^2}$ in (III.14) comes from differentiation of subtractions for subgraphs containing the special vertex. The presence of the term $r_4\Gamma^{(N)}_{2,A^2}$ in (III.15), on the other hand, is due to subtractions for graphs containing both the Δ_o and the N_2A^2 vertices. In the same way as before we can write now

$$\left[m^2 \frac{\partial}{\partial m^2} + \mu^2 \frac{\partial}{\partial \mu^2} + \beta \frac{\partial}{\partial g} -(N-2)\gamma \right]\Gamma^{(N)}_{2,A^2} = \alpha m^2 \Delta_o \Gamma^{(N)}_{2,A^2} +$$

$$+ u\Gamma^{(N)}_{2,A^2} \quad ; \; u = -\alpha \; m^2 r_4 + \mu^2 \alpha_4 \qquad \text{(III.17)}$$

$$\left[\mu^2 \frac{\partial}{\partial \mu^2} + \sigma \frac{\partial}{\partial g} - (N-2)\; \tau\right] \Gamma^{(N)}_{2,A^2} = v \; \Gamma^{(N)}_{2,A^2}$$

$$v = \mu^2 \alpha_4 \qquad \text{(III.18)}$$

$$(m^2 \frac{\partial}{\partial m^2} + \lambda_1 \frac{\partial}{\partial g} - (N-2)\lambda_2)\Gamma^{(N)}_{2,A^2} = \alpha m^2 \Delta_o \Gamma^{(N)} - \alpha m^2 \alpha_4 \Gamma^{(N)}_{2,A^2}$$

$$\text{(III.19)}$$

If now we accept the vality of power counting arguments(6) for determining the leading power of the zero mass limit then, as $m\to 0$,$\Gamma^{(N)}$, $\Gamma^{(N)}_{2,A^2}$, $\Delta_o \Gamma^{(N)}$ and $\Delta_o \Gamma^{(N)}_{2,A^2}$ are at worst logarithmically divergent so that from

$$\lambda_1 ,\; \lambda_2 ,\; \alpha_4 \;\sim 0\; (m^2 \lg^x m^2)$$

$$\alpha \sim 0\; (\lg^x m^2) \qquad \text{(III.20)}$$

and from (III.12) and (III.19) results

$$\lim_{m\to 0} m^2 \frac{\partial}{\partial m^2} \Gamma^{(N)} = 0 \text{ and } \lim_{m\to 0} m^2 \frac{\partial}{\partial m^2} \Gamma^{(N)}_{2,A^2} = 0 \qquad \text{(III.21)}$$

giving that as $m\to 0$ both $\Gamma^{(N)}$ and $\Gamma^{(N)}_{2,A^2}$ **stay finite.**

The discussion of the infrared behaviour of vertex functions containing one normal product of degree higher than two is simpler if all normalization conditions satisfied by these functions are at non zero momenta. If this is the case the proof of infrared finiteness is as follows. First we **note** that minimally subtracted normal **products** can, by power counting, be only logarithmically divergent. Suppose now that we have proved the infrared finiteness of vertex functions of normal products up to those of degree δ. Then Zimmermann identities relating $N_{\delta+1}[\mathcal{O}]$ with $N_\delta[\mathcal{O}]$ tell us that the vertex functions of $N_{\delta+1}[\mathcal{O}]$, with $\delta+1>$ canonical dimension of $\mathcal{O}$, can be only logarithmically divergent also. Following similar steps to these that produced (III.19) we derive the equation

$$\left[m^2 \frac{\partial}{\partial\mu^2} + \lambda_1 \frac{\partial}{\partial g} - N\lambda_2\right]\Gamma^{(N)}_{\delta+1,\mathcal{O}_i} = \alpha m^2 \Delta_o \Gamma^{(N)}_{\delta+1,\mathcal{O}_i} +$$

$$+ \sum_j M_{ij}\,\Gamma^{(N)}_{\delta+1,\mathcal{O}_j} \qquad \text{(III.22)}$$

where, because of the normalization conditions, in the zero mass limit

$$M_{ij} \sim O(m^2 \lg^x m^2)$$

From this and from (III.22) follows then

$$\lim_{m\to 0} m^2 \frac{\partial}{\partial m^2} \Gamma^{(N)}_{\delta+1,\mathcal{O}_i} = 0$$

giving the desired result that as $m\to 0$ $\Gamma^{(N)}_{\delta+1,\mathcal{O}_i}$ stay finite .

As a concrete example of the construction above we well consider now the case of normal product vertex functions, $\bar{\Gamma}^{(N)}_{4}$, of degree four, subtracted according to the scheme specified by (II.6). In this case power counting arguments give that as $m\to 0$ $\bar{\Gamma}^{(N)}_{4,A\partial^2 A}$, $\bar{\Gamma}^{(N)}_{4,\partial_\mu A\partial^\mu A}$ and $\bar{\Gamma}^{(N)}_{4,A^4}$ are for non-exceptional momenta logarithmically divergent at worst. Then, as argued previously, $\bar{\Gamma}^{(N)}_{4,A^2}$ is also logarithmically divergent at worst because of Zimmermann's identity (II.16). The differential equations to be used in this case are of the form

$$\left[m^2 \frac{\partial}{\partial m^2} + \lambda_1 \frac{\partial}{\partial g} - (N-2t)\,\lambda_2\right] \bar{\Gamma}^{(N)}_{4,\mathcal{O}_i} = \alpha m^2 \Delta_o \bar{\Gamma}^{(N)}_{4,\mathcal{O}_i}$$

$$+ \Sigma\, M_{ij}\, \bar{\Gamma}^{(N)}_{4,\mathcal{O}_j} \qquad \text{(III.23)}$$

where

$t = 1$ for $\mathcal{O}_i = A\partial^2 A,\ \partial_\mu A\partial^\mu A,\ A^2$ and

$t = 2$ for $\mathcal{O}_i = A^4$

The coefficients M_{ij} are easily obtained from the normalization conditions. Thus as m goes to zero we get from (III.23)

$$m^2 \frac{\partial}{\partial m^2} \bar{\Gamma}^{(N)}_{4,\mathcal{O}_i} = O(m^2 \lg^X m^2) \text{ for } \mathcal{O}_i = A\partial^2 A,\ \partial_\mu A\partial^\mu A,\ A^4 \tag{III.24}$$

and

$$m^2 \frac{\partial}{\partial m^2} \bar{\Gamma}^{(N)}_{4,A^2} = -\alpha m^2 \Delta_o \bar{\Gamma}^{(N)}_{4,A^2}(0,0)\bar{\Gamma}^{(N)}_{4,A^2} + O(m^2 \lg^X m^2) \tag{III.25}$$

so that $\bar{\Gamma}^{(N)}_{4,\mathcal{O}_i}$ ($\mathcal{O}_i = A\partial^2 A,\ \partial_\mu A\partial^\mu A,\ A^4$) are actually infrared finite and $\bar{\Gamma}^{(N)}_{4,A^2}$ although logarithmically divergent can be made finite by multiplication by the factor u as Zimmermann's idendity (II.16) now shows.

IV. WARD IDENTITY

As an application we derive the Ward identity for a charged scalar field. The main problem is the unavailability of the simple rule, obeyed by normal products subtracted at $p=0$: $\partial_\mu N_\delta[\mathcal{O}(X)] = N_{\delta+1}[\partial_\mu \mathcal{O}(x)]$. Instead the subtraction scheme of equ. (II.14) leads to the rule

$$(P_i-n)_\mu <0[TN_\delta\ ]\,\mathcal{O}(-\Sigma P_i)\ \tilde{A}(P_i)\ \ldots\ \tilde{A}^+(P_n)|0> =$$

$$= <0|TN_{\delta+1}\ [\ (P_i-n)_\mu\,\mathcal{O}(-\Sigma P_i)]\ \tilde{A}(P_1)\ \ldots\tilde{A}^+(P_n)\ |0> \qquad (IV.1)$$

where P_i is one of the independent momenta.

In this section we are interested in an object of the type $N_3\ [\ A^+\overleftrightarrow{\partial}_\mu A\ ](X)$, whose two-point function is the only one needing subtractions.Thus,in order to get rid of the n-dependent terms in $<0|TN_3[A^+\overleftrightarrow{\partial}_\mu A\ ](0)\ \tilde{A}(P_1)\ \tilde{A}(P_2)|0>$, we subtract in such a way, that the momentum $Q= -P_1 - P_2$ flowing into the graph at the special vertex is zero at the subtraction point. Since $Q = - P_1 - P_2=-(P_1 -n) - (P_2+n)$, we subtract at $P_1=n, P_2=-n$. The n-dependent terms in equ. (IV.1) now add up to zero.

We are thus motivated to introduce the current operator

$$j_\mu(X) = N_3\ [\ A^+\ \overleftrightarrow{\partial}_\mu\ A\ ](X) \qquad (IV.2)$$

defined by imposing the following subtraction scheme with degree function $\delta(\gamma)$ given by equ. (II.13):

Whenever the special vertex belongs to a proper divergent graph with two legs, we subtract at the two independent ingoing momenta P_1 and P_2 at $P_1=\eta$ and $P_2=-\eta$, applying the Taylor operator of equ. (II.14); otherwise we subtract at all $P_i=\eta$, using the scheme of equ. (II.6)[7]. In every theory this prescription has to be checked for its infrared finiteness, since in the subtractions no momentum is flowing into the special vertex.

The Ward identity for the current (IV.2) can be established directly by graphical analysis or using the equation of motion

$$<0\,|T\,N_4\left[A^+\overleftrightarrow{\partial}^2 A\right](X)Y|0> \;=\; <0|TN_4\{-\frac{1}{3!}\,g(A(X)A^+(X))^2 + \tag{IV.3}$$

$$+\,(a-m^2)\,A^+(X)A(X)\}\,Y|0> -i\sum_j\left[\delta(X-Z_j)-\delta(X-W_j)\right]<0|TY|0>$$

where $Y=\prod_{j=1}^{m}A(Z_j)A^+(W_j)$ and the N_4's occuring in the above equation are defined in the same way as the current. Although these normal products are not infrared finite, this divergence cancels in $N_4[A^+\overleftrightarrow{\partial}^2 A](X)$ producing the infrared finite equation

$$\partial^\mu_X<0|TN_3[A^+\overleftrightarrow{\partial}_\mu A](X)Y|0> \;=\; <0|TN_4[A^+\overleftrightarrow{\partial}^2 A](X)Y> \;= \tag{IV.4}$$

$$=\sum_j\left[\delta(X-Z_j)-\delta(X-W_j)\right]<0|TY|0>$$

The current $j_\mu(X)$ is obviously infrared finite.

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FOOTNOTES AND REFERENCES

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