Instituto de Física
Universidade de São Paulo

CLASSICAL RE-INTERPRETATION OF THE SCHRÖDINGER EQUATION ACCORDING TO STOCHASTIC ELECTRODYNAMICS

Dechoum, K.; França, H. M.; Malta, C. P.

Instituto de Física, Universidade de São Paulo, SP, Brasil

Publicação IF - 1387/99

CLASSICAL RE-INTERPRETATION OF THE SCHRÖDINGER EQUATION ACCORDING TO STOCHASTIC ELECTRODYNAMICS

K. Dechoum[1], H.M. França* [2], and C.P. Malta[2]

[1] Departamento de Física Moderna, Universidad de Cantábria, 39005, Santander, Spain.
[2] Instituto de Física, Universidade de São Paulo, C.P. 66318, 05315-970 São Paulo, SP, Brazil

ABSTRACT

We study the statistical evolution of a charged particle moving in phase space under the action of the vacuum fluctuations of the zero-point electromagnetic field. Our starting point is the Liouville equation, from which we derive a classical stochastic Schrödinger like equation for the probability amplitude in configuration space. The standard Schrödinger equation used in Quantum Mechanics is obtained as a particular case of the classical stochastic Schrödinger like equation. An inconsistency appearing in the standard Schrödinger equation, when we take into account the vacuum electromagnetic fluctuations and the radiation reaction, is clearly identified by means of two examples using different sources of electromagnetic noise. The classical stochastic Schrödinger like equation, however, is consistently interpreted within the realm of Stochastic Electrodynamics. A simple application with a prediction that can be confirmed experimentally is presented.

Keywords: Zero-point fluctuations; Stochastic electrodynamics.

---

* Corresponding author; e-mail: hfranca@fma.if.usp.br

1 Introduction

The classical electromagnetic theory has been largely extended by the program of the Stochastic Electrodynamics (SED) [1, 2], due to the inclusion of the effects of the real electromagnetic zero-point radiation. According to the SED picture, there is a clear correspondence between the nonrelativistic Heisenberg equations of motion, for a spinless charged particle interacting with the quantized electromagnetic field of Quantum Electrodynamics (QED), and the classical (Langevin type) Abraham-Lorentz equation with real vacuum fluctuation forces. Therefore, the role of the random radiation field reservoir, and the radiation reaction force, are naturally incorporated into the SED approach.

A mathematical tool widely used in SED is the Fokker-Planck equation, which is derived from the stochastic Liouville equation for describing the Brownian motion of the microscopic charged particles [3]. Unfortunately, however, this method has a restricted use due to the mathematical difficulties for solving the Fokker-Planck equation, mainly in the cases associated with the motion under nonlinear forces.

Our purpose here is to show that the stochastic Liouville equation can be put in a mathematical form that is easier to manipulate even in the case of nonlinear forces. We shall derive a classical Schrödinger like equation from the Liouville equation, using a procedure similar to that introduced by Wigner [4], in order to describe Quantum Mechanics in phase space. Our approach introduces a free parameter $\hbar'$ in the Wigner type transform [5]. We shall show that this procedure enables us to make to make a clear distinction between the free parameter $\hbar'$ and the Planck's constant $\hbar$. Only the vacuum electromagnetic fluctuations will depend on the numerical value of the Planck's constant $\hbar$. We shall see that this distinction will be of great help in order to clarify the physical meaning of the Schrödinger like equation and its interpretation within the realm of a purely classical theory.

2 Connecting the stochastic Liouville equation to a Schrödinger like classical stochastic equation

The description of classical phenomena by classical statistical mechanics is based on the concept of phase space. The mean value of any dynamical variable $A(\mathbf{x}, \mathbf{p}, t)$ is calculated according to the relation

$$\langle A \rangle = \int A(\mathbf{x}, \mathbf{p}, t)\, W(\mathbf{x}, \mathbf{p}, t)\, \mathbf{d^3 x d^3 p} \quad , \tag{1}$$

and the probability density distribution in phase space, $W(\mathbf{x}, \mathbf{p}, t)$, evolves in time according to the Liouville equation

$$\frac{\partial W}{\partial t} + \dot{\mathbf{x}} \cdot \frac{\partial W}{\partial \mathbf{x}} + \dot{\mathbf{p}} \cdot \frac{\partial W}{\partial \mathbf{p}} = 0 \quad , \tag{2}$$

where $\dot{\mathbf{x}}$ and $\dot{\mathbf{p}}$ are obtained from the classical Hamilton's equations of motion.

Consider an ensemble of systems which consist of a nonrelativistic spinless charged particle interacting only with the electromagnetic field. The Hamiltonian which describes the time evolution of the whole system (particle plus field) is

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + e\phi + H_{\mathrm{R}} \quad , \tag{3}$$

where e and m are the charge and mass of the particle, repectively, $\phi(\mathbf{x}, \mathbf{t})$ is the scalar potential, and

$$\mathbf{A}(\mathbf{x}, t) = \mathbf{A}_{\mathrm{ext}} + \mathbf{A}_{\mathrm{VF}} + \mathbf{A}_{\mathrm{RR}} \quad . \tag{4}$$

is the vector potential. The term $\mathbf{A}_{\rm ext}$ is an external deterministic disturbance. The term $\mathbf{A}_{\rm VF}$ is the vector potential associated with the real vacuum fluctuations, and can be written as

$$\mathbf{A}_{\rm VF}(\mathbf{x},t) = \sum_{\lambda=1}^{2}\sum_{\mathbf{k}}\sqrt{\frac{2\pi\hbar c^2}{\omega_{\mathbf{k}}V}}\hat{\epsilon}(\mathbf{k},\lambda)\left[a_{\mathbf{k}\lambda}\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)} + a^{*}_{\mathbf{k}\lambda}\, e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)}\right] \quad , \tag{5}$$

where V is the volume containing the particle and the radiation field, $\mathbf{k}$ is the wave vector, $\omega_{\mathbf{k}} = c|\mathbf{k}|$, λ is the polarization index, and $\hat{\epsilon}(\mathbf{k},\lambda)$ are the polarization vectors. The amplitudes $a_{\mathbf{k}\lambda}$ are taken to be random variables. The random character of the field is contained in these variables which are such that $\langle a_{\mathbf{k}\lambda}\rangle = 0$ and $\langle |a_{\mathbf{k}\lambda}|^2\rangle = 1/2$ ($\langle\ \rangle$ denotes the ensemble average). The term $\mathbf{A}_{\rm RR}$ is the vector potential that describes the radiation reaction [1, 2] and $H_{\rm R}$ is the Hamiltonian of the background radiation field (contains only variables of the field). In the case of zero temperature, $H_{\rm R}$ can be written as [2]

$$H_{\rm R} = \frac{1}{8\pi}\int \mathbf{d^3r}\ (\mathbf{E}^2_{\rm VF} + \mathbf{B}^2_{\rm VF}) \quad , \tag{6}$$

where

$$\mathbf{E}_{\rm VF} = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}_{\rm VF} \quad , \quad \mathbf{B}_{\rm VF} = \nabla \times \mathbf{A}_{\rm VF} \quad , \tag{7}$$

so that

$$\langle H_{\rm R}\rangle = \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} \quad . \tag{8}$$

The extension to a non-zero temperature T is obtained by introducing the factor $\coth(\hbar\omega_{\mathbf{k}}/2kT)$.

Each particle of the ensemble evolves in time according to the Hamilton's equations

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{p}} = \frac{1}{m}(\mathbf{p} - \frac{e}{c}\mathbf{A}) \quad , \tag{9}$$

and

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{x}}\left[\frac{e}{mc}\mathbf{p}\cdot\mathbf{A} - \frac{e^2}{2mc^2}\mathbf{A}^2 - e\phi\right] \quad . \tag{10}$$

Substituting the equations (9), (10) into (2) we get the Liouvillian form of the equation governing the time evolution of the ensemble of particles for each realization of the stochastic field $\mathbf{A}_{\text{VF}}$, namely

$$\frac{\partial W}{\partial t} + \frac{1}{m}(\mathbf{p} - \frac{e}{c}\mathbf{A})\cdot\frac{\partial W}{\partial \mathbf{x}} + \frac{\partial W}{\partial \mathbf{p}}\cdot\frac{\partial}{\partial \mathbf{x}}\left[\frac{e}{mc}\left(\mathbf{p} - \frac{e}{2c}\mathbf{A}\right)\cdot\mathbf{A} - e\phi\right] = 0 \quad . \tag{11}$$

It is important to stress that, after obtaining the solution of (11), it is necessary to calculate the ensemble average over all possible realizations of the field $\mathbf{A}_{\text{VF}}$ in order to obtain the average distribution $\langle W(\mathbf{x},\mathbf{p},t)\rangle$. This is done by considering the average over the random Gaussian amplitudes $a_{\mathbf{k}\lambda}$ in (5). Notice that in (11) $\mathbf{A} = \mathbf{A}(\mathbf{x}, \mathbf{t})$ and $\phi = \phi(\mathbf{x}, t)$ are explicit functions of the variables $\mathbf{x}$ and t.

Consider the Fourier transform defined by

$$\widetilde{W}(\mathbf{x},\mathbf{y},t) \equiv \int \mathbf{d}^3\mathbf{p}\; W(\mathbf{x},\mathbf{p},t)\; \exp\left(-\frac{2i\mathbf{p}\cdot\mathbf{y}}{\hbar'}\right) \quad , \tag{12}$$

where $\mathbf{y}$ is a point in the configuration space and $\hbar'$ is a free parameter having dimension of action. The meaning of the free parameter $\hbar'$ will be discussed further below. Notice that (12) corresponds to the well known Wigner transform [4] if $\hbar' = \hbar$. Using the definition (12) the Liouville equation (11) assumes the following form

$$\int \mathbf{d}^3\mathbf{y} \exp\left(\frac{2i\mathbf{p}\cdot\mathbf{y}}{\hbar'}\right)\left\{i\hbar'\frac{\partial \widetilde{W}}{\partial t} - \frac{\hbar'^2}{2m}\frac{\partial^2 \widetilde{W}}{\partial \mathbf{x}\cdot\partial \mathbf{y}} - \frac{ie\hbar'}{mc}\mathbf{A}(\mathbf{x},t)\cdot\frac{\partial \widetilde{W}}{\partial \mathbf{x}} \quad + \right.$$

$$\left. -\; 2\mathbf{y}\cdot\frac{\partial}{\partial \mathbf{x}}\left[\frac{e}{mc}\left(\mathbf{p} - \frac{e}{2c}\mathbf{A}(\mathbf{x},t)\right)\cdot\mathbf{A}(\mathbf{x},t) - e\phi\right]\widetilde{W}\right\} \;=\; 0 \quad . \tag{13}$$

In what follows we will concentrate our attention in the particular case of very small $\hbar'$ ($\hbar' \ll \hbar$). In this case $\widetilde{W}(\mathbf{x},\mathbf{y},t)$ is different from zero only if $|\mathbf{y}|$ is small, as can be seen from equation (12). Therefore, the functions $\mathbf{A}(\mathbf{x},t)$ and $\phi(\mathbf{x},t)$ in the last term of equation (13) can be replaced by the expressions

$$\mathbf{y}\cdot\frac{\partial}{\partial\mathbf{x}}\phi(\mathbf{x},t) \simeq \phi(\mathbf{x}+\mathbf{y},t)-\phi(\mathbf{x},t) \quad , \tag{14}$$

and

$$\mathbf{y}\cdot\frac{\partial}{\partial\mathbf{x}}\mathbf{A}(\mathbf{x},t) \simeq \mathbf{A}(\mathbf{x}+\mathbf{y},t)-\mathbf{A}(\mathbf{x},t) \quad . \tag{15}$$

Consequently, one can take

$$\int \mathbf{d^3y}\; 2\mathbf{y}\cdot\frac{\partial}{\partial\mathbf{x}}\left(\frac{e}{mc}\mathbf{p}\cdot\mathbf{A}(\mathbf{x},t)\right)\widetilde{W}(\mathbf{x},\mathbf{y},t)\;\exp\left(\frac{2i\mathbf{p}\cdot\mathbf{y}}{\hbar'}\right) =$$

$$= \int \mathbf{d^3y}\;\widetilde{W}(\mathbf{x},\mathbf{y},t)\frac{e}{mc}\left[\mathbf{A}(\mathbf{x}+\mathbf{y},t)-\mathbf{A}(\mathbf{x}-\mathbf{y},t)\right]\cdot$$

$$\cdot\left(\frac{\hbar'}{2i}\right)\frac{\partial}{\partial\mathbf{y}}\exp\left(\frac{2i\mathbf{p}\cdot\mathbf{y}}{\hbar'}\right) \quad . \tag{16}$$

Therefore, after integration by parts, eq.(13) can be written as

$$i\hbar'\frac{\partial\widetilde{W}}{\partial t}-\frac{\hbar'^2}{2m}\frac{\partial^2\widetilde{W}}{\partial\mathbf{x}\cdot\partial\mathbf{y}}-\frac{ie\hbar'}{mc}\left[\frac{\mathbf{A}(\mathbf{x}+\mathbf{y},t)+\mathbf{A}(\mathbf{x}-\mathbf{y},t)}{2}\right]\cdot\frac{\partial\widetilde{W}}{\partial\mathbf{x}}+$$

$$-\;\frac{ie\hbar'}{2mc}\left[\mathbf{A}(\mathbf{x}+\mathbf{y},t)-\mathbf{A}(\mathbf{x}-\mathbf{y},t)\right]\cdot\frac{\partial\widetilde{W}}{\partial\mathbf{y}}-\frac{ie\hbar'}{2mc}\widetilde{W}\frac{\partial}{\partial\mathbf{y}}\cdot\left[\mathbf{A}(\mathbf{x}+\mathbf{y},t)-\mathbf{A}(\mathbf{x}-\mathbf{y},t)\right]+$$

$$+\;\frac{e^2}{2mc^2}\left[\mathbf{A}^2(\mathbf{x}+\mathbf{y},t)-\mathbf{A}^2(\mathbf{x}-\mathbf{y},t)\right]\widetilde{W}+e\left[\phi(\mathbf{x}+\mathbf{y},t)+\phi(\mathbf{x}-\mathbf{y},t)\right]\widetilde{W}=0 \quad . \tag{17}$$

In what follows, we shall study the case in which the Fourier transform $\widetilde{W}(\mathbf{x},\mathbf{y},t)$ can be written in the form [4]

$$\widetilde{W}(\mathbf{x},\mathbf{y},t) = \psi^*(\mathbf{x}+\mathbf{y},t)\psi(\mathbf{x}-\mathbf{y},t) \equiv \psi^*(\mathbf{r},t)\psi(\mathbf{s},t) \quad . \tag{18}$$

Substituting (18) in (17) we obtain the following Schrödinger type equation for the functions $\psi(\mathbf{r},t)$

$$i\hbar'\frac{\partial\psi}{\partial t} = \frac{1}{2m}\left(-i\hbar'\frac{\partial}{\partial\mathbf{r}} - \frac{e}{c}\mathbf{A}(\mathbf{r},t)\right)^2\psi + e\phi(\mathbf{r},t)\psi \quad , \tag{19}$$

and the corresponding equation for $\psi^*(\mathbf{r},t)$, with the vector potential $\mathbf{A}$ as given in (4) and (5).

Therefore, the Schrödinger type equation depends on the Planck's constant only due to its presence in $\mathbf{A}_{\mathrm{VF}}$ defined in (5). In other words, equation (19) has terms which are proportional to $\sqrt{\hbar}$ and $\hbar$. Moreover, the solutions of (19) must be interpreted by considering that the limit $\hbar' \to 0$ must be taken in the end of the calculations.

3 Incompatibility of the standard Schrödinger equation with the zero-point field

The above derivation shows a clear correspondence between the quantum Schrödinger equation for spinless particles, and the classical stochastic Schrödinger like equation given by (19). The case of neutral spinning particle has been already discussed by Dechoum, França and Malta [6].

The limit $\hbar' \to 0$ of the solution of the classical stochastic Schrödinger like equation corresponds, physically, to classical (non-Heisenberg) states of motion as shown by Dechoum and França [5]. Nevertheless, we shall observe several effects, arising from the vacuum fluctuations, which depend non-trivially on the Planck's constant $\hbar$. This is better understood by means of very simple examples. One interesting example, discussed in reference [6], is the derivation of the Pauli-Schrödinger equation in the spinorial form, starting from the Liouville equation. The experimental results of the Stern-Gerlach experiment, and also the Rabi type molecular beam experiments, were appropriately described and interpreted classically, in the limit $\hbar' \to 0$, that is, in the classical limit where the particles have well-defined trajectory, and also continuous orientation of the spin vector.

The best example, however, is the one-dimensional harmonic oscillator discussed in many details in previous works [5, 7]. In order to apply equation (19) to the charged harmonic oscillator, we shall assume that the scalar potential ϕ is the simple static function satisfying $e\phi = (1/2)m\omega_0^2 x^2$, ω_0 being the natural frequency of the osccillator. We have shown in ref. [5] that by introducing the function $\Psi(x,t) \equiv \exp\left[i\frac{ex}{\hbar'}A_x(t)\right]\psi(x,t)$ we obtain for (19) the equivalent equation

$$i\hbar'\frac{\partial\Psi}{\partial t} = \left[-\frac{(\hbar')^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega_0^2 x^2}{2} - ex\left(E_{\rm RR} + E_{\rm VF}\right)\right]\Psi(x,t) \quad ., \tag{20}$$

where $E_{\rm VF}$ and $E_{\rm RR}$ depend only on t (dipole approximation). In this equation $-m\omega_0^2 x$ is the harmonic force, $eE_{\rm RR} = -\frac{e}{c}\frac{\partial}{\partial t}(\mathbf{A}_{\rm RR})_x$ is the radiation reaction force, and $eE_{\rm VF} = -\frac{e}{c}\frac{\partial}{\partial t}(\mathbf{A}_{\rm VF})_x$ is the random force. The exact solution of (20), in the form of a coherent state Ψ_{cs}, can be easily constructed [5, 7]. It is possible to show that

$$\Psi_{cs}(x,t) = \left(\frac{\hbar'}{m\omega_0}\right)^{-\frac{1}{4}} \exp\left\{\frac{i}{\hbar'}\left[xp_c(t) - g(t)\right] - \frac{m\omega_0}{\hbar'}(x - x_c(t))^2\right\} \quad , \tag{21}$$

where $p_c(t) \equiv m\dot{x}_c(t)$, $\quad 2m\dot{g}(t) \equiv p_c^2(t) - m^2\omega_0^2 x_c^2(t) + m\hbar'\omega_0$, and

$$m\ddot{x}_c(t) = -m\omega_0^2 x_c(t) + e\left[E_{RR}(t) + E_{VF}(t)\right] \quad , \tag{22}$$

so that $x_c(t)$ is the classical stochastic trajectory obtained from the equation of motion (22). At equilibrium (or stationary state) we have $\langle x_c(t)\rangle = 0$, and

$$\langle x_c^2\rangle = \frac{\hbar}{2m\omega_0} \quad , \tag{23}$$

as is well known [1, 2]. However, using the exact solution (21) of the Schrödinger type equation (20), we obtain

$$\langle x^2\rangle = \langle \int_{-\infty}^{\infty} dx |\Psi_{cs}(x,t)|^2 x^2\rangle = \frac{\hbar'}{2m\omega_0} + \langle x_c^2\rangle = \frac{\hbar' + \hbar}{2m\omega_0} \quad . \tag{24}$$

This gives the correct value at zero temperature, namely $\langle x^2\rangle = \hbar/2m\omega_0$, in the limit $\hbar' \to 0$. Only in this limit the solutions of equations (19) and (20) are physically acceptable. This is an important result that is very easy to understand within the realm of SED, if we recall the derivation (see eqs. (13) to (18)) of the classical Schrödinger like stochastic equation (19). The inevitable conclusion is that the standard Schrödinger equation, namely equation (19) with $\hbar' = \hbar$, does not give consistent results if the zero-point electromagnetic field $\mathbf{A}_{VF}$ is fully considered.

In order to further illustrate the advantages of the stochastic Schrödinger like equation we shall consider the system consisting of a harmonic oscillator (electric dipole) interacting with an anisotropic source of noise as for instance the solenoid of a simple RLC circuit without battery [7]. The fluctuating current in the solenoid generates a random electric field ($\mathbf{E}_{sol}$) that affects the charge oscillating in the x direction as is illustrated in the Fig.1. If the radius of the solenoid is large enough an anomalous contribution to the Nyquist noise becomes very important as first suggsted by França and Santos [8]. It is generated by the flux of $\mathbf{B}_{\rm VF}(t)$ through the solenoid coils. Therefore, it is possible to show that the total spectral distribution of the random voltage at zero temperature is given by [8]

$$\langle\tilde{\varepsilon}(\omega)\tilde{\varepsilon}(\omega')\rangle = \frac{\hbar\omega}{2\pi}\left[R + \frac{2\pi^2N^2}{3c}\left(\frac{a\omega}{c}\right)^4\right]\delta(\omega+\omega') \quad , \tag{25}$$

where R is the resistance of the circuit, N is the number of coils, and a is the radius of the solenoid. The second term in equation (25), namely the anomolous Nyquist noise, is due to $\mathbf{B}_{\rm VF}(t)$, and it contributes significantly only if a is large enough. This term was neglected in ref. [7] because the radius of the solenoid considered therein was very small ($a \simeq 7\times\ 10^{-4}$ cm). Here we shall assume that a^4 and N^2 are large enough so that the second term in (25) becomes significant.

Following the steps of the calculation presented by Dechoum et al.[7], it is possible to obtain the average oscillator energy using (20), (21) and (22). Notice that in equation (22) $E_{\rm VF}$ should be replaced by $E_{\rm VF} + (\mathbf{E}_{\rm sol})_x$. The result is

$$\begin{aligned} \epsilon &= m\omega_0^2\langle x_c^2(t)\rangle = \\ &= \frac{\hbar\tau\omega^2}{\pi}\int_0^\infty \frac{d\omega\ \omega^3\left(1+\beta(\omega,y)\left[1+\frac{2\pi^2N^2}{3cR}\left(\frac{a\omega}{c}\right)^4\right]\right)}{(\omega^2-\omega_0^2)^2+\tau^2\omega^6\left[1+\beta(\omega,y)\right]^2} \quad , \end{aligned} \tag{26}$$

where $\tau \equiv 2e^2/3mc^3$. The function $\beta(\omega,y)$ is given by (see [7])

$$\beta(\omega,y) = \frac{\frac{3}{2}R\left(\frac{2\pi Na^2}{\ell y}\right)^2}{c|Z(\omega)|^2} \quad , \tag{27}$$

where $Z(\omega)$ is the impedance of the RLC circuit, ℓ is the length of the solenoid and y is the distance from the dipole to the solenoid axis (see Fig.1). In the case $\tau\omega_0 \ll 1$ it is possible to show that the integral in (27) gives

$$\epsilon \simeq \frac{\hbar\omega_0}{2}\left(\frac{1+\beta(\omega_0,y)\left[1+\frac{2}{3}\frac{\pi^2N^2}{cR}\left(\frac{a\omega_0}{c}\right)^4\right]}{1+\beta(\omega_0,y)}\right) \quad , \tag{28}$$

which differs from the result obtained in [7] due to the contribution of the anomolous Nyquist noise (factor within square brackets). The result for $y < a$ is obtained by replacing a^2/y by y in expression (27).

Notice that $\epsilon \to \hbar\omega_0/2$ if the solenoid and the oscillator are far apart ($y \to \infty$). The result can be extended to finite temperature T by introducing the factor $\coth(\hbar\omega_0/2kT)$ in (28). The factor multiplying $\hbar\omega_0/2$ in (28) depends on various parameters characterizing the interaction of the electric dipole with the RLC circuit. We shall estimate this factor in the case of small $\beta(\omega_0, y = a)$ with $\beta N^2(a\omega_0/c)^4/cR > 1$. For simplicity, we shall assume that the circuit is in resonance with the dipole oscillator ($\sqrt{1/LC} \simeq \omega_0$) so that $\beta \simeq \beta_{\max} = 3(2\pi Na/\ell)^2/2cR$. Moreover, in deriving (28), it has been assumed that $c \gg \omega_0\ell \gg \omega_0 a$ (see ref. [7]), so we shall take $(a/\ell) \simeq 10^{-1}$, and $(a\omega_0/c) \simeq 10^{-2}$ in equation (28) obtaining

$$\begin{aligned}\frac{2\epsilon}{\hbar\omega_0} - 1 &\simeq \frac{4\pi^4N^4}{(cR)^2}\left(\frac{a\omega_0}{c}\right)^4\left(\frac{a}{\ell}\right)^2 \simeq \frac{4\times 10^{-10}N^4}{(cR)^2} \\ &\simeq \frac{3}{(cR)^2} \quad ,\end{aligned} \tag{29}$$

if we take $N = 300$. Notice that $cR \simeq 0.03$ for $R \simeq 1$ ohm.

From this numerical result one can conclude that the effect is large enough to be measured. We suggest to surround (or to fill) the solenoid with a solid material of cristaline structure. The anomalous electromagnetic noise generated by the solenoid will affect significantly the specific heat of the cristaline substance as indicated by the estimate (29). The calculation of the specific heat can be done using the procedure explained in the work by Blanco et al. [9].

4 Discussion

Dalibard *et al* [10] and França, Franco and Malta [11] provided an identification of the contribution of the radiation reaction and the vacuum fluctuation forces to the processes of radiation emission and atomic stability. Using the Heisenberg picture and perturbative QED calculations Dalibard *et al* [10] have shown that

$$P_{\text{Larmor}}(a) = \frac{4e^2}{3c^3} \sum_{b\ (\epsilon_b<\epsilon_a)} \langle a\,|\ddot{\mathbf{r}}|\,b\rangle \cdot \langle b\,|\ddot{\mathbf{r}}|\,a\rangle \quad . \tag{30}$$

This equation is the quantum generalization of the Larmor formula $(2e^2\ddot{\mathbf{r}}^2)/(3c^3)$ for the rate of radiation emission, including the zero-point field effects, of an electron in the quantum state $|a\rangle$ (Dirac notation) with energy ϵ_a. We see that the inclusion of the zero-point electromagnetic field simply doubles the rate of the radiation emission, being thus very important for obtaining agreement with experiment. Dechoum and França [5] extended this result to the SED picture using the harmonic oscillator and the classical stochastic Schrödinger like equation. Further insight on the general connection between SED and QED, for the free electromagnetic fields and for dipole oscillator system, is provided by T.H.Boyer [12].

Suárez Barnes *et al* [13] have studied the one-dimensional motion of the electron in the Coulomb field using the simple equation

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \frac{e^2}{|x|}\right)\psi(x,t) \quad . \tag{31}$$

The Coulomb potential $V(x)$ was approximated by

$$V(x) = -\frac{e^2}{|x|} \simeq V(q_t) + (x - q_t)V'(q_t) + \frac{(x - q_t)^2}{2}V''(q_t) \quad , \tag{32}$$

where q_t is the *classical trajectory*. A coherent state solution was obtained from (31) and (32).

Equations (31) and (32) allowed Suárez Barnes *et al* to obtain a remarkable reproduction of the hydrogen spectrum, using classical reference trajectories that have a *continuous* energy range. No quantization conditions were imposed on these classical reference trajectories. For the reader's convenience the spectrum calculated in [13] is reproduced in the Fig.2. As far as we know, this constitutes the first accurate *classical* calculation of the atomic spectrum since the advent of Quantum Mechanics. This calculation can be interpreted classically due to the approximation (32). For potentials of this form, the Schrödinger equation is equivalent to the Liouville equation as was pointed out by many authors [1, 4, 14, 15].

Finally we would like to stress that in our picture, based on the classical stochastic Schrödinger like equation, the wave-particle duality hypothesis plays no role. Therefore, it is desirable to extend our calculations so as to make possible its application for describing the diffraction pattern observed in many experiments with electron beams.

ACKNOWLEDGMENT

We acknowledge financial support from FAPESP and CNPq.

References

[1] T.W. Marshall, *Proc. Royal Soc. (London) A* **276**, 475 (1963).

[2] L. de la Peña and A.M. Cetto in *"The Quantum Dice: An Introduction to Stochastic Electrodynamics"* (Kluwer Academic Publishers, 1996).

[3] L. de la Peña and A.M. Cetto, *J. Math. Phys.* **18**,1612 (1977).

[4] E. Wigner, *Phys. Rev.* **40**, 749 (1932). See also R. F. O'Connell and E. P. Wigner, *Phys. Lett. A* **83**, 145 (1981).

[5] K. Dechoum and H.M. França, *Found. Phys.*, **25**, 1599 (1995) and *Found. Phys.*, **26**, 1573 (1996).

[6] K. Dechoum, H.M. França and C.P. Malta, *Phys. Lett. A* **248**, 93 (1998). This paper discuss the various classical aspects of the Pauli-Schrödinger equation, for a neutral spinning particle, including the classical interpretation of the Stern-Gerlach experiments.

[7] K. Dechoum, H. M. França and A. Maia Jr., *N. Cim B* 113, 455 (1998). See also R. Blanco, K. Dechoum, H.M. França and E. Santos, *Phys. Rev. A* **57** , 724 (1998) for a detailed calculation of the Casimir force acting on the oscillator.

[8] H.M. França and R.B. B. Santos, *Phys. Lett A* **251**, 100 (1999).

[9] R. Blanco, H.M. França and E. Santos, *Phys. Rev. A* **43** , 693 (1991).

[10] J. Dalibard, J. Dupont-Roc and C. Cohen-Tannoudji, *J. Physique* **43**, 1617 (1982).

[11] H.M. França, H. Franco and C.P. Malta, *Eur. J. Phys* **18**, 343 (1997).

[12] T. H. Boyer, *Phys. Rev. D* **11**, 809 (1975).

[13] I.M. Suárez Barnes, M. Nauenberg, M. Nockleby and S. Tomsobic, *Phys. Rev. Lett.* **71**, 1961 (1993); *J. Phys. A: Math. Gen.* **27**, 3299 (1994).

[14] G. Manfredi, S. Mola and M.R. Feix, *Eur. J. Phys.* **14**, 101 (1993).

[15] T. Takabayasi, *Prog. Theor. Phys.* **11**, 341 (1954).

Figures Caption and Figures

Figure 1: Schematic picture of the electric dipole at a distance y from the solenoid axis. The relevant electromagnetic fields generated by the solenoid ($\mathbf{E}_{\text{sol}}$) and the oscillating dipole ($\mathbf{B}_{\text{dip}}$) are indicated.

Figure 2: Spectrum generated by the classical motion in the Coulomb potential, according to the parametric oscillator approximation. The continuous energy is denoted by ϵ, and the circles correspond to the exact quantum results. The units are such that $e = m = \hbar = 1$.

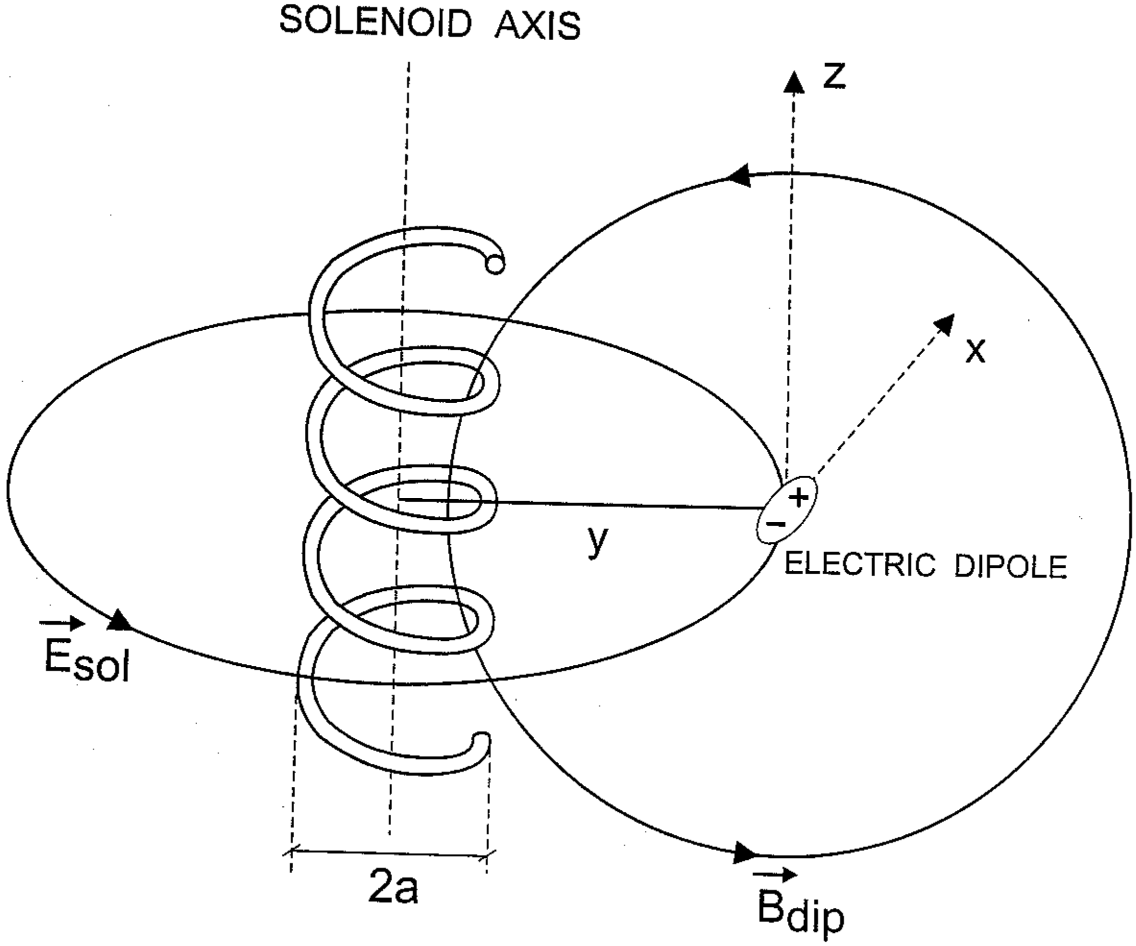

FIGURE 1

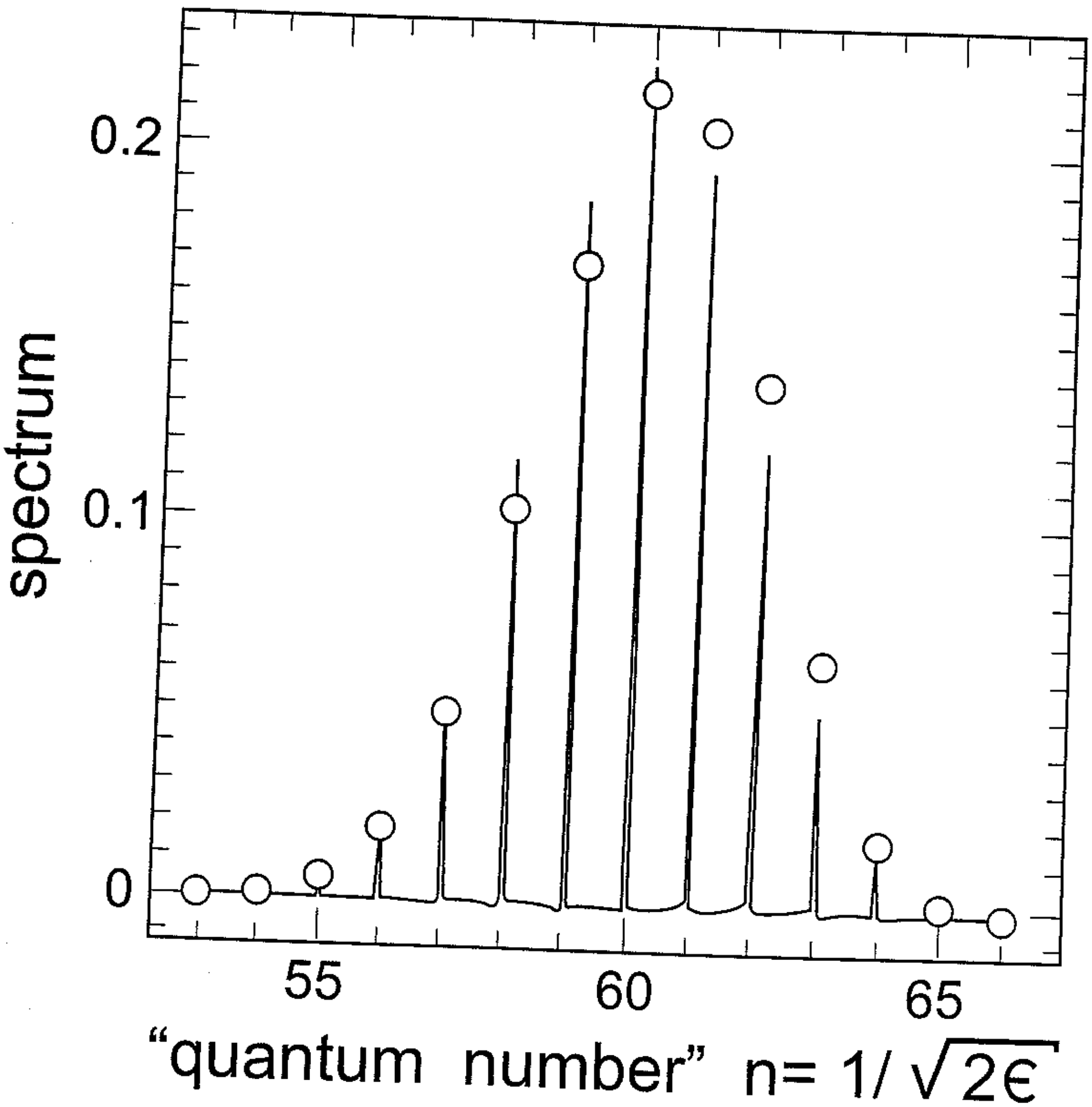

FIGURE 2