IFUSP/P-16

ON POSITIVITY PRESERVATION FOR FREE QUANTUM FIELDS

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* Supported by BNDE
To be submitted: Journal of Mathematical Analysis and Applications.

February/74

A B S T R A C T

Let A be a real Bose or Fermi one-particle operator with $||A|| \leq I$. Using Kaplansky's density theorem, a simple proof is given of the fact that $\Gamma(A)$, the operator in Fock space induced by A, is positivity preserving in the relevant L^2-space.

INTRODUCTION

A powerful tool in the methods of constructive quantum field theory has been the use of positivity properties of certain operators (- usually, the semigroups generated by the free and spatially cutoff $P(\Phi)_2$ - Hamiltonians [3,4,5,6,9,14,15,17,18] and also the semigroups and resolvents generated by some fermion Hamiltonians [2,4]).

For the free case, most proofs tend to be complicated by the approximations used [4,17] (see, however, [10]). We present an alternative proof which follows essentially from Kaplansky's density theorem [7,8]. Moreover, we are able to treat both the Bose and Fermi cases almost simultaneously. Our result seems to be new in the Fermi case with one-particle operator not self-adjoint. (The self-adjoint case was proved by Gross [5]).

§ 1. Notation

Let H' be a real Hilbert space, and let H be its complexification. Let $K = \bigoplus_{n=0}^{\infty} K_n$ be the Hilbert space tensor algebra over H, and let $\mathcal{S}, \mathcal{A}$ and K_s and K_a be the symmetrization and antisymmetrization projections in K, and their respective ranges. Then $K_s \cap K_a = K_0 \oplus K_1$, where $K_0 = \mathbb{C}$, and $K_1 = H$, the tensors of rank zero and one, respectively.

For $z \in H$, the creation operator $C(z): K_n \to K_{n+1}$ is defined by $k \mapsto C(z)k = \sqrt{n+1}\; z \otimes k$, $k \in K_n$. By linearity, $C(z)$ defines an operator on the algebraic tensor algebra over H.

Let $h \in H'$. The boson field, $\Phi(h)$, is the densely defined symmetric operator in K_s given by

$$\Phi(h) = \mathcal{S}C(h)\mathcal{S} + (\mathcal{S}C(h)\mathcal{S})^*$$

and the fermion field, $\Psi(h)$, is the densely defined symmetric operator on K_a given by

$$\Psi(h) = \mathcal{A}C(h)\mathcal{A} + (\mathcal{A}C(h)\mathcal{A})^*$$

It is well-known that $\Phi(h)$ is essentially self-adjoint, and that, for h_1, $h_2 \in H'$, we have

$$\exp i\,\overline{\Phi(h_1)}\,\exp i\,\overline{\Phi(h_2)} = \exp i\,\overline{\Phi(h_1+h_2)}$$

on K_s (the bar denotes the operator closure). $\Psi(h)$ defines a bounded operator on K_a, and, for h_1, $h_2 \in H'$, we have

$$\Psi(h_1)\,\Psi(h_2) + \Psi(h_2)\,\Psi(h_1) = 2\,(h_1,h_2)\mathbb{1} \text{ on } K_a \quad (1).$$

Denote $\exp i\,\overline{\Phi(h)} \upharpoonright K_s$ by $U(h)$, and $\Psi(h) \upharpoonright K_a$ by $\Psi(h)$, $h \in H'$, and let Ω denote the element $1 \in \mathbb{C} = K_0$ considered as an element of $K_s \cap K_a$. Let $\mathfrak{A}_0$ and $\mathcal{L}_0$ be the self-adjoint algebras generated by polynomials in $\{U(h) \mid h \in H'\}$ and $\{\Psi(h) \mid h \in H'\}$, respectively, and let $\mathfrak{A}$ and $\mathcal{L}$ be their strong closures in $\mathcal{B}(K_s)$ and $\mathcal{B}(K_a)$, respectively. Since $\mathfrak{A}$ is commutative, it is algebraically isomorphic, as a C*-algebra, to the uniform algebra, $C(Q)$, of continuous complex-valued functions on its spectrum, Q. The positive linear functional $A \mapsto (A\Omega,\Omega)$, $A \in \mathfrak{A}$, thus defines a regular probability measure, μ, on Q. Since Ω is cyclic for $\mathfrak{A}$, the map $D : \mathfrak{A} \to K_s$, given by $A \mapsto A\Omega$, extends to a unitary operator $D : L^2(Q,\mu) \to K_s$. Furthermore, since Ω is cyclic, $\mathfrak{A}$ is maximal abelian and so $\mathfrak{A} = L^\infty(Q,\mu)$ (16).

For the fermion case, we note that the map $D : \mathcal{L} \to K_a$, given by $A \mapsto A\Omega$, extends to a unitary operator $D : L^2(\mathcal{L}) \to K_a$, where $L^2(\mathcal{L})$ is the completion of $\mathcal{L}$ with respect to the norm $\|A\| = (A^*A\Omega,\Omega)^{1/2}$ (5,13). If $m(\cdot)$ denotes the functional $(\cdot\Omega,\Omega)$ on $\mathcal{L}$, then $(K_a,\mathcal{L},m)$ is a regular probability gage space (5,11,12). $L^2(\mathcal{L})$ can therefore be considered as a set of possibly unbounded operators on K_a. The notion of positivity is thus well-defined in $L^2(\mathcal{L})$: $A \in L^2(\mathcal{L})$, $A \geq 0$ if and only if A is a non-negative self-adjoint operator.

§ 2. Positivity preservation

Let (L^2,L^∞) denote either $(L^2(Q,\mu),\ L^\infty(Q,\mu))$ or $(L^2(\mathcal{L}),\mathcal{L})$.

Definition A linear operator T, on L^2 is said to be positivity preserving if T maps non-negative elements into non-negative elements, and if $\theta \in L^2$, $\theta \geq 0$ and $T\theta = 0$ implies that $\theta = 0$.

Let $A : H \to H$, be a linear operator and suppose $\|A\| \leq 1$. Then $\Gamma(A) : K \to K$ is given on K_n by $A \otimes \ldots \otimes A$ (n factors), $\Gamma(A)\Omega \equiv \Omega$. $\Gamma(A)$ defines a bounded operator on K (in fact it is a contraction), and K_s and K_a are invariant under $\Gamma(A)$.

Theorem Let $A : H \to H$, $\|A\| \leq 1$. Suppose $A : H' \to H'$. Then $D^{-1}\Gamma(A)D$ is positivity preserving on L^2.

Proof Let $\theta \in L^2$, $\theta \geq 0$. Then $(D^{-1}\Gamma(A)D\theta,\ 1)_{L^2} =$

$$= (\Gamma(A)D\theta,\Omega)_K$$
$$= (D\theta,\Gamma(A^*)\Omega)_K = (D\theta,\Omega)_K$$
$$= (\theta,1)_{L^2}$$

Hence $D^{-1}\Gamma(A)D\theta = 0 \Rightarrow (\theta,1)_{L^2} = 0$

$\Rightarrow \theta = 0$ (11,12).

Let $\theta \in L^2$, $\theta \geq 0$. We have now only to show that $D^{-1}\Gamma(A)D\theta \geq 0$. But in any regular gage space, $\chi \in L^2$ is non-negative if and only if $(\chi, P)_{L^2} \geq 0$ for all projections P in the ring of the gage space. Therefore we need only show that $(D^{-1}\Gamma(A)D\theta, P)_{L^2} \geq 0$ for all projections $P \in L^\infty$, i.e. that

$$(\Gamma(A)D\theta, DP)_K = (\Gamma(A)D\theta, P\Omega)_K \geq 0, \text{ for all } P = P^* = P^2 \in L^\infty$$

Now, if $\mathfrak{R}$ denotes either $\mathfrak{A}_0$ or $\mathfrak{B}_0$, then we have that $\mathfrak{R}$ is strongly dense in L^∞. Therefore, by Kaplansky's density theorem [(7,8)], there is a uniformly bounded net $\{S_\nu\}$ in $\mathfrak{R}$ converging strongly to P. But then $S_\nu^* S_\nu$ converges strongly to $P^*P = P$, and, in particular, $S_\nu^* S_\nu \Omega$ converges to $P\Omega$ strongly in K. So we see, by continuity, that the proof is complete if we can show that $(\Gamma(A)D\theta, S^*S\Omega)_K \geq 0$ for any $S \in \mathfrak{R}$. It is here that we are forced to consider the bose and fermi cases separately.

(i) <u>Boson case:</u> Let $S \in \mathfrak{A}_0$. Then S has the form

$$S = \sum_{j=1}^{N} a_j \, U(h_j)$$

for some $N < \infty$, $a_j \in \mathbb{C}$, $h_j \in H'$, $1 \leq j \leq N$. We have

$$(\Gamma(A)D\theta, S^*S\Omega)_K = (D\theta, \Gamma(A^*)S^*S\Omega)_K.$$

But one can show (e.g. 17) that

$$\Gamma(A^*)S^*S\Omega = \sum_{j,k=1}^{N} \bar{a}_j a_k \, U(A^*h_k - A^*h_j)\exp\left[((h_k-h_j), (\mathbb{1}-AA^*)(h_k-h_j))_{H'}\right]\Omega$$

and so $D^{-1}\Gamma(A^*)S^*S\Omega$

$$= \sum_{j,k=1}^{N} \bar{a}_j a_k \, U(A^*h_k - A^*h_j)\exp\left[((h_k-h_j), (\mathbb{1}-AA^*)(h_k-h_j))\right].$$

Now, for almost all $q \in Q$, and for h in any fixed finite dimensional subspace of H', the map $h \mapsto U(A^*h)(q)$ is positive semi-definite, as is the map $h \mapsto \exp[(h,(\mathbb{1}-AA^*)h)_{H'}]$. It follows that their product is also positive semi-definite; i.e. $D^{-1}\Gamma(A^*)S^*S\Omega \geq 0$ almost everywhere. Hence

$$(\Gamma(A)D\theta, S^*S\Omega)_K = (\theta, D^{-1}\Gamma(A^*)S^*S\Omega)_{L^2} \geq 0 \quad \text{as required.}$$

(ii) Fermion case: Let $S \in \mathcal{b}_o$. Then S has the form $S = \mathcal{P}(\Psi(h_1),\ldots,\Psi(h_n))$ where $\mathcal{P}(x_1,\ldots,x_n)$ is some complex polynomial in n-variables, and $h_1,\ldots,h_n \in H'$. Let $\{E_m\}$ be a sequence of finite dimensional projections on H', such that E_m increases to $\mathbb{1}$, and $E_m h_i = h_i, \forall\, m,\ 1 \leq j \leq N$. E_m defines a finite dimensional projection in H, and $E_m A E_m$ converges strongly to A in H, as $m \to \infty$ and therefore $\Gamma(E_m A E_m)$ converges strongly to $\Gamma(A)$ in K, so we need only consider $\Gamma(E_m A E_m)$. We must show that

$$(\Gamma(E_m A E_m)D\theta, S^*S\Omega)_K = (D^{-1}\Gamma(E_m A E_m)DD^{-1}\Gamma(E_m)D\theta,\ S^*S)_{L^2} \geq 0$$

(we have used $\Gamma(AB) = \Gamma(A)\ \Gamma(B)$ in order to write $\Gamma(E_m A E_m) = \Gamma(E_m A E_m)\Gamma(E_m)$). But $D^{-1}\Gamma(E_m)D\theta$ is the conditional expectation of θ with respect to $\mathcal{b}(E_m H')$, the algebra generated by $\{\Psi(h) \mid h \in E_m H'\}$ [5,11,12,20]. Hence, $D^{-1}\Gamma(E_m)D\theta \in L^2(\mathcal{b}(E_m H'))$, and $\theta \geq 0$ implies that $D^{-1}\Gamma(E_m)D\theta \geq 0$. Also, $S^*S \in \mathcal{b}(E_m H')$, and

$D^{-1}\ \Gamma(E_m A E_m)D : L^2(\mathcal{b}(E_m H')) \to L^2(\mathcal{b}(E_m H'))$, and so we have reduced the problem to a finite dimensional one.

Let H_m' and H_m denote $E_m H'$ and $E_m H$, respectively, and let $B = E_m A E_m$. Then B is an operator in H_m. Let $B = U|B|$ be its polar decomposition, and let N be the projection onto the null space of U. Then U is unitary from $(NH_m)^{\perp}$ onto UH_m, and so $\dim(NH_m)^{\perp} = = \dim UH_m$. Since $\dim H_m < \infty$, it follows that $\dim NH_m = \dim(UH_m)^{\perp}$, and so there exists a unitary $V : NH_m \to (UH_m)^{\perp}$. By setting $V \upharpoonright (NH_m)^{\perp} = 0$, we see that $(U+V)$ is unitary from H_m onto H_m. Since $U : H_m' \to H_m'$, we can choose V so that $V : H_m' \to H_m'$, and therefore $U+V : H_m' \to H_m'$. Furthermore, $U = (U+V)N^{\perp}$, and so $B = (U+V)N^{\perp}|B|$. Now, $\Gamma(A_1A_2) = \Gamma(A_1)\Gamma(A_2)$, and it is known (5) that $\Gamma(A)$ is positivity preserving if A is self-adjoint, so we need only consider the case B = W, unitary, with $W : H_m' \to H_m'$. But in this case, it is not difficult to see that

$$\Gamma(W)\mathcal{P}(\Psi(h_1),\dots,\Psi(h_n))^*\mathcal{P}(\Psi(h_1),\dots,\Psi(h_n))\Omega =$$

$$= \mathcal{P}(\Psi(Wh_1),\dots,\Psi(Wh_n))^*\,\mathcal{P}(\Psi(Wh_1),\dots,\Psi(Wh_n))\Omega$$

(This can be proved, for example, by using Wick's theorem (19)).

Thus $$D^{-1}\Gamma(W^*)DS^*S = T^*T \geq 0$$

where $T = \mathcal{P}(\Psi(W^*h_1),\dots,\Psi(W^*h_n))$, and so $(D^{-1}\Gamma(W)D\theta', S^*S) \geq 0$ if $\theta' \geq 0$ and the proof is complete.

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