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IFUSP/P-16

ON POSITIVITY PRESERVATION FOR FREE QUANTUM FIELDS

by

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To be submitted: Journal of Mathematical Analysis and
Applications.

February/74

A B S T R A C T

Let A be a real Bose or Fermi one-particle operator with $\|A\| \leq I$. Using Kaplansky's density theorem, a simple proof is given of the fact that $\Gamma(A)$, the operator in Fock space induced by A , is positivity preserving in the relevant L^2 -space.

INTRODUCTION

A powerful tool in the methods of constructive quantum field theory has been the use of positivity properties of certain operators (- usually, the semigroups generated by the free and spatially cutoff $P(\Phi)_2$ - Hamiltonians (3,4,5,6, 9,14,15,17,18) and also the semigroups and resolvents generated by some fermion Hamiltonians (2,4)).

For the free case, most proofs tend to be complicated by the approximations used (4,17) (see, however, (10)). We present an alternative proof which follows essentially from Kaplansky's density theorem (7,8). Moreover, we are able to treat both the Bose and Fermi cases almost simultaneously. Our result seems to be new in the Fermi case with one-particle operator not self-adjoint. (The self-adjoint case was proved by Gross (5)).

§ 1. Notation

Let H' be a real Hilbert space, and let H be its complexification. Let $K = \bigoplus_{n=0}^{\infty} K_n$ be the Hilbert space tensor algebra over H , and let \mathcal{L} , \mathcal{A} and K_s and K_a be the symmetrization and antisymmetrization projections in K , and their respective ranges. Then $K_s \cap K_a = K_0 \oplus K_1$, where $K_0 = \mathbb{C}$, and $K_1 = H$, the tensors of rank zero and one, respectively.

For $z \in H$, the creation operator $C(z): K_n \rightarrow K_{n+1}$ is defined by $k \mapsto C(z)k = \sqrt{n+1} z \otimes k$, $k \in K_n$. By linearity, $C(z)$ defines an operator on the algebraic tensor algebra over H .

Let $h \in H'$. The boson field, $\phi(h)$, is the densely defined symmetric operator in K_s given by

$$\phi(h) = \mathcal{L} C(h) \mathcal{L} + (\mathcal{L} C(h) \mathcal{L})^*$$

and the fermion field, $\Psi(h)$, is the densely defined symmetric operator on K_a given by

$$\Psi(h) = \mathcal{A}C(h)\mathcal{A} + (\mathcal{A}C(h)\mathcal{A})^*$$

It is well-known that $\Phi(h)$ is essentially self-adjoint, and that, for $h_1, h_2 \in H'$, we have

$$\overline{\exp i \Phi(h_1)} \overline{\exp i \Phi(h_2)} = \overline{\exp i \Phi(h_1+h_2)}$$

on K_S (the bar denotes the operator closure). $\Psi(h)$ defines a bounded operator on K_a , and, for $h_1, h_2 \in H'$, we have

$$\Psi(h_1) \Psi(h_2) + \Psi(h_2) \Psi(h_1) = 2 (h_1, h_2) \mathbb{1} \text{ on } K_a \quad (1).$$

Denote $\overline{\exp i \Phi(h)} \upharpoonright K_S$ by $U(h)$, and $\Psi(h) \upharpoonright K_a$ by $\psi(h)$, $h \in H'$, and let Ω denote the element $1 \in \mathbb{C} = K_0$ considered as an element of $K_S \cap K_a$. Let \mathcal{A}_0 and \mathcal{B}_0 be the self-adjoint algebras generated by polynomials in $\{U(h) \mid h \in H'\}$ and $\{\psi(h) \mid h \in H'\}$, respectively, and let \mathcal{A} and \mathcal{B} be their strong closures in $\mathcal{B}(K_S)$ and $\mathcal{B}(K_a)$, respectively. Since \mathcal{A} is commutative, it is algebraically isomorphic, as a C^* -algebra, to the uniform algebra, $C(Q)$, of continuous complex-valued functions on its spectrum, Q . The positive linear functional $A \mapsto (A\Omega, \Omega)$, $A \in \mathcal{A}$, thus defines a regular probability measure, μ , on Q . Since Ω is cyclic for \mathcal{A} , the map $D : \mathcal{A} \rightarrow K_S$, given by $A \mapsto A\Omega$, extends to a unitary operator $D : L^2(Q, \mu) \rightarrow K_S$. Furthermore, since Ω is cyclic, \mathcal{A} is maximal abelian and so $\mathcal{A} = L^\infty(Q, \mu)$ (16).

For the fermion case, we note that the map $D : \mathcal{B} \rightarrow K_a$,

given by $A \mapsto A\Omega$, extends to a unitary operator $D : L^2(\mathcal{L}) \rightarrow K_a$, where $L^2(\mathcal{L})$ is the completion of \mathcal{L} with respect to the norm $\|A\| = (A^*A\Omega, \Omega)^{1/2}$ (5,13). If $m(\cdot)$ denotes the functional (\cdot, Ω) on \mathcal{L} , then (K_a, \mathcal{L}, m) is a regular probability gage space (5,11,12). $L^2(\mathcal{L})$ can therefore be considered as a set of possibly unbounded operators on K_a . The notion of positivity is thus well-defined in $L^2(\mathcal{L}) : A \in L^2(\mathcal{L}), A \geq 0$ if and only if A is a non-negative self-adjoint operator.

§ 2. Positivity preservation

Let (L^2, L^∞) denote either $(L^2(Q, \mu), L^\infty(Q, \mu))$ or $(L^2(\mathcal{L}), \mathcal{L})$.

Definition A linear operator T , on L^2 is said to be positivity preserving if T maps non-negative elements into non-negative elements, and if $\theta \in L^2, \theta \geq 0$ and $T\theta = 0$ implies that $\theta = 0$.

Let $A : H \rightarrow H$, be a linear operator and suppose $\|A\| \leq 1$. Then $\Gamma(A) : K \rightarrow K$ is given on K_n by $A \otimes \dots \otimes A$ (n factors), $\Gamma(A)\Omega \equiv \Omega$. $\Gamma(A)$ defines a bounded operator on K (in fact it is a contraction), and K_s and K_a are invariant under $\Gamma(A)$.

Theorem Let $A : H \rightarrow H, \|A\| \leq 1$. Suppose $A : H' \rightarrow H'$. Then $D^{-1}\Gamma(A)D$ is positivity preserving on L^2 .

Proof Let $\theta \in L^2, \theta \geq 0$. Then $(D^{-1}\Gamma(A)D\theta, 1)_{L^2} =$
 $= (\Gamma(A)D\theta, \Omega)_K$
 $= (D\theta, \Gamma(A^*)\Omega)_K = (D\theta, \Omega)_K$
 $= (\theta, 1)_{L^2}$

Hence $D^{-1}\Gamma(A)D\theta = 0 \Rightarrow (\theta, 1)_{L^2} = 0$
 $\Rightarrow \theta = 0$ (11,12).

Let $\theta \in L^2$, $\theta \geq 0$. We have now only to show that $D^{-1}\Gamma(A)D\theta \geq 0$. But in any regular gage space, $\chi \in L^2$ is non-negative if and only if $(\chi, P)_{L^2} \geq 0$ for all projections P in the ring of the gage space. Therefore we need only show that $(D^{-1}\Gamma(A)D\theta, P)_{L^2} \geq 0$ for all projections $P \in L^\infty$, i.e. that

$$(\Gamma(A)D\theta, DP)_K = (\Gamma(A)D\theta, P\Omega)_K \geq 0, \text{ for all } P = P^* = P^2 \in L^\infty$$

Now, if \mathcal{R} denotes either \mathcal{A}_0 or \mathcal{B}_0 , then we have that \mathcal{R} is strongly dense in L^∞ . Therefore, by Kaplansky's density theorem (7,8), there is a uniformly bounded net $\{S_\nu\}$ in \mathcal{R} converging strongly to P . But then $S_\nu^*S_\nu$ converges strongly to $P^*P = P$, and, in particular, $S_\nu^*S_\nu\Omega$ converges to $P\Omega$ strongly in K . So we see, by continuity, that the proof is complete if we can show that $(\Gamma(A)D\theta, S^*S\Omega)_K \geq 0$ for any $S \in \mathcal{R}$. It is here that we are forced to consider the bose and fermi cases separately.

(i) Boson case: Let $S \in \mathcal{A}_0$. Then S has the form

$$S = \sum_{j=1}^N a_j U(h_j)$$

for some $N < \infty$, $a_j \in \mathbb{C}$, $h_j \in H'$, $1 \leq j \leq N$. We have

$$(\Gamma(A)D\theta, S^*S\Omega)_K = (D\theta, \Gamma(A^*)S^*S\Omega)_K.$$

But one can show (e.g. 17) that

$$\Gamma(A^*)S^*S\Omega = \sum_{j,k=1}^N \bar{a}_j a_k U(A^*h_k - A^*h_j) \exp\left[\left((h_k - h_j), (\mathbb{1} - AA^*)(h_k - h_j)\right)_{H'}\right] \Omega$$

and so $D^{-1}\Gamma(A^*)S^*S\Omega$

$$= \sum_{j,k=1}^N \bar{a}_j a_k U(A^*h_k - A^*h_j) \exp\left[\left((h_k - h_j), (\mathbb{1} - AA^*)(h_k - h_j)\right)\right].$$

Now, for almost all $q \in Q$, and for h in any fixed finite dimensional subspace of H' , the map $h \mapsto U(A^*h)(q)$ is positive semi-definite, as is the map $h \mapsto \exp[(h, (\mathbb{1} - AA^*)h)_{H'}]$. It follows that their product is also positive semi-definite; i.e. $D^{-1}\Gamma(A^*)S^*S\Omega \geq 0$ almost everywhere. Hence

$$(\Gamma(A)D\theta, S^*S\Omega)_K = (\theta, D^{-1}\Gamma(A^*)S^*S\Omega)_{L^2} \geq 0 \quad \text{as required.}$$

(ii) Fermion case: Let $S \in \mathcal{L}_0$. Then S has the form $S = \mathcal{P}(\Psi(h_1), \dots, \Psi(h_n))$ where $\mathcal{P}(x_1, \dots, x_n)$ is some complex polynomial in n -variables, and $h_1, \dots, h_n \in H'$. Let $\{E_m\}$ be a sequence of finite dimensional projections on H' , such that E_m increases to $\mathbb{1}$, and $E_m h_i = h_i, \forall m, 1 \leq j \leq n$. E_m defines a finite dimensional projection in H , and $E_m A E_m$ converges strongly to A in H , as $m \rightarrow \infty$ and therefore $\Gamma(E_m A E_m)$ converges strongly to $\Gamma(A)$ in K , so we need only consider $\Gamma(E_m A E_m)$. We must show that

$$(\Gamma(E_m A E_m)D\theta, S^*S\Omega)_K = (D^{-1}\Gamma(E_m A E_m)D D^{-1}\Gamma(E_m)D\theta, S^*S)_{L^2} \geq 0$$

(we have used $\Gamma(AB) = \Gamma(A)\Gamma(B)$ in order to write $\Gamma(E_m A E_m) = \Gamma(E_m A E_m)\Gamma(E_m)$). But $D^{-1}\Gamma(E_m)D\theta$ is the conditional expectation of θ with respect to $\mathcal{L}(E_m H')$, the algebra generated by $\{\Psi(h) \mid h \in E_m H'\}$ (5, 11, 12, 20). Hence, $D^{-1}\Gamma(E_m)D\theta \in L^2(\mathcal{L}(E_m H'))$, and $\theta \geq 0$ implies that $D^{-1}\Gamma(E_m)D\theta \geq 0$. Also, $S^*S \in \mathcal{L}(E_m H')$, and

$D^{-1}\Gamma(E_m A E_m)D : L^2(\mathcal{L}(E_m H')) \rightarrow L^2(\mathcal{L}(E_m H'))$, and so we have reduced the problem to a finite dimensional one.

Let H'_m and H_m denote $E_m H'$ and $E_m H$, respectively, and let $B = E_m A E_m$. Then B is an operator in H_m . Let $B = U|B|$ be its polar

decomposition, and let N be the projection onto the null space of U . Then U is unitary from $(NH_m)^\perp$ onto UH_m , and so $\dim(NH_m)^\perp = \dim UH_m$. Since $\dim H_m < \infty$, it follows that $\dim NH_m = \dim(UH_m)^\perp$, and so there exists a unitary $V : NH_m \rightarrow (UH_m)^\perp$. By setting $V \upharpoonright (NH_m)^\perp = 0$, we see that $(U+V)$ is unitary from H_m onto H_m . Since $U : H'_m \rightarrow H'_m$, we can choose V so that $V : H'_m \rightarrow H'_m$, and therefore $U+V : H'_m \rightarrow H'_m$. Furthermore, $U = (U+V)N^\perp$, and so $B = (U+V)N^\perp |B|$. Now, $\Gamma(A_1 A_2) = \Gamma(A_1) \Gamma(A_2)$, and it is known (5) that $\Gamma(A)$ is positivity preserving if A is self-adjoint, so we need only consider the case $B = W$, unitary, with $W : H'_m \rightarrow H'_m$. But in this case, it is not difficult to see that

$$\begin{aligned} \Gamma(W) \mathcal{P}(\Psi(h_1), \dots, \Psi(h_n)) * \mathcal{P}(\Psi(h_1), \dots, \Psi(h_n)) \Omega &= \\ &= \mathcal{P}(\Psi(Wh_1), \dots, \Psi(Wh_n)) * \mathcal{P}(\Psi(Wh_1), \dots, \Psi(Wh_n)) \Omega \end{aligned}$$

(This can be proved, for example, by using Wick's theorem (19)).

$$\text{Thus} \quad D^{-1} \Gamma(W^*) D S^* S = T^* T \geq 0$$

where $T = \mathcal{P}(\Psi(W^* h_1), \dots, \Psi(W^* h_n))$, and so $(D^{-1} \Gamma(W) D \theta', S^* S) \geq 0$ if $\theta' \geq 0$ and the proof is complete.

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