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Aharonov-Bohm Effect in Synchrotron Radiation

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Abstract

Synchrotron radiation of a charged particle in a constant uniform magnetic field and in the presence of the Aharonov-Bohm solenoid field is studied in the frame of the relativistic quantum theory. First, to this end exact solutions of the Klein-Gordon and Dirac equations are found. Using such solutions, all characteristics of one photon spontaneous irradiation, such as its intensity and angular distribution and polarization were calculated and analyzed. It is shown that usual spectrum of the synchrotron radiation is essentially affected by the presence of the solenoid (the Aharonov-Bohm effect). We believe that this deformation may be observed by spectroscopic methods of measurement. It is shown that

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I. INTRODUCTION

Aharonov-Bohm (AB) effect attracts until present days a great attention due to its importance for understanding of the role of electromagnetic potentials in the quantum theory. In the work [1] by Aharonov and Bohm it was demonstrated a non-trivial scattering of a charged particle off a potential of an infinitely long and infinitesimally thin magnetic solenoid (further AB field)[1]. The effect may be related to the phase shift $\exp(ie\oint \mathbf{A}\, d\mathbf{r}/\hbar c)$ in the the wave function of the particle [3], which is moving along a closed path embracing the solenoid. In spite of the fact that the wave function vanishes in the area, where the magnetic field is non-zero, the phase shift is proportional to the magnetic flux created by the solenoid. A number of theoretical works and convinced experiments was done to prove the existence of this phenomenon. A detailed exposition of this activity may be met in some reviews and books [4–7]. In particular, it was shown [8–10] that AB scattering is accompanied by an electromagnetic radiation, and its angular distribution and polarization were calculated. A pair creation by a single photon in the presence of the AB potential was calculated in [11]. An example of the AB effect for bond states is a splitting of Landau levels in a uniform magnetic field and in the AB field [12]. The interaction between spin and AB field leads to Dirac wave functions, which do not vanish on the magnetic solenoid. Thus, for spinning particles the AB field does not create a very convinced situation to observe the effect in its initial form. The issue of spin changes slightly the interpretation of the AB effect. The theoretical study of the AB scattering for spinning particles was presented in many papers, see for example [13] and [14]. AB effect was also studied in connection with fractional spin and statistics in [15] and with its contribution to cosmic strings in [16,17]. The AB effect in an anyon scattering was considered in [18], and radiative corrections to the effect in $3+1$ dimensions were calculated in [19]. The AB scattering within the Chern-Simons theory of scalar particles was studied in [20]. There exist impressive applications of the AB effect in

[1]Essentially the same effect was discussed earlier by Ehrenberg and Siday [2]

solid state physics [21,22].

In the present article we are going to study the quantum behavior of a charged relativistic particle in a combination of a constant uniform magnetic field with the AB field (further magnetic-solenoid field). Such a field combination creates exactly a situation in which AB effect for bond states may be observed. Namely, it allows one to demonstrate that the particle, rotating in the magnetic field, feels the presence of the solenoid, in spite of the fact that it moves out of the solenoid field (its wave function vanishes in the area where the solenoid is present). That means that the particle feels potentials of the solenoid field, which do not vanish in the area of the particle motion. First, exact solutions of the Schrödinger equation in the magnetic-solenoid field (nonrelativistic case) were found in [12]. Then they where analyzed in [23–25] from the AB effect point of view. For example, coherent states were constructed in [24], they correspond most closely to a classical motion along a circle embracing the solenoid. It was, in particular, demonstrated that classical relations between the radius and energy of motion of the particle are affected by the solenoid presence.

It is well-known that a charged particle moving in a uniform magnetic field irradiates, as any accelerated charged particle. In the latter magnetic such a radiation is called synchrotron one (SR). In the present article we are going to investigate how the presence of the AB field affects the SR. SR may be studied by spectroscopic methods of measurement, which are extremely sensitive, thus, a real situation to observe the AB effect may be discovered.

The article is organized in the following way: In the first Sect. we analyze classical motion of a charge in the magnetic-solenoid field. In Sections II and III we find exact solutions of Klein-Gordon and Dirac equations and present an analysis of spectra and quantum motions of spinless and spinning particles in such a field. In Section IV we consider spontaneous radiation of a spinless relativistic particle in the magnetic-solenoid field in the frame of the quantum theory. We analyze here an influence of the solenoid field on SR. We summarize new results in the Conclusion. In the Appendix we derive some important for calculations properties of Lagerr functions.

II. CLASSICAL CHARGED PARTICLE IN MAGNETIC-SOLENOID FIELD

Consider a constant uniform magnetic field of the strength H directed along the axis z. Besides, along this axis is placed an infinitely long and infinitesimal thin solenoid, which creates a finite magnetic flux Φ along the axis (the AB field). It may be given by the electromagnetic potentials of the form

$$A_x = -y\left(\frac{\Phi}{2\pi r^2}+\frac{H}{2}\right),\; A_y = x\left(\frac{\Phi}{2\pi r^2}+\frac{H}{2}\right),\; A_z=0,\; r^2=x^2+y^2,\; A_0=0\,. \tag{2.1}$$

The potentials (2.1) define just the magnetic-solenoid field mentioned in the Introduction

$$\mathbf{H}=(0,0,H_z),\; H_z = H+\Phi\delta(x)\delta(y)\,. \tag{2.2}$$

First of all, we are going to analyze the classical motion of the electron (a point-like particle with a charge $-e$, $(e>0)$) in the field under consideration. It is defined by the Lorentz equations

$$\frac{d\mathbf{P}}{dt} = -\frac{e}{c}[\mathbf{v},\mathbf{H}]\,,\; \mathbf{P}=\frac{m_0\mathbf{v}}{\sqrt{1-\beta^2}}\,,\; \beta=\frac{\mathbf{v}}{c}\,,\; \mathbf{v}=\frac{d\mathbf{r}}{dt}\,,\;\; \mathbf{r}=(x,y,z)\,, \tag{2.3}$$

where m_0 stands for the rest mass.

Taking into account a relation

$$\mathbf{P}=\frac{E}{c^2}\mathbf{v}=\frac{E}{c^2}\frac{d\mathbf{r}}{dt}\,, \tag{2.4}$$

which follow from (2.3), and conservation of energy in the field under consideration,

$$E=\frac{m_0c^2}{\sqrt{1-\beta^2}}=const \Rightarrow \beta^2=const\,, \tag{2.5}$$

we may rewrite the Lorentz equations for trajectories, which do not have intersections with the axis z, in the following form

$$\frac{d\mathbf{P}}{dt}=-\frac{\omega}{H}[\mathbf{P},\mathbf{H}]\,,\; \frac{dP_x}{dt}=-\omega P_y,\; \frac{dP_y}{dt}=\omega P_x,\; \frac{dP_z}{dt}=0,\; \omega=\frac{ecH}{E}\,. \tag{2.6}$$

Integrating these equations by use of (2.4), we arrive to the well-known formulas, which describe the classical relativistic motion of the electron in the constant and uniform magnetic field

$$
\begin{aligned}
& x = x_0 + R\cos(\omega t + \varphi_0),\ y = y_0 + R\sin(\omega t + \varphi_0),\ z = z_0 + c\beta_3 t\ , \\
& P_x = -\hbar\gamma R\sin(\omega t + \varphi_0),\ P_y = \hbar\gamma R\cos(\omega t + \varphi_0),\ P_z = \frac{E}{c}\beta_3\ .
\end{aligned} \tag{2.7}
$$

Here $x_0, y_0, z_0, R, \varphi_0, \beta_3$ are integration constants, and

$$
\gamma = \frac{eH}{c\hbar} > 0\ . \tag{2.8}
$$

One ought to remark that the classical motion is not affected by the presence of the AB field.

As a consequence of (2.7) we find the following relations

$$
\begin{aligned}
& P_x^2 + P_y^2 = \hbar^2\gamma^2 R^2,\ E^2 = m_0^2c^4 + e^2H^2R^2 + c^2P_z^2 = m_0^2c^4 + c^2\hbar^2\gamma^2R^2 + c^2P_z^2\ , \\
& E^2(1-\beta_3^2) = m_0^2c^4 + c^2\hbar^2\gamma^2R^2,\ e^2H^2R^2 = m_0^2c^4(\beta^2 - \beta_3^2)(1-\beta^2)^{-1}\ ,
\end{aligned} \tag{2.9}
$$

and an expression for z-projection of the angular momentum $\mathbf{L} = [\mathbf{r}, \mathbf{p}]$,

$$
L_z = -\hbar\bar{\mu} + \frac{1}{2}\hbar\gamma(R^2 - R_0^2),\ R_0^2 = x_0^2 + y_0^2,\ \bar{\mu} = \frac{e\Phi}{2\pi c\hbar}\ , \tag{2.10}
$$

where the generalized momentum $\mathbf{p}$ is related to the kinetic one $\mathbf{P}$ by the usual relation $\mathbf{P} = \mathbf{p} + \frac{e}{c}\mathbf{A}$ (remember that $e > 0$). The constant $\bar{\mu}$ is dimensionless. It follows from (2.10) that L_z is an integral of motion.

In turn, one can get from (2.9) an important relation

$$
\mathcal{K}^2 = m^2 + \gamma^2R^2 + k_3^2\ , \tag{2.11}
$$

where new notations are introduced

$$
E = c\hbar\mathcal{K},\ m = m_0c/\hbar,\ P_z = \hbar k_3\ . \tag{2.12}
$$

The equations (2.9) imply that classical trajectories are spiral lines, with the radius R. Their projections on the $z = 0$ plane have the form

$$
(x - x_0)^2 + (y - y_0)^2 = R^2\ . \tag{2.13}
$$

Let us present $\bar{\mu}$ in the following form

$$\bar{\mu} = l_0 + \mu, \;\; 0 \le \mu < 1, \;\; l_0 = \begin{cases} [\bar{\mu}], & \bar{\mu} > 0 \\ [\bar{\mu} - 1], & \bar{\mu} < 0 \end{cases} . \tag{2.14}$$

Then, taking into account the definition of $\bar{\mu}$, we may write

$$\Phi = \Phi_0(l_0 + \mu), \quad \Phi_0 = 2\pi c\hbar/e \, . \tag{2.15}$$

We are going to call μ mantissa of the magnetic flux Φ, whereas l_0 gives an integer number of quanta Φ_0, which contains the total flux Φ. We would like to stress that the mantissa, being always positive, may not be equal to one.

The projection L_z may be presented in the form

$$L_z = \hbar(l - l_0), \quad l = L_z/\hbar + l_0 \, , \tag{2.16}$$

where l is a dimensionless quantity. It follows from (2.10) that

$$l + \mu = L_z/\hbar + \bar{\mu} = \frac{\gamma}{2}(R^2 - R_0^2) \, . \tag{2.17}$$

Thus, one can see that at

$$l > -\mu, \; (R > R_0) \, , \tag{2.18}$$

a trajectory embraces the solenoid, whereas at

$$l < -\mu, \; (R < R_0) \, , \tag{2.19}$$

it does not.

Introducing dimensionless complex quantities a_1 and a_2 by relations,

$$\hbar\sqrt{2\gamma}a_1 = P_x - iP_y, \quad \hbar\sqrt{2\gamma}a_2 = iP_x - P_y + \hbar\gamma(x + iy) \, , \tag{2.20}$$

we may express coordinates and momenta in their terms

$$\begin{aligned} &2P_x = \hbar\sqrt{2\gamma}(a_1 + a_1^+), \; 2P_y = i\hbar\sqrt{2\gamma}(a_1 - a_1^+) \, , \\ &\sqrt{2\gamma}x = a_2^+ + a_2 + i(a_1 - a_1^+), \; \sqrt{2\gamma}y = i(a_2^+ - a_2) - a_1 - a_1^+ \, . \end{aligned} \tag{2.21}$$

Then we get a representation of the classical motion in terms of a_1 and a_2

$$a_1 = -i\sqrt{\frac{\gamma}{2}}Re^{-i(\omega t+\varphi_0)},\; a_2 = \sqrt{\frac{\gamma}{2}}(x_0 + iy_0)\,,$$
$$a_1^+ a_1 + a_1 a_1^+ = \gamma R^2 = (P_x^2 + P_y^2)/\hbar^2\gamma\,,$$
$$a_1^+ a_1 + a_1 a_1^+ - a_2^+ a_2 - a_2 a_2^+ = 2\left(L_z/\hbar + \bar{\mu}\right) = 2(l+\mu)\,. \tag{2.22}$$

III. SOLUTIONS OF KLEIN-GORDON EQUATION IN MAGNETIC-SOLENOID FIELD

Here we are going to study solutions of the Klein-Gordon equation

$$\left(\hat{P}^2 - m_0^2 c^2\right)\Psi(x) = 0,\;\; \hat{P}_\mu = i\hbar\frac{\partial}{\partial x^\mu} + \frac{e}{c}A_\mu(x) \tag{3.1}$$

in the magnetic-solenoid field. Here three mutual commutative operators $\hat{P}_0$, $\hat{P}_3$, and $\hat{L}_z$, are integrals of motion. The latter operator $\hat{L}_z$ corresponds to the classical quantity (2.16). We look for solutions of (3.1), which are eigenvectors for these operators,

$$\hat{P}_0\Psi = \hbar\mathcal{K}\Psi,\; \hat{P}_z\Psi = \hbar k_3\Psi,\; \hat{L}_z\Psi = \hbar(l - l_0)\Psi\,. \tag{3.2}$$

Since l_0 are integer the quantum numbers l have to be integer as well to provide uniqueness of the wave function. Depending of the value of this quantum number there are two type of solutions:

$$\Psi^{(1)}_{n,l,k_3}(x) = N_1 exp\left(-ic\mathcal{K}_1 t + ik_3 z\right)\psi^{(1)}_{n,l}(r,\varphi),\; l \geq 0\,, \tag{3.3}$$
$$\Psi^{(2)}_{n,l,k_3}(x) = N_2 exp\left(-ic\mathcal{K}_2 t + ik_3 z\right)\psi^{(2)}_{n,l}(r,\varphi),\; l < 0\,, \tag{3.4}$$

where φ is a polar angle. Thus, we meet here a partial breaking of degeneracy of the energy with respect to the quantum number l (compare with solutions in pure magnetic field [26,27]).

The first type of solutions (3.3), where $l \geq 0$, corresponds to the validity of the inequality (2.18) on the classical level, since l are integer only. We may interpret these states as ones, which correspond to classical trajectories embracing the solenoid. In this case

$$\psi^{(1)}_{n,l}(r,\varphi) = \exp\left[i(l-l_0)\varphi\right] I_{n+\mu,n-l}(\rho)$$
$$= \sqrt{\frac{\Gamma(1+n-l)}{\Gamma(1+\mu+n)}} \exp\left[i(l-l_0)\varphi - \frac{\rho}{2}\right] \rho^{\frac{l+\mu}{2}} L^{l+\mu}_{n-l}(\rho)\,, \quad 0 \le l \le n\,, \tag{3.5}$$

where $\rho = \gamma r^2/2$. Via $I_{k,s}(\rho)$ we have denoted Lagerr functions. Their exact definition and some important for us properties can be found in the Appendix. $L^{\alpha}_n(x)$ are Lagerr polynomials, see [31] (Eqs. 8.970, 8.972.1). Energy levels in this case are defined by the relation

$$\mathcal{K}_1^2 = m^2 + k_3^2 + 2\gamma(n+\mu+\frac{1}{2}) = m^{*2} + k_3^2 + 2\gamma(n+\mu), \quad m^{*2} = m^2 + \gamma\,. \tag{3.6}$$

They depend on the quantum number n, which will be called principle quantum number. Each energy level is $n+1$ times degenerated in accordance with possible number of admitted l at a given n. We will call m^* effective mass of a spinless particle.

The second type of solutions (3.4), where $l < 0$, corresponds to the validity of the inequality (2.19) on the classical level. We may interpret these states as ones, which correspond to classical trajectories, which do not embrace the solenoid. In this case

$$\psi^{(2)}_{n,l}(r,\varphi) = \exp\left[i(l-l_0)\varphi\right] I_{n-l-\mu,n}(\rho)$$
$$= \sqrt{\frac{\Gamma(1+n)}{\Gamma(1+n-l-\mu)}} \exp\left[i(l-l_0)\varphi - \frac{\rho}{2}\right] \rho^{-\frac{l+\mu}{2}} L^{-l-\mu}_{n}(\rho)\,, \quad l = -1, -2, \ldots\,. \tag{3.7}$$

Energy levels are defined here by the relation

$$\mathcal{K}_2^2 = m^2 + k_3^2 + 2\gamma(n+\frac{1}{2}) = m^{*2} + k_3^2 + 2\gamma n\,. \tag{3.8}$$

Each level is infinitely degenerated with respect to the quantum number l.

The energy levels (3.6) and (3.8) differ if $0 < \mu < 1$ and coincide at $\mu = 0$. In the latter case the wave functions (3.5) and (3.7) may be written in the same form due to the property (A19), and coincide with the wave functions [26,27] of the particle in the magnetic field only. All the differences in the energy levels and wave functions due to the presence of the magnetic flux are defined only by the mantissa of the flux.

One can introduce a more compact form of the wave functions, which will be convenient in concrete calculations of the electron radiation, see Sect.V. To this end let us introduce an effective angular quantum number $\bar{l}$, and an effective principle quantum number $\bar{n}$,

$$\bar{l} = l + \mu\,, \quad \bar{n} = n + \mu(2 - j) = \begin{cases} n + \mu, & j = 1, \\ n, & j = 2, \end{cases} . \tag{3.9}$$

Then the equations (3.6) and (3.8), which define the energy levels, take the form

$$\mathcal{K}_j^2 = m^{*2} + k_3^2 + 2\gamma\bar{n}\,, \tag{3.10}$$

and the wave functions may be written as

$$\psi_{n,l}^{(1)}(r,\varphi) = exp[i(l - l_0)\varphi] I_{\bar{n},\bar{n}-\bar{l}}(\rho)\,, \tag{3.11}$$

$$\psi_{n,l}^{(2)}(r,\varphi) = exp[i(l - l_0)\varphi] I_{\bar{n}-\bar{l},\bar{n}}(\rho)\,. \tag{3.12}$$

We may see that the difference in the states (3.11) and (3.12) is connected with a non-symmetry of the Lagerr functions in their lower indices. At $\mu = 0$ they become symmetric (see Eqs. (A19)) and solutions do not feel the magnetic flux.

One ought to remark that the wave functions vanish on the z-axis at any μ,

$$\psi_{n,l}^{(j)}(r = 0, \varphi) = 0\,, \quad j = 1, 2\,.$$

That means that the probability to find the particle in the area of the magnetic flux is zero. Nevertheless, this flux affects the behavior of the particle (the AB effect) in all the space (it modifies the form of the wave functions).

If the normalization factors in (3.3), and (3.4) have the form

$$N_j = (8\pi L\mathcal{K}_j/\gamma)^{-1/2}\,, \tag{3.13}$$

then the wave functions obey the following orthonormality relations

$$(\Psi_{n',l',k_3'}, \Psi_{n,l,k_3}) = \delta_{n,n'}\delta_{l,l'}\delta_{k_3,k_3'}\,, \tag{3.14}$$

with respect to the common Klein-Gordon scalar product (we use a cut-off in z-direction, $-L < z < L$, $L \to \infty$)

$$(\Psi, \Psi') = \frac{1}{\hbar} \int \left[\Psi^\dagger \hat{P}_0 \Psi' + (\hat{P}_0 \Psi)^\dagger \Psi' \right] d\mathbf{r} \, . \quad (3.15)$$

Considering quantities P_x, P_y in the relations (2.20) as operators defined in (3.1), we may construct corresponding operators to the classical quantities a_1, a_2. Let us denote such operators by the same letters without hats. Then these operators have the form

$$\begin{aligned}
a_1 &= -i\sqrt{\rho} e^{-i\varphi} \left[(l_0 + \mu + \rho - i\partial_\varphi)/2\rho + \partial_\rho \right] \, , \\
a_1^+ &= i\sqrt{\rho} e^{i\varphi} \left[(l_0 + \mu + \rho - i\partial_\varphi)/2\rho - \partial_\rho \right] \, , \\
a_2 &= -\sqrt{\rho} e^{i\varphi} \left[(l_0 + \mu - \rho - i\partial_\varphi)/2\rho - \partial_\rho \right] \, , \\
a_2^+ &= -\sqrt{\rho} e^{-i\varphi} \left[(l_0 + \mu - \rho - i\partial_\varphi)/2\rho + \partial_\rho \right] \, . \quad (3.16)
\end{aligned}$$

Besides, the relations (2.20), an analog of the relations (2.22) takes place

$$\hat{P}_x^2 + \hat{P}_y^2 = \gamma \hbar^2 (a_1 a_1^+ + a_1^+ a_1), \; \hat{L}_z = \frac{\hbar}{2} (a_1 a_1^+ + a_1^+ a_1 - a_2 a_2^+ - a_2^+ a_2) - \hbar \bar{\mu} \, . \quad (3.17)$$

Using the commutation relations for the momentum operators

$$\hat{P}_x \hat{P}_y - \hat{P}_y \hat{P}_x = -i \frac{e\hbar}{c} H_z \, , \quad (3.18)$$

and definition of the magnetic field (2.2), we arrive to the following commutation relations for the operators a_1, α_2

$$[a_1, a_1^+] = 1 + f, \quad [a_2, a_2^+] = 1 - f, \quad [a_1, a_2] = if, \quad [a_1, a_2^+] = 0 \, , \quad (3.19)$$

where dimensionless singular function f has the form

$$f = (\Phi/H)\delta(x)\delta(y) \, . \quad (3.20)$$

If D is an arbitrary connected area on $z = 0$ plane, which includes the origin, then one can easily see that $\int_D f \, dxdy = \Phi/H = \int_D f \, rdrd\varphi = \frac{1}{\gamma} \int_D f \, d\rho d\varphi$, where the variable ρ was defined above, see (3.5). Thus, we get

$$f = (\Phi/H)\delta(x)\delta(y) = \frac{\Phi \delta(r)}{\pi H r} = \frac{\gamma \Phi}{\pi H} \delta(\rho) = 2 \frac{\Phi}{\Phi_0} \delta(\rho) \, . \quad (3.21)$$

It is clear that the operators a_k, a_k^+, $k=1,2$, may be considered as ones of creation and annihilation only at $\Phi = f = 0$. However, they act as operators of creation and annihilation on functions, which are continuous in the point $\rho = 0$ and vanish in this point together with their first derivative with respect to ρ.

Using formulas (A8)-(A11) from the Appendix, one can find the action of the operators a_k, a_k^+, $k = 1,2$ on the wave functions (3.5), (3.7),

$$\begin{aligned} &a_1\psi_{n,l}^{(j)} = (-1)^j i\sqrt{\bar{n}}\psi_{n-1,l-1}^{(j)},\ a_1^+\psi_{n,l}^{(j)} = (-1)^{j-1} i\sqrt{\bar{n}+1}\psi_{n+1,l+1}^{(j)}\,, \\ &a_2\psi_{n,l}^{(j)} = (-1)^j\sqrt{\bar{n}-\bar{l}}\psi_{n,l+1}^{(j)},\ a_2^+\psi_{n,l}^{(j)} = (-1)^j\sqrt{\bar{n}-\bar{l}+1}\psi_{n,l-1}^{(j)}\,,\ \ j=1,2\,. \end{aligned} \tag{3.22}$$

These formulas show that the functions $\psi_{n,l}^{(1)}$ may be created by an action of the operators a_k^+ on $\psi_{0,0}^{(1)}$, and the functions $\psi_{n,l}^{(2)}$ may be created by an action of the operators a_k^+ on $\psi_{0,-1}^{(2)}$. Namely,

$$\begin{aligned} &\psi_{n,l}^{(1)} = (-1)^l i^n\sqrt{\frac{\Gamma(1+\mu)}{\Gamma(1+\mu+n)\Gamma(1+n-l)}}(a_2^+)^{n-l}(a_1^+)^n\psi_{0,0}^{(1)}\,, \\ &\psi_{n,l}^{(2)} = i^n\sqrt{\frac{\Gamma(2-\mu)}{\Gamma(1+n)\Gamma(1-\mu+n-l)}}(a_1^+)^n(a_2^+)^{n-l-1}\psi_{0,-1}^{(2)}\,. \end{aligned} \tag{3.23}$$

It is natural to interpret $\psi_{0,0}^{(1)}$ as a vacuum state for the set of states $\psi_{n,l}^{(1)}$, and $\psi_{0,-1}^{(2)}$ as a vacuum state for the set of states $\psi_{n,l}^{(2)}$. Thus, we have two vacuum states in the problem at $0<\mu<1$.

At $\mu = 0$ the situations changes. Due to (A19)

$$\psi_{n,l}^{(1)} = (-1)^l\psi_{n,l}^{(2)}\,,\ \ \mu = 0\,, \tag{3.24}$$

and $\psi_{0,0}^{(1)}$ is connected to $\psi_{0,-1}^{(2)}$ for any $l<n$,

$$a_2^+\psi_{0,0}^{(1)} = \psi_{0,-1}^{(2)}\,,\quad a_2\psi_{0,-1}^{(2)} = \psi_{0,0}^{(1)}\,. \tag{3.25}$$

Thus, we have only one vacuum in the problem, one series of energy (3.8), and all the wave functions are created from the vacuum $\psi_{0,0}^{(1)}$.

The definitions (3.5) and (3.7) for the functions $\psi^{(1)}_{n,l}$ and $\psi^{(2)}_{n,l}$, and relations (3.22) remain valid even if the values of l differ from that which are pointed out in (3.5) and (3.7). For example, we may get from (3.22) the following relations

$$a_1^+\psi^{(2)}_{n,-1} = -i\sqrt{n+1}\psi^{(2)}_{n+1,0} = -i(1+n)\sqrt{\frac{\Gamma(1+n)}{\Gamma(2-\mu+n)}}exp[-il_0\varphi - \frac{\rho}{2}]\rho^{-\frac{\mu}{2}}L^{-\mu}_{n+1}(\rho)\ ,$$

$$a_1\psi^{(1)}_{n,0} = -i\sqrt{n+\mu}\psi^{(1)}_{n-1,-1} = -i(n+\mu)\sqrt{\frac{\Gamma(1+n)}{\Gamma(1+\mu+n)}}exp[-i(1+l_0)\varphi - \frac{\rho}{2}]\rho^{-\frac{1-\mu}{2}}L^{\mu-1}_{n}(\rho)\ ,$$

$$a_2^+\psi^{(1)}_{n,0} = -\sqrt{n+1}\psi^{(1)}_{n,-1} = -(1+n)\sqrt{\frac{\Gamma(1+n)}{\Gamma(1+\mu+n)}}exp[-i(1+l_0)\varphi - \frac{\rho}{2}]\rho^{-\frac{1-\mu}{2}}L^{\mu-1}_{n+1}(\rho)\ ,$$

$$a_2\psi^{(2)}_{n,-1} = \sqrt{1-\mu+n}\psi^{(2)}_{n,0} = (1-\mu+n)\sqrt{\frac{\Gamma(1+n)}{\Gamma(2-\mu+n)}}exp(-il_0\varphi - \frac{\rho}{2})\rho^{-\frac{\mu}{2}}L^{-\mu}_{n}(\rho)\ , \quad (3.26)$$

selecting corresponding values for l. The functions $\psi^{(1)}_{s,-1}$, $\psi^{(2)}_{s,0}$ are still defined by the formulas (3.5) and (3.7), however they do not present physical solutions of the problem. Thus, in the general case ($0 < \mu < 1$) the action of the operators a_k^+, a_k on the wave functions may lead them out of the class of physical solutions. That does not happen at $\mu = 0$ due to (3.24) and (3.25). The right sides of (3.25) take infinite values in the point $r = 0$ (only at $0 < \mu < 1$), but the corresponding functions remain quadratically integrable.

IV. SOLUTIONS OF DIRAC EQUATION IN THE MAGNETIC-SOLENOID FIELD

Similar to the Klein-Gordon equation the Dirac equation

$$(\gamma^\mu \hat{P}_\mu - m_0 c)\Psi(x) = 0 \quad (4.1)$$

in the field (2.1) admits as integrals of motion the energy, the momentum projection on the axis z

$$\hat{P}_0\Psi = \hbar\mathcal{K}\Psi,\ \hat{P}_z\Psi = \hbar k_3\Psi\ , \quad (4.2)$$

as well as the angular momentum projection on the same axis,

$$\hat{J}_z\Psi = \hbar(l - l_0 - \frac{1}{2})\Psi,\ \hat{J}_z = \hat{L}_z + \frac{\hbar}{2}\Sigma_3\ . \tag{4.3}$$

Here $\Sigma_3 = \mathrm{diag}(\sigma_3, \sigma_3)$, where σ_3 is a corresponding Pauli matrix.

We are going to look for the solutions of the equations (4.1), (4.2), and (4.3) in the form

$$\Psi(x) = Nexp(-ic\mathcal{K}t + ik_3 z)\psi(r,\varphi)\ . \tag{4.4}$$

Then the bispinor $\psi = (\psi_\alpha),\ \alpha = 1,2,3,4$, has to obey the equations

$$\left[\gamma^0\mathcal{K} - m - \gamma^3 k_3 - \frac{\sqrt{2\gamma}}{2}(\gamma^1 + i\gamma^2)a_1 - \frac{\sqrt{2\gamma}}{2}(\gamma^1 - i\gamma^2)a_1^+\right]\psi = 0\ , \tag{4.5}$$

$$\left[\frac{1}{2}(1+\Sigma_3) - i\partial_\varphi - l + l_0\right]\psi = 0\ . \tag{4.6}$$

We remember that

$$P_x - iP_y = \hbar\sqrt{2\gamma}a_1,\ P_x + iP_y = \hbar\sqrt{2\gamma}a_1^+\ .$$

As in the Klein-Gordon case we get here two type of solutions $\psi^{(j)},\ j = 1,2,$

$$\begin{aligned} &\psi_1^{(j)} = C_1\psi^{(j)}_{n-1,l-1},\ \psi_2^{(j)} = (-1)^{j-1}iC_2\psi^{(j)}_{n,l}\ , \\ &\psi_3^{(j)} = C_3\psi^{(j)}_{n-1,l-1},\ \psi_4^{(j)} = (-1)^{j-1}iC_4\psi^{(j)}_{n,l}\ ,\ \ j = 1,2\ , \end{aligned} \tag{4.7}$$

where the constant bispinor C_α $(\alpha = 1,2,3,4)$ is subjected to the following algebraic system of equations (we use standard representation for γ-matrices)

$$AC = 0,\ A = \mathcal{K} - m\rho_3 - k_3\rho_1\Sigma_3 - Q\rho_1\Sigma_1\ ,\ \ Q = \sqrt{2\gamma\bar{n}}\ . \tag{4.8}$$

The system (4.8) has a nontrivial solution if

$$detA = 0\ . \tag{4.9}$$

Thus, we get the energy spectrum

$$\mathcal{K}_j^2 = m^2 + k_3^2 + Q^2\ . \tag{4.10}$$

As in the Klein-Gordon case, states with $j = 1$ correspond to classical trajectories embracing the solenoid,

$$\mathcal{K}_1^2 = m^2 + k_3^2 + 2\gamma(n+\mu), \quad 0 \le l \le n \,. \tag{4.11}$$

In the case $j = 2$ they do not embrace it,

$$\mathcal{K}_2^2 = m^2 + k_3^2 + 2\gamma n, \quad l < 0 \,. \tag{4.12}$$

If the condition (4.9) takes place, then the rank of the matrix A is always equal to 2. Thus, nontrivial solutions of the equations (4.8) contain two arbitrary constants. General solution for the constant bispinor C may be written in a block form via an arbitrary two component spinor v,

$$C = \begin{pmatrix} [\mathcal{K}+m]v \\ [k_3\sigma_3 + Q\sigma_1]v \end{pmatrix} , \tag{4.13}$$

where σ_1, σ_3 are Pauli matrices.

The spinor v describes spinning degrees of freedom of the electron. If one selects an additional to (4.2), and (4.3) spinning integral of motion (which always exists [26,27]), then the spinor may be specified.

For the set (4.7) with $j = 2$ the states with $n = 0$, $l < 0$ are special. They differ from zero only if

$$v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \sigma_3 v = -v \,, \tag{4.14}$$

and if

$$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \sigma_3 v = v \,, \tag{4.15}$$

they vanish. The total wave function (4.4) in this case (at $j = 2$) is an eigenvector of the operator Σ_3,

$$\Sigma_3 \Psi^{(2)} = -\Psi^{(2)} \,, \tag{4.16}$$

with the eigenvalue -1 (the electron spin may only be directed against the magnetic field). That fact is well-known in the absence of the magnetic flux [26,27].

For the set (4.7) with $j = 1$ the states with $n = 0$ ($l = 0$) are also special. In this case, and at $0 < \mu < 1$, there exist two nontrivial solutions for spinors v,

$$\sigma_3 v = \pm v \, . \tag{4.17}$$

However, such solutions contain, in particular, functions

$$\psi^{(1)}_{-1,-1} = \frac{exp[-i(1+l_0)\varphi - \rho/2]}{\sqrt{\Gamma(\mu)}}\rho^{-\frac{1-\mu}{2}} \, , \tag{4.18}$$

which grow infinitely at $\rho = 0$, $\psi^{(1)}_{-1,-1}(r = 0, \varphi) = \infty$. Nevertheless, even such solutions have finite norms. One may see that the probability to find an electron on the axis z is not zero (moreover, it is infinite). That fact is known and it is related to superstrong interaction between the electron and the solenoid, since the former is moving maximally close to the solenoid, embracing the latter ($n = 0$). The radius of the orbit R in such a case is minimal and related to μ by the relation $R^2 = 2\mu/\gamma$. Such a behavior of the wave functions is a characteristic of the spinor case only. Thus, for states of electron with $n = l = 0$ the interpretation of the AB effect has to be slightly changed [13].

The wave functions (4.4) obey the orthonormality relations

$$(\Psi_{n',l',k_3'}, \Psi_{n,l,k_3}) = \int \Psi^+_{n',l',k_3'}\Psi_{n,l,k_3}d\mathbf{r} = \delta_{n,n'}\delta_{l,l'}\delta_{k_3,k_3'}(v'^+ v) \, , \tag{4.19}$$

if the factors N_j are (as in (4.15) here $-L < z < L$, $L \to \infty$),

$$N_j = [8\pi L\mathcal{K}_j(\mathcal{K}_j + m)/\gamma]^{-1/2} \, .$$

V. SYNCHROTRON RADIATION OF SPINLESS PARTICLES

A. Introduction

As was already stressed in the beginning of the article classical trajectories, which do not intersect the axis z, do not feel the presence of the solenoid field. Thus, from the classical point of view radiation of a charge in the magnetic-solenoid field, has to be the same as in pure magnetic field. The corresponding classical theory is well developed, see for example [26].

Here we consider the spontaneous radiation of a spinless relativistic particle in the magnetic-solenoid field in the frame of the quantum theory. We will call radiation in such a field synchrotron one as well. Ordinary SR in the frame of relativistic quantum theory was investigated in detail in numerous works (see [26] and Refs. there). We have seen in previous Sect. that the presence of the solenoid affects the quantum motion of the charge. As a consequence of that characteristics of the corresponding radiation have to differ from those of ordinary SR, calculated in the frame of a quantum theory.

Study of SR in magnetic-solenoid field will be done in the same manner as in [26]. However, concrete calculations are much more complicated and contain many new aspects and technical tricks. Below we will often refer to general formulas from [26] in the form, which was used in these works, and to results of calculations in pure magnetic field, sometimes without additional citations.

B. Matrix elements of transitions and frequencies of radiation

Consider here spontaneous radiation of a spinless particle in the magnetic-solenoid field and compare it with the SR in the pure magnetic field. Detailed analysis for the latter case was done in the works [28–30]. All characteristics of SR are expressed via corresponding matrix elements of the operator

$$\exp\left[-i(\boldsymbol{\kappa}\mathbf{r})\right]\hat{\mathbf{P}}\ , \tag{5.1}$$

where $\hat{\mathbf{P}}$ is the operator of kinetic momentum (3.1), and $\boldsymbol{\kappa}$ is the wave vector of a photon. In the Cartesian reference frame this vector may be presented as follows

$$\boldsymbol{\kappa} = \kappa\left(\sin\theta\cos\varphi', \sin\theta\sin\varphi', \cos\theta\right)\ . \tag{5.2}$$

The spherical angles θ, φ' define an angle distribution of the emitted photons, whereas κ defines energy $E_{ph} = c\hbar\kappa$ of a photon.

The polarization of the SR will be defined similar to [26]. To this end one constructs σ and π components of the operator $\hat{\mathbf{P}}$, namely,

$$
\begin{aligned}
\hat{P}_\sigma &= -\hat{P}_x \sin\varphi' + \hat{P}_y \cos\varphi' \, , \\
\hat{P}_\pi &= (\hat{P}_x \cos\varphi' + \hat{P}_y \sin\varphi') \cos\theta - \hat{P}_z \sin\theta \, .
\end{aligned} \tag{5.3}
$$

These components may be written in terms of operators a_1, a_1^+, using Eqs. (2.21),

$$
\begin{aligned}
\hat{P}_\sigma &= i\hbar\sqrt{\frac{\gamma}{2}}\left[a_1 \exp(i\varphi') - a_1^+ \exp(-i\varphi')\right] \, , \\
\hat{P}_\pi &= i\hbar\sqrt{\frac{\gamma}{2}}\left[a_1 \exp(i\varphi') + a_1^+ \exp(-i\varphi')\right] \cos\theta - \hat{P}_z \sin\theta \, .
\end{aligned} \tag{5.4}
$$

One can exactly calculate matrix elements of the operators (5.4) between the initial and final states $\Psi^{(j)}_{n,l,k_3}$ and $\Psi^{(j')}_{n',l',k'_3}$. Let us remark on some details of such calculations.

Integration over the angle φ leads to Bessel functions similar to usual SI case. In the latter case and in course of a subsequent integration over ρ one meets integrals of the type (A33). Here one meets integrals of the type (A32) as well. That is related to the fact that in the present case the condition (A19) is absent. The matrix elements of transition amplitude do not contain the quantity l_0, they depend only on the mantissa μ of the magnetic flux.

Similar to the usual SR case there appear two conservation lows: one for z-component of the momentum,

$$
k_3 - k'_3 = \kappa \cos\theta \, , \tag{5.5}
$$

and another one for the energy,

$$
K_j - K'_{j'} = \kappa \, . \tag{5.6}
$$

These relations together with (2.9) or (3.11) define the frequency of the radiation κ as a function of initial and final quantum numbers and the angle θ. Due to the axial symmetry of the problem it does not depends on φ'. At $\mu = 0$ the frequency κ is a function of the principle quantum number n, of the number of the harmonics irradiated ν,

$$
\nu = n - n' \, , \tag{5.7}
$$

and on the angle θ. It does not depends on the orbital quantum numbers l, l'. At $\mu > 0$ the degeneracy of the frequency κ with respect to the quantum numbers l, l' is partially broken. The frequency depends on the type of the initial and final states, namely on the quantum numbers j, j', in accordance with Eq. (5.6). Thus, $\kappa = \kappa_{jj'}$. Introducing an effective number $\bar{\nu} = \bar{\nu}_{jj'}$ of the irradiated harmonic,

$$\bar{\nu} = \bar{n} - \bar{n}' = \nu + \mu(j' - j) = \begin{cases} \nu, \; j = j' \,, \\ \nu + \mu, \; j = 1, \; j' = 2 \,, \\ \nu - \mu, \; j = 2, \; j' = 1, \; \bar{\nu} > 0 \,, \end{cases} \, , \tag{5.8}$$

one can easily get for spinless particles

$$\kappa_{jj'} = \frac{K_j}{\sin^2\theta}\left(1 - \sqrt{1 - \beta_j^2 \frac{2\bar{\nu}}{2\bar{n}+1}\sin^2\theta}\right) \, , \tag{5.9}$$

where

$$\beta_j^2 = 1 - \left(\frac{m}{K_j}\right)^2 = 1 - \left(\frac{m_0 c^2}{E_j}\right)^2 \, . \tag{5.10}$$

Similar formula takes place for spinning particles,

$$\kappa_{jj'} = \frac{K_j}{\sin^2\theta}\left(1 - \sqrt{1 - \beta_j^2 \frac{\bar{\nu}}{\bar{n}}\sin^2\theta}\right) \, . \tag{5.11}$$

The expressions (5.9) and (5.11) are obtained for initial states with $k_3 = 0$. Both expressions may be written as

$$\kappa_{jj'} = \frac{2\gamma\bar{\nu}}{K_j + \sqrt{K_j^2 - 2\gamma\bar{\nu}\sin^2\theta}} \, . \tag{5.12}$$

Introducing a new dimensionless variable $q = q_{jj'}$,

$$q = \frac{\kappa^2 \sin^2\theta}{2\gamma} \, , \quad 0 \le q < \bar{\nu} \, , \tag{5.13}$$

one may easily get the following expressions

$$\kappa = \frac{\gamma}{K_j}(\bar{\nu} + q), \quad \sin\theta = \sqrt{\frac{2}{\gamma}} K_j \frac{\sqrt{q}}{\bar{\nu} + q} \, ,$$
$$\sqrt{K_j^2 - 2\gamma\bar{\nu}\sin^2\theta} = K_j \frac{\bar{\nu} - q}{\bar{\nu} + q} \, . \tag{5.14}$$

Thus, at $\mu > 0$ there appear two spectral series, one from the transitions without change of the quantum number j, and another one with the change. In the former case $\bar{\nu} = \nu$, whereas in the latter case $\bar{\nu} = \nu \pm \mu$, so that the effective number of the emitted harmonic becomes non-integer. Fixing $\nu > 0$ and a principal quantum number of the initial state, one gets an inequality

$$\kappa_{21} < \kappa_{11} < \kappa_{22} < \kappa_{12} , \tag{5.15}$$

which becomes an equality at $\mu = 0$. The difference between the frequencies κ_{11} and κ_{22} may be easily estimated in the magnetic fields $H \ll H_0$, where $H_0 = m_0^2 c^3/e\hbar \approx 4,41 \times 10^{13}$gauss is a critical field, which outnumbers any magnetic field produced at present in the laboratory. It is proportional to μ,

$$\kappa_{22} - \kappa_{11} \approx \kappa_{22}\delta\mu \, , \quad \delta = \frac{\gamma}{K_1^2} = \frac{\gamma}{m^2}\frac{m^2}{K_1^2} = \frac{H}{H_0}\left(\frac{m_0 c^2}{E_1}\right)^2 . \tag{5.16}$$

At typical fields $H \approx 10^4$gauss, which are realized in accelerators, and at not very high electron energies, we get $\delta < 10^{-9}$. The frequency difference has in this case the form

$$\kappa_{12} - \kappa_{21} \approx 2\kappa_{12}\mu/\nu = 2\frac{\omega}{c}\mu \, , \tag{5.17}$$

where ω is the classical synchrotron frequency (2.6). Thus, this difference becomes visible for harmonics with small numbers.

Especially remarkable becomes a possibility at $\mu > 0$ to emit the harmonic $\nu = 0$ in course of transitions from $j = 1$ to $j' = 2$. In pure magnetic field such harmonic is forbidden. For the fields $H \ll H_0$ the frequency of the radiation, which corresponds to such a harmonic is

$$\omega_{12} = \omega\mu = \frac{ecH}{E_1}\mu \, . \tag{5.18}$$

In course of transitions from $j = 2$ to $j' = 1$ (at $\nu = 1$) the frequency

$$\omega_{21} = \omega(1 - \mu) \tag{5.19}$$

is emitted. Both these frequencies are less than the one ω in contrast to the fact that in usual SR, where $\mu = 0$, the least frequency is the one ω. That is the most essential and in principle controllable consequence of the AB effect in SR.

C. Exact expression for intensity of radiation

As was already said we are able to calculate the matrix elements of the operator (5.1) exactly. Thus, there is a possibility to get exact expressions for the probability and for the intensity of the one photon emission. Using expressions (A8)-(A11), one may get the following expression for the intensity of the radiation (taking into account its polarization):

$$W_j = W_0 \frac{H}{H_0} \left(\frac{\gamma}{K_j^2}\right)^2 \sum_{\nu,j'} \frac{1}{2\pi} \int_0^{2\pi} d\varphi' \frac{1}{2} \int_0^{\pi} d\theta \, \sin\theta \frac{(\bar{\nu}+q)^3}{\bar{\nu}-q} Q_{jj'} |F_{jj'}|^2 \,. \tag{5.20}$$

Here

$$W_0 = \frac{e^2 m_0^2 c^3}{\hbar^2} \tag{5.21}$$

is a constant, which has dimension of the intensity, and $F_{jj'}$ has the form

$$\begin{aligned} F_{jj'} &= 2l_2\sqrt{q} I'_{jj'}(q) + l_3 \cot\theta \sqrt{\frac{2K_j^2}{\gamma}} I_{jj'}(q) \,, \\ I_{1j'}(q) &= I_{\bar{n},\bar{n}'}(q), \quad I_{2j'}(q) = I_{\bar{n}',\bar{n}}(q) \,. \end{aligned} \tag{5.22}$$

The latter quantity does not depend on the orbital quantum number l, it is completely defined by the effective principle quantum number $\bar{n}$ and by the polarization of the radiation. The polarization is characterized by the quantities l_2 and l_3, see [26]. At $l_2 = 1, \ l_3 = 0$ we get so called σ-component of linear polarization; $l_2 = 0, \ l_3 = 1$ we get so called π-component of linear polarization; at $l_2 = \pm l_3 = 1/\sqrt{2}$ we get right (left) circular polarization; and, finally, at $l_2^2 = l_3^2 = 1, \ l_2 l_3 = 0$ we get total intensity of non-polarized radiation.

The quantity $Q_{jj'}$ depends on the initial orbital quantum number l, it is a sum over the final quantum numbers l',

$$Q_{1j'} = \sum_{l'} I^2_{\bar{n}'-\bar{l}',\bar{n}-\bar{l}}(q), \quad Q_{2j'} = \sum_{l'} I^2_{\bar{n}-\bar{l},\bar{n}'-\bar{l}'}(q) \,. \tag{5.23}$$

The limits of the summation over l' depend on j'. At $j' = 1$ they are $0 \le l' \le n - \nu$, and at $j' = 2$ they are $-\infty < l' \le -1$.

The integrand in (5.22) does not depend on φ'. Thus, the integration over the latter variable may be done, that will be taken into account in further expressions.

Integrating over θ we get zero for total circular polarization due to the fact that the mainly circular polarizations in the upper $(0 \leq \theta \leq \pi/2)$ and in the lower $(\pi/2 \leq \theta \leq \pi)$ half-spaces, having opposite signs, compensate themselves exactly.

If we are only interested in linear polarization, then one can always count formally that $l_2 l_3 = 0$.

D. Summation over final orbital quantum numbers

If $\mu = 0$ then the quantity $|F|^2$ in Eq. (5.20) does not depend on the type of the final state (the property (A19) is valid in this case). The effective quantum numbers coincide in this case with ordinary ones, $\bar{n} = n$, $\bar{l} = l$. Thus, one can get from (5.23), taking into account (A19) and (A38),

$$\sum_{j'} Q_{jj'} = \sum_{k=0}^{\infty} I_{k,s}^2(q) = 1, \; s = n - l \,. \tag{5.24}$$

That is a well-known from SR theory result. In other words, the intensity of SR does not depend on the initial orbital quantum numbers. At $0 < \mu < 1$ the quantity $Q_{jj'}$ depends on l, n, ν, and the degeneracy disappears completely. Physically that result may be interpreted as follows: According to the Eq. (2.17) the quantity l characterizes the distance between the classical trajectory and the solenoid. It is natural that the intensity of the radiation depends on this distance as well as on the type of the state. In the absence of the solenoid the origin of the reference frame is not fixed and physics does not depend on l.

Let return to the equations (5.23), which define $Q_{jj'}$. Using formulas (A34), (A35), (A40), and (A41), one can get from them the following expression for the derivative of $Q_{jj'}$

$$\frac{d}{dq} Q_{jj'}(q) = (-1)^{1+j+j'} \sqrt{\frac{k+1}{q}} \left[(2-j) I_{k+1,s}(q) I_{k,s}(q) \right.$$
$$\left. + (j-1) I_{s,k+1}(q) I_{s,k}(q) \right] , \;\; k = \bar{n}' - \mu, \;\; s = \bar{n} - \bar{l} \,. \tag{5.25}$$

Then, taking into account a behavior of $Q_{jj'}$ at $q = 0$ and $q = \infty$, one can integrate (5.25) to obtain

$$Q_{jj'}(q) = j' - 1 + (-1)^{j'-1}\left[(2-j)\int_q^\infty \sqrt{\frac{k+1}{y}} I_{k+1,s}(y) I_{k,s}(y) dy \right.$$
$$\left. +(j-1)\int_0^q \sqrt{\frac{k+1}{y}} I_{s,k+1}(y) I_{s,k}(y) dy\right] . \tag{5.26}$$

The result (5.24) follows from (5.26) at $\mu = 0$.

E. Week magnetic field approximation

Consider here the magnetic field which obeys the condition $H \ll H_0$, see a discussion above. Besides, we believe that the inequality

$$2\gamma\bar{\nu}K_j^{-2} = 2\frac{H}{H_0}\left(\frac{m_0c^2}{E_j}\right)^2 \bar{\nu} \ll 1 \tag{5.27}$$

takes place. We know that the number of a harmonic, which is effectively emitted in the relativistic case, is $\bar{\nu} \sim (E_j/m_0c^2)^3$. For such a harmonic the equation (5.27) results in

$$2\frac{H}{H_0}\frac{E_j}{m_0c^2} \ll 1 . \tag{5.28}$$

As it is known [26] the condition (5.28) implies that quantum corrections are small in the relativistic case. That is not true in the nonrelativistic approximation due to the fact that here only a harmonic $\bar{\nu} \sim 1$ is emitted effectively and the condition (5.27) takes place if $H \ll H_0$. Practically, the condition (5.28 always takes place for real laboratory magnetic fields and energies of electrons.

In the above suppositions with respect to the magnetic field we find from (5.13)

$$q = \frac{1}{2}\frac{H}{H_0}\left(\frac{m_0c^2}{E_j}\right)^2 \bar{\nu}^2 \sin^2\theta . \tag{5.29}$$

If the numbers of harmonics emitted are not very high, then

$$q \ll 1 . \tag{5.30}$$

We are going to suppose that namely this condition takes place, and at the same time will call (5.30) the condition of the week magnetic field. In the ultra-relativistic case, when $\nu \sim (E_j/m_0c^2)^3$, we find

$$q \sim \frac{H}{H_0}\left(\frac{E_j}{m_0c^2}\right)^4 . \tag{5.31}$$

Thus, in this case q may be not small. Such a situation will be considered in a next article.

At $q \ll 1$ we may present the intensity of the radiation in the following form

$$W_j = W_j^{\rm cl}\bar{W}_j\,, \quad W_j^{\rm cl} = \frac{2}{3}\frac{e^4H^2\beta_j^2(1-\beta_j^2)}{m_0^2c^3}\,, \tag{5.32}$$

$$\bar{W}_j = \sum_{j'}\bar{W}_{jj'}\,, \quad \bar{W}_{jj'} = \frac{3}{4(2\bar{n}+1)}\int_0^\pi \sin\theta d\theta S_{jj'}^2 \sum_\nu \bar{\nu}^2 R_{jj'}\,, \tag{5.33}$$

$$S_{11} = S_{22} = S_{12} = l_2 + l_3\cos\theta = S,\ S_{21} = l_2 - l_3\cos\theta = \bar{S}\,. \tag{5.34}$$

We have selected here the quantity $W_j^{\rm cl}$, which is the intensity of the first harmonic radiation in semiclassical case (see [26]). The radiation polarization is characterized by the factor $S_{jj'}$. In particular, it indicates that the transition $2 \to 1$ has an opposite sign of circular polarization in comparison to all other transitions. That may serve as a way to identify the part of radiation related to such transitions. The quantities $R_{jj'}$ may be calculated in lowest order with respect to q, using exact expressions (5.20),(5.22), and (5.28).

1. We find for the $(1 \to 1)$ transitions

$$R_{11} = \frac{\Gamma(n+\mu+1)q^{\nu-1}}{\Gamma(n+\mu+1-\nu)\Gamma^2(\nu)}\,, \quad 1 \le \nu \le l\,,$$
$$R_{11} = \frac{\Gamma(n-l+1)\Gamma(n+\mu+1)q^{2\nu-l-1}}{\Gamma(n-\nu+1)\Gamma(n-\nu+\mu+1\Gamma^2(\nu-l+1))\Gamma^2(\nu)}\,, \quad l \le \nu \le n\,. \tag{5.35}$$

These formulas mean that effectively only harmonics with $\nu = 1$ are emitted, and at the same time the intensity of radiation does not depend on l if $1 \le l \le n$. We get easily for the quantity $\bar{W}_{11}$

$$\bar{W}_{11} = \frac{2(n+\mu)}{2(n+\mu)+1}\bar{S}^2\,, \quad \bar{S}^2 = \frac{3}{4}l_2^2 + \frac{1}{4}l_3^2\,. \tag{5.36}$$

We see that the radiation is polarized similar to the usual SR in nonrelativistic case [26]. The quantity $\bar{W}_{11}$ grows slowly with n, so that $\lim_{n\to\infty}\bar{W}_{11} = \bar{S}^2$.

The probability for transitions with $l = 0$ in initial states is of second order (in q) in comparison with all other transitions. For the former transitions we find at $\nu = 1$

$$\bar{W}_{11} = \frac{H}{H_0}\left(\frac{m_0c^2}{E_1}\right)^2 \frac{6n(n+\mu)}{5(2n+2\mu+1)}\left(\frac{5}{6}l_2^2 + \frac{1}{6}l_3^2\right) . \tag{5.37}$$

One can see that in the case under consideration the rate of linear polarization is greater than for (5.36)

2. For transitions $(2 \to 2)$ we find

$$R_{22} = \frac{\Gamma(n+1)q^{\nu-1}}{\Gamma(n+1-\nu)\Gamma^2(\nu)} . \tag{5.38}$$

As before, we see that effectively only the first harmonic is emitted, and at the same time the intensity of radiation does not depend on $l < 0$ in leading order in q. For this harmonic $R_{22} = n$, and

$$\bar{W}_{22} = \frac{2n}{2n+1}\bar{S}^2 . \tag{5.39}$$

The latter result is physically obvious and may be derived from (5.36) at $\mu = 0$.

3. For transitions $(2 \to 1)$ we find

$$R_{21} = \frac{\Gamma(n+|l|+1-\mu)\Gamma(n-\nu+1+\mu)\Gamma^2(1+\nu-\mu)\mu^2(1-\mu)^2q^{|l|-1}}{\Gamma(n+1-\nu)\Gamma(n+1)\Gamma^2(|l|+\nu+1-\mu)} f^2(\mu) . \tag{5.40}$$

We have introduced here a function $f(\mu)$,

$$f(\mu) = \frac{\sin\mu\pi}{\mu(1-\mu)\pi} , \quad 0 \le \mu \le 1 , \tag{5.41}$$

with the following properties

$$f(\mu) = f(1-\mu),\ f(0) = f(1) = 1,\ f_{\max}(\mu = 1/2) = 4/\pi > 1,\ 1 \le f(\mu) \le 4/\pi . \tag{5.42}$$

Thus, the function vary very little from the unit at $0 \le \mu \le 1$.

In the transitions under consideration, we meet a situation which is completely different from ones considered before. Here the emission takes effectively place only in course of transitions from the state with $l = -1$. That has natural physical explanation: At $l = -1$, $j = 2$ the classical trajectory is maximal close to the solenoid, which is out of the trajectory. Namely from such a state it is most easy to pass to the situation when the solenoid is inside the trajectory. Remarkable is the fact that there are not any restrictions on the numbers of the harmonics emitted. At $l = -1$ we get

$$R_{21} = \frac{\Gamma(n+2-\mu)\Gamma(n-\nu+1+\mu)\mu^2(1-\mu)^2}{\Gamma(n+1)\Gamma(n-\nu+1)(\nu+1-\mu)^2}\, f^2(\mu) \,, \tag{5.43}$$

and

$$\begin{aligned} \bar{W}_{21} &= \frac{2\Gamma(n+2-\mu)\mu^2(1-\mu)^2 M^{21} f^2(\mu)}{(2n+1)\Gamma(n+1)}\, \bar{S}^2 \,, \\ M^{21} &= \sum_{\nu=1}^{n} M_\nu^{21} \,, \quad M_\nu^{21} = \frac{\Gamma(n-\nu+1+\mu)}{\Gamma(n-\nu+1)} \left(\frac{\nu-\mu}{\nu-\mu+1}\right)^2 \,. \end{aligned} \tag{5.44}$$

For example, at $n = 1$ we get

$$\bar{W}_{21}(n=1) = \frac{2\mu^2(1-\mu)^4}{3(2-\mu)}\, f(\mu)\bar{S}^2 \,. \tag{5.45}$$

One can see that M_ν^{21} does not vary rapidly when ν does. That means that at least some of the first harmonics have equal probabilities of emission. One can derive an estimation for M. It follows from (5.44) that

$$M^{21} \le \sum_{\nu=1}^{n} \frac{\Gamma(n-\nu+1+\mu)}{\Gamma(n-\nu+1)} = \frac{\Gamma(n+1+\mu)}{(1+\mu)\Gamma(n)};. \tag{5.46}$$

Then

$$\begin{aligned} \bar{W}_{21} &= (n+1-\mu)\frac{\mu^2(1-\mu)^2 f^2(\mu)}{1+\mu}\, \bar{M}^{21}\bar{S}^2 \,, \\ \bar{M}^{21} &\le \bar{M}_n^{21} = \frac{2n}{2n+1}\frac{\Gamma(n+1+\mu)\Gamma(n+1-\mu)}{\Gamma^2(n+1)} \,. \end{aligned} \tag{5.47}$$

It is easy to find

$$\lim_{n\to\infty} \bar{M}_n^{21} = 1 \,. \tag{5.48}$$

At $\mu = 0$ the quantity $\bar{W}_{21}$ vanishes. Thus, if the mantissa of the magnetic flux is not zero there appears a possibility for transitions, which where forbidden in SR. These transitions give approximately equal contributions (at least first of them) in the intensity of the radiation.

Especially interesting is the case when $\nu = 1$. As was already remarked in this case we get an emitted frequency (5.19), which is less then the basic synchrotron one. From (5.45) we find

$$\bar{W}_{21}(\nu=1) = \left[\frac{\mu(1-\mu)^2 f(\mu)}{2-\mu}\right]^2 \bar{S}^2 A_n^{21}\,,$$
$$A_n^{21} = \frac{2(n-\mu)(n+1-\mu)}{n(2n+1)}\frac{\Gamma(n+\mu)\Gamma(n-\mu)}{\Gamma^2(n)}\,. \tag{5.49}$$

Since $\lim_{n\to\infty} A_n^{21} = 1$, the radiation intensity is approximately equal to the classical one multiplied by the factor

$$\left[\frac{\mu(1-\mu)^2 f(\mu)}{2-\mu}\right]^2 . \tag{5.50}$$

4. Consider finally the transitions $(1 \to 2)$. In this case we get

$$R_{12} = \frac{\Gamma(n-\nu-\mu+2)\Gamma(n+\mu+1)(\nu+\mu)^2 q^l}{\Gamma(n-l+1)\Gamma(n-\nu+1)\Gamma^2(l-\nu+2-\mu)\Gamma^2(\nu+1+\mu)}\,. \tag{5.51}$$

We see that the transitions take effectively place only from the state with $l = 0$. Since this state corresponds to a classical trajectory which passes maximally close to the solenoid (embracing it), a physical interpretation is similar to the one given in the previous case. Thus, for the case $l = 0$ we get

$$\bar{W}_{12} = \frac{2\Gamma(n+1+\mu)\mu^2(1-\mu)^2 M^{12} f^2(\mu)}{(2n+2\mu+1)\Gamma(n+1)}\bar{S}^2\,,$$
$$M^{12} = \sum_{\nu=0}^{n} M_\nu^{12}\,,\quad M_\nu^{12} = \frac{\Gamma(n-\nu+2-\mu)}{\Gamma(n-\nu+1)}\left(\frac{\nu+\mu}{\nu+\mu-1}\right)^2 . \tag{5.52}$$

As before we observe here a possibility for emission of $\nu = 0$ harmonic, which is forbidden in the usual SR. The frequency of the corresponding radiation is given by the expression (5.18). For $\nu = 0$ one finds

$$\bar{W}_{12}(\nu=0) = \mu^4 f^2(\mu)\bar{S}^2 A_n^{12}\,,$$
$$A_n^{12} = \frac{2(n+\mu+1)}{2n+2\mu+1}\frac{\Gamma(n+\mu+1)\Gamma(n-\mu+1)}{\Gamma^2(n+1)}\,,\quad \lim_{n\to\infty} A_n^{12} = 1\,. \tag{5.53}$$

In particular,

$$\bar{W}_{12}(\nu = n = 0) = \frac{2\mu^4 f(\mu)}{2\mu + 1}\bar{S}^2 \,. \tag{5.54}$$

Thus, the intensity of the emission of that harmonic may be obtained multiplying the corresponding classical intensity by the factor $\mu^4 f(\mu)$.

In the transitions under consideration ($1 \to 2$) all harmonics with different numbers ν contribute quasy equally to the intensity, since M_ν^{12} does not vary rapidly when ν does. One may find the following estimation for the M_ν^{12}

$$M^{12} \geq \frac{\Gamma(n+3-\mu)}{(2-\mu)\Gamma(n+1)} \,. \tag{5.55}$$

Thus, we get

$$\begin{aligned} \bar{W}_{12} &= \frac{(n+\mu-1)}{2-\mu}\left[\mu(1-\mu)f(\mu)\right]^2 \bar{M}^{12}\bar{S}^2 \,, \\ \bar{M}^{12} \geq \bar{M}_n^{12} &= \frac{2(n+2-\mu)}{2n+2\mu+1}\frac{\Gamma(n+1+\mu)\Gamma(n+1-\mu)}{\Gamma^2(n+1)} \,. \end{aligned} \tag{5.56}$$

It is easy to find as before that

$$\lim_{n\to\infty} \bar{M}_n^{12} = 1 \,. \tag{5.57}$$

At $\mu = 0$ these transitions are allowed only in the higher orders in q.

F. Semiclassical approximation and AB effect

It was shown in theory of usual SR [26] that semiclassical decomposition for radiation intensity corresponds to a decomposition in small parameter ν/n. Technically to this end the formula (A28) was used. It is natural to believe that in the case under consideration we may try to use similar parameter $\bar{\nu}/n$ to get semiclassical decomposition in spite of the fact that $\bar{\nu}$ is not integer. Acting in such a way, we get a classical part of the intensity

$$W_j^{cl} = \frac{e^4 H^2(1-\beta_j^2)}{m_0^2 c^3}\sum_{\nu,j'}\int_0^\pi \sin\theta \, d\theta \bar{\nu}^2 Q_{jj'}^{cl} |F^{cl}|^2 \,, \tag{5.58}$$

where the corresponding classical quantities have the form

$$F^{cl} = l_2\beta_j J'_{jj'}(z) + l_3 \cot\theta J_{jj'}(z)\,, \quad z = \bar{\nu}\beta_j \sin\theta\,,$$

$$J_{11} = J_{22} = J_{12} = J_{\bar{\nu}}(z)\,, \quad J_{21} = J_{-\bar{\nu}}(z)\,, \tag{5.59}$$

$$Q^{cl}_{jj'} = \frac{1}{2} + (-1)^{j+j'} \int_z^\infty dy \left[(2-j)J_{l-\bar{\nu}+1}(y)J_{l-\bar{\nu}}(y) \right.$$
$$\left. +(j-1)J_{|l|+\bar{\nu}}(y)J_{|l|+\bar{\nu}-1}(y)\right]\,. \tag{5.60}$$

Here $J_s(y)$ is a Bessel function.

At $\mu = 0$ the quantity $|F^{cl}|^2$ does not depend on j', and $\sum_{j'=1,2} Q^{cl}_{jj'} = 1$. Thus, we arrive to well-known [26] classical expression for spectral-angular distribution of SR.

For concrete jj' the expressions for $Q^{cl}_{jj'}$ may be written in more convenient forms,

$$Q^{cl}_{11} = \begin{cases} 1 - \int_0^z J_{l-\nu+1}(y)J_{l-\nu}(y)dy, \; l \geq \nu\,, \\ \int_0^z J_{\nu-l-1}(y)J_{\nu-l}(y)dy, \; l < \nu\,, \end{cases}$$

$$Q^{cl}_{12} = \begin{cases} \int_0^z J_{l-\nu+1-\mu}(y)J_{l-\nu-\mu}(y)dy, \; l \geq \nu\,, \\ \frac{1}{2} - \int_0^z J_{l-\nu+1-\mu}(y)J_{l-\nu-\mu}(y)dy, \; l < \nu\,, \end{cases}$$

$$Q^{cl}_{22} = 1 - \int_0^z J_{|l|+\nu}(y)J_{|l|+\nu-l}(y)dy\,,$$

$$Q^{cl}_{21} = \int_0^z J_{|l|+\nu-\mu}(y)J_{|l|+\nu-l-\mu}(y)dy\,. \tag{5.61}$$

In particular, in the nonrelativistic approximation, one gets

$$W^{cl}_j = W^{cl}\bar{S}^2 \sum_{j'} \bar{W}^{cl}_{jj'}\,,$$

$$\bar{W}^{cl}_{jj} = 1,\; \bar{W}^{cl}_{12} = \mu^4 f^2(\mu),\; \bar{W}^{cl}_{21} = \left[\frac{\mu(1-\mu)^2 f(\mu)}{2-\mu}\right]^2\,. \tag{5.62}$$

We have selected $l = \nu = 0$ for $(1 \to 2)$ transitions and $|l| = \nu = 1$ for $(2 \to 1)$ transitions. The expressions (5.61) follow from the exact quantum expressions (5.37), (5.40), (5.49), and (5.52) in the limit $n \to \infty$.

Semiclassical expressions (5.61) depend essentially on the initial quantum number l if μ is not zero. Taking into account the Eq. (2.17), which relates $R^2 - R_0^2$ (minimal distance between the classical trajectory and the solenoid) with l, we may express l in terms of this pure classical characteristic to make the formulas (5.61) formally completely classical as well.

What seams really important for us to stress is the fact that both quantum expressions and semiclassical ones do not have any degeneracy in parameters, which define the initial state, in the presence of the magnetic flux with nonzero μ.

It follows from the equation (2.17) that at a fixed $R^2 - R_0^2$ we get $|l| \sim 1/\hbar$. Formal limit to pure classical case demands $|l| \to \infty$. Doing such a limit in the expressions (5.61) we get

$$Q^{cl}_{jj'} = \delta_{jj'} \,. \tag{5.63}$$

That result is natural. In pure classical theory the trajectory of a particle is given (there is not any back reaction from emitted photons on particle motion) and transitions with change of the initial state are forbidden.

G. Concluding remarks

Unfortunately, it is difficult to proceed further with exact analytic calculations. For example, one cannot simplify sums of the form

$$\sum_{\nu=1}^{\infty} J_\nu^2(\nu x) J_\nu^2(\nu y) = f(xy)\,,\; 0 < x < 1,\; 0 < y < 1\,,$$
$$\sum_{\nu=1}^{\infty} J_{\bar\nu}(\bar\nu x) = \varphi_\mu(x)\,,\; \bar\nu = \nu + \mu,\; 0 < x < 1\,.$$

which appear in such calculations. Such sums are known only at $\mu = 0$.

However, one may sometimes estimate a behavior of the quantities $Q_{jj'}$ in the ultra-relativistic approximation. Using an experience obtained in usual SR calculations [26], we believe that $J_\nu(z) \sim \sqrt{\epsilon}$, $\epsilon = 1 - \beta^2 \sin^2\theta$. In the ultra-relativistic case $\epsilon \sim \epsilon_0 = (m_0 c^2/E)^2$. From the identity

$$\int_0^z J_\nu(y) J_{\nu-1}(y) dy = \nu \int_0^z \frac{J_\nu^2(y)}{y} dy + \frac{1}{2} J_\nu^2(z)$$

we get

$$\int_0^z J_\nu(y) J_{\nu-1}(y) dy \sim \epsilon \,.$$

Finally, taking into account that the angle integration reduces the order of ϵ,

$$\int_0^\pi \epsilon^{-s} \sin\theta \, d\theta \sim \epsilon_0^{-s+1/2} \,,$$

we get from (5.60) in the ultra-relativistic case

$$Q_{jj'} = \delta_{jj'} + \sqrt{\epsilon_0}\alpha_{jj'} \,,$$

where $\alpha_{jj'}$ are some finite quantities. That means that only transitions without a change of j (change of the type of states) take place. We believe that the effective number of the harmonics does not change as well due to quasi-continuous character of the spectrum. Thus, the evidence of the AB effect in such a case are negligible in comparison with the leading terms. These qualitative considerations may be wrong at small $|l| = 0, 1, 2, \ldots$. The peculiarity of the behavior of SR in nonrelativistic limit may serve as a justification of the latter consideration.

The exact expressions for intensity of SR, which we have obtained, may be analyzed numerically in concrete cases. For example, transitions from lower levels in strong fields etc. Analysis of SR for spinning particles will be done by us in a next article.

VI. DISCUSSION

First of all, we would like to stress that several new exact results related to the magnetic-solenoid field are obtained. Among them exact solutions of Klein-Gordon and Dirac equations in such a field. Besides, matrix elements of transitions with a photon emission are calculated and on this base all characteristics of the SR in the field are obtained and analyzed.

It is demonstrated that only the mantissa of the AB field affects characteristics of usual SR (that is consistent with the fact that the AB effect is related to the shift of the phase factor in wave functions). The main physically interesting peculiarities of such an influence are the following:

Each spectral line (harmonic) of usual SR is split into two lines. One of them remains very close to the initial SR line. The corresponding to it relative frequency shift is small, namely, it is proportional to H/H_0 and may be estimated as $10^{-9}\div 10^{-6}$ at present laboratory strengths of magnetic fields. The relative shift of the second harmonic is equal to μ or to $1-\mu$ (depending of the type of the initial state). Experimental observation of such shifts (especially in nonrelativistic case) seams to be not difficult.

As it is known, the intensity of usual SR does not depend on the orbital quantum number of the initial state. The presence of the magnetic flux with non zero μ breaks completely such a degeneracy. The quantum number enters in a nontrivial way in all the characteristic of the SR.

Besides, in usual SR, in the nonrelativistic case ($\beta^2 \ll 1$), only the first harmonic ($\nu = 1$) is practically emitted. The emission of all others harmonics is proportional to $\beta^{2\nu}$. The presence of the magnetic flux with non zero μ changes this situation drastically. If initial states of charged particles have $l = 0, -1$ then in the nonrelativistic case a set of harmonics has the same probability of irradiation. That may increase the total intensity of the radiation.

Finally, there appear principle new lines related to the existence of two type of states in the problem. Namely, there exist a radiation of harmonics with frequencies, which are less then the basic synchrotron frequency ω. These frequencies are $\mu\omega$ or $(1-\mu)\omega$, depending on the type of the initial state. Important that the intensities of the radiation for these harmonics are comparable with that for the basic synchrotron frequency. Significant relative shift between these new frequencies and the basic synchrotron frequency allows one to think about their possible experimental observation.

APPENDIX A: SOME PROPERTIES OF LAGERR FUNCTIONS

1. We define Lagerr functions $I_{n,m}$ by a relation

$$I_{n,m}(x) = \sqrt{\frac{\Gamma(1+n)}{\Gamma(1+m)}} \frac{\exp(-x/2)}{\Gamma(1+n-m)} x^{\frac{n-m}{2}} \Phi(-m, n-m+1; x) \, . \tag{A1}$$

Here $\Phi(\alpha, \gamma; x)$ is the degenerate hyper-geometric function in standard definition ([31], p.1085, Eq. (9.210)). At $\gamma \neq -s$ (s is an integer nonnegative number) the function may be given by a convergent at any complex x series

$$\Phi(\alpha,\gamma;x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\gamma)_k} \frac{x^k}{k!} = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)}{\Gamma(\gamma+k)} \frac{x^k}{k!}\,, \tag{A2}$$

where Pohgammer symbols $(\alpha)_k$ are defined for any complex α as follows

$$(\alpha)_k = \alpha(\alpha+1)...(\alpha+k-1) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}\,. \tag{A3}$$

The Lagerr functions are defined for any complex x and for any numbers n, m, for which the right side (A1) has sense.

2. If m is an integer nonnegative number, then the Lagerr functions are connected to the Lagerr polynomials $L_m^\alpha(x)$ by the relation

$$I_{n,m}(x) = \sqrt{\frac{\Gamma(1+m)}{\Gamma(1+n)}} \exp(-x/2) x^{\frac{n-m}{2}} L_m^{n-m}(x)\,, \quad m = 0, 1, 2, \dots\,. \tag{A4}$$

We use a standard definition ([31], p.1061, Eqs. (8.970), (8.972.1))

$$L_n^\alpha(x) = \frac{1}{n!} e^x x^{-\alpha} \frac{d^n}{dx^n} e^{-x} x^{n+\alpha} = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} = \binom{n+\alpha}{n} \Phi(-n, 1+\alpha; x)\,, \tag{A5}$$

where

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)...(\alpha-n+1)}{n!} \tag{A6}$$

are binomial coefficients, and n is an integer nonnegative number. At the same time

$$\binom{\alpha}{n} = \frac{\Gamma(1+\alpha)}{\Gamma(1+n)\Gamma(1+\alpha-n)}\,. \tag{A7}$$

The equation (A7) defines $\binom{\alpha}{n}$ for any complex α and n, for which the right side of (A7) has sense.

3. Using well-known ([31], p.1086; Eqs. (9.212), (9.213), (9.216)) properties of the degenerate hyper-geometric function, one can find the following relations for the Lagerr functions

$$2\sqrt{x(n+1)}I_{n+1,m}(x) = (n-m+x)I_{n,m}(x) - 2xI'_{n,m}(x)\ , \tag{A8}$$

$$2\sqrt{x(m+1)}I_{n,m+1}(x) = (n-m-x)I_{n,m}(x) + 2xI'_{n,m}(x)\ , \tag{A9}$$

$$2\sqrt{xn}I_{n-1,m}(x) = (n-m+x)I_{n,m}(x) + 2xI'_{n,m}(x)\ , \tag{A10}$$

$$2\sqrt{xm}I_{n,m-1}(x) = (n-m-x)I_{n,m}(x) - 2xI'_{n,m}(x)\ , \tag{A11}$$

$$2\sqrt{nm}I_{n-1,m-1}(x) = (n+m-x)I_{n,m}(x) - 2xI'_{n,m}(x)\ , \tag{A12}$$

$$2\sqrt{(n+1)(m+1)}I_{n+1,m+1}(x) = (n+m+2-x)I_{n,m}(x) + 2xI'_{n,m}(x)\ , \tag{A13}$$

and a differential equation

$$4x^2I''_{n,m}(x) + 4xI'_{n,m}(x) - [x^2 - 2x(1+n+m) + (n-m)^2]I_{n,m}(x) = 0\ . \tag{A14}$$

Its general solution I has the form

$$I = AI_{n,m}(x) + BI_{m,n}(x)\ , \tag{A15}$$

in case if the functions $I_{n,m}(x)$ and $I_{m,n}(x)$ are linearly independent. Besides the cases when the equation (A19) holds, they are alway linearly independent. The relations (A8)-(A13) and the equation(A14) have sense at any $n, m = 0$. Straightforward calculations based on the Eqs. (A1) and (A2) lead to the following formulas

$$\lim_{n\to 0}\sqrt{n}I_{n-1,m}(x) = -\frac{\sin m\pi}{\pi}\sqrt{\Gamma(1+m)}x^{-\frac{1+m}{2}}\exp(x/2)\ , \tag{A16}$$

$$\lim_{m\to 0}\sqrt{m}I_{n,m-1}(x) = 0\ . \tag{A17}$$

4. Using some additional properties of the degenerate hyper-geometric function, one can derive the following representation for the Lagerr functions

$$I_{n,m}(x) = \sqrt{\frac{\Gamma(1+n)}{\Gamma(1+m)}}\frac{\exp(x/2)}{\Gamma(1+n-m)}x^{\frac{n-m}{2}}\Phi(1+n, 1+n-m; -x)\ , \tag{A18}$$

and a relation ([31], p.1086, (9.214))

$$I_{n,m}(x) = (-1)^{n-m}I_{m,n}(x),\quad n-m \text{ integer}\ . \tag{A19}$$

5. Using (A1), (A18), and (A2), one can derive series representations for $I_{n,m}(x)$,

$$I_{n,m}(x) = -\frac{\sin m\pi}{\pi}\sqrt{\Gamma(1+n)\Gamma(1+m)}e^{-\frac{x}{2}}x^{\frac{n-m}{2}}\sum_{k=0}^{\infty}\frac{\Gamma(k-m)}{\Gamma(k-m+n+1)}\frac{x^k}{k!}$$

$$= \sqrt{\Gamma(1+n)\Gamma(1+m)}e^{-\frac{x}{2}}x^{\frac{n-m}{2}}\sum_{k=0}^{\infty}\frac{(-x)^k}{\Gamma(m-k+1)\Gamma(n-m+k+1)k!}\ . \quad \text{(A20)}$$

$$I_{n,m}(x) = \frac{exp(x/2)x^{\frac{n-m}{2}}}{\sqrt{\Gamma(1+n)\Gamma(1+m)}}\sum_{k=0}^{\infty}\frac{\Gamma(n+k+1)}{\Gamma(n-m+k+1)}\frac{(-x)^k}{k!}\ . \quad \text{(A21)}$$

6. At $Re\,x \to +\infty$ an asymptotic formula takes place

$$\Phi(a,c;x) \approx \frac{\Gamma(c)}{\Gamma(a)}e^x x^{a-c}\ . \quad \text{(A22)}$$

That results in the following asymptotic behavior of $I_{n,m}(x)$ at $Re\,x \to +\infty$

$$I_{n,m}(x) \approx -\frac{\sin m\pi}{\pi}\sqrt{\Gamma(1+n)\Gamma(1+m)}x^{-\frac{n+m+2}{2}}\exp(x/2)\ , \quad \text{(A23)}$$

(under the supposition that m is not integer), and to the representation

$$I_{n,m}(x) \approx (-1)^m\frac{x^{\frac{n+m}{2}}exp(-x/2)}{\sqrt{\Gamma(1+n)\Gamma(1+m)}}\ , \quad \text{(A24)}$$

if m is integer.

7. Below we present some asymptotics of $I_{n,m}(x)$.

At $n \to \infty$, and m, x are fixed we get

$$I_{n,m}(x) \approx \frac{n^m x^{\frac{n-m}{2}}\exp(-x/2)}{\sqrt{\Gamma(1+n)\Gamma(1+m)}}\ . \quad \text{(A25)}$$

At $m \to \infty$, and n, x are restricted ($n-m$ is not integer)

$$I_{n,m}(x) \approx \frac{\sin\pi(n-m)}{\pi m^{n+1}}\sqrt{\Gamma(n+1)\Gamma(m+1)}e^{\frac{x}{2}}x^{\frac{n-m}{2}}\ . \quad \text{(A26)}$$

If $n-m$ is integer, then taking into account (A19), one may reduce the problem to the case (A25).

One can get an asymptotic behavior of $I_{\alpha+n,n}(x)$ at $n \to \infty$ and α, x fixed,

$$I_{\alpha+n,n}(x) \approx \frac{e^{-\frac{x}{2}}}{(\pi^2 nx)^{1/4}}\cos(2\sqrt{nx} - \frac{\pi}{2}\alpha - \frac{\pi}{4})\ . \quad \text{(A27)}$$

Finally, one may calculate the following limit

$$\lim_{p\to\infty} I_{p+\alpha,p+\beta}\left(\frac{x^2}{4p}\right) = J_{\alpha-\beta}(x)\ . \tag{A28}$$

8. At $\alpha > -1$ and $n \geq 0$ all are integer, the Lagerr functions $I_{\alpha+n,n}(x)$ may be written as (see (A4))

$$I_{\alpha+n,n}(x) = \sqrt{\frac{n!}{\Gamma(n+\alpha+1)}} e^{-\frac{x}{2}} x^{\frac{\alpha}{2}} L_n^{\alpha}(x)\ . \tag{A29}$$

One can prove, using (A23) and (A24), that in this case they form a complete and orthogonal set on the interval $(0, \infty)$,

$$\int_0^\infty I_{\alpha+n,n}(x) I_{\alpha+m,m}(x) dx = \delta_{m,n}\ . \tag{A30}$$

9. Substituting (A4) into (A19) at $n, m \geq 0$ and all are integer, we get for Lagerr polynomials

$$(-1)^m \Gamma(1+m) x^n L_m^{n-m}(x) = (-1)^n \Gamma(1+n) x^m L_n^{m-n}(x)\ . \tag{A31}$$

10. An important role in the radiation theory play the following two integrals containing Lagerr functions

$$\int_0^\infty I_{\alpha+m,m}(x) I_{\beta+n,n}(x) J_{\alpha+\beta}(2\sqrt{qx}) dx = (-1)^{n+m} I_{\beta+n,m}(q) I_{\alpha+m,n}(q),\ \Re\left(\alpha+\beta+1\right) > 0\ . \tag{A32}$$

$$\int_0^\infty I_{\alpha+m,m}(x) I_{\beta+n,n}(x) J_{\alpha-\beta}(2\sqrt{qx}) dx = (-1)^{n+m} I_{n,m}(q) I_{\alpha+m,\beta+n}(q),\ \Re(\alpha+1) > 0\ , \tag{A33}$$

where $n, m \geq 0$ and all are integer. An integral similar to (A33) is present in ([31], p.853, eq.(7.422.2)), but the answer contains a mistake, which is present in other tables.

11. Consider a sum

$$R_\mu^{m,n}(x) = \sum_{k=0}^\infty I_{k+\mu,m}(x) I_{k+\mu,n}(x)\ . \tag{A34}$$

Then

$$\frac{d}{dx} R_\mu^{m,n}(x) = \frac{1}{2}\sqrt{\frac{\mu}{x}} \left[I_{\mu-1,m}(x) I_{\mu,n}(x) + I_{\mu,m}(x) I_{\mu-1,n}(x)\right]\ . \tag{A35}$$

If $Re\,(2\mu - m - n) > 0$, then it is easy to see that $R_\mu^{m,n}(0) = 0$ and we get

$$R_\mu^{m,n}(x) = \frac{1}{2}\int\limits_0^x \sqrt{\frac{\mu}{y}}\left[I_{\mu-1,m}(y)I_{\mu,n}(y) + I_{\mu,m}(y)I_{\mu-1,n}(y)\right]dy, \quad Re\,(2\mu - m - n) > 0\,. \quad \text{(A36)}$$

If $n, m \geq 0$ and all are integer, then the latter restriction may be omitted

$$R_\mu^{m,n}(x) = \delta_{m,n} - \frac{1}{2}\int\limits_x^\infty \sqrt{\frac{\mu}{y}}[I_{\mu-1,m}(y)I_{\mu,n}(y) + I_{\mu,m}(y)I_{\mu-1,n}(y)]dy\,. \quad \text{(A37)}$$

In this case one can also get

$$R_0^{m,n}(x) = \delta_{m,n} = \sum_{k=0}^\infty I_{k,m}(x)I_{k,n}(x) = \sum_{k=0}^\infty I_{m,k}(x)I_{n,k}(x)\,. \quad \text{(A38)}$$

12. Similar results may be derived for the sum $G_{\mu,s}^{m,n}(x)$,

$$G_{\mu,s}^{m,n}(x) = \sum_{k=0}^s I_{m,k+\mu}(x)I_{n,k+\mu}(x), \quad \text{(A39)}$$

$$\frac{d}{dx}G_{\mu,s}^{m,n}(x) = \frac{1}{2}\sqrt{\frac{\mu+s+1}{x}}\left[I_{m,\mu+s}(x)I_{n,\mu+s+1}(x) + I_{m,\mu+s+1}(x)I_{n,\mu+s}(x)\right]$$
$$-\frac{1}{2}\sqrt{\frac{\mu}{x}}\left[I_{m,\mu-1}(x)I_{n,\mu}(x) + I_{m,\mu}(x)I_{n,\mu-1}(x)\right]\,. \quad \text{(A40)}$$

One can integrate the latter equation to get $G_{\mu,s}^{m,n}(x)$. In particular, at $\mu = 0$ and at arbitrary m, n, one gets

$$G_{0,s}^{m,n}(x) = \sum_{k=0}^s I_{m,k}(x)I_{n,k}(x) = -\frac{1}{2}\int_x^\infty \sqrt{\frac{s+1}{y}}\left[I_{m,s}(y)I_{n,s+1}(y) + I_{m,s+1}(y)I_{n,s}(y)\right]dy\,.$$
$$\text{(A41)}$$

At integer m, $\mu = 0$ and arbitrary $n > m$, we get $G_{0,\infty}^{m,n}(x) = 0$, that results in

$$\sum_{k=0}^\infty I_{m,k}(x)I_{n,k}(x) = 0, \quad n > m, \ m - \text{integer}\,. \quad \text{(A42)}$$

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