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ON THE COMMUTATION PROPERTIES OF NORMAL MODE OPERATORS AND VERTICES IN THE THEORY OF THE RELATIVISTC QUANTUM STRING.

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ABSTRACT

We study the commutation properties of the creation and annihilation operators of the theory of the relativistic quantum string with the vertex of interaction that describes the splitting and joining of strings in the transverse gauge.

As a result of the change of boundary conditions, annihilation operators acting on the vertex are partially transmitted through the vertex and partially reflected as creation operators. A compact formalism is established to describe this effect, and an application is made to study the singularities of the string operators at the point of splitting. The results of this approach are compared with the previous results obtained on the three-string interaction vertex.

1. INTRODUCTION

During the last few years much progress has been made in understanding the dynamical structure underlying the dual resonance models. It has been established that the dual amplitudes are transition amplitudes for well defined relativistic extended objects, the "relativistic quantum strings" which propagate in space time with a spectrum of excitations identical to the spectrum of intermediate resonances of the dual models.

An important operator of the theory is the vertex operator which describes the combination of two strings into a third one or the decay of a string into two fragments. The first evaluation of the matrix elements of this operator was made by Ademollo, Del Giudice, Di Vecchia and Fubini[1], but its physical meaning became apparent after the works of Mandelstam[2] Cremmer and Gervais[3] Kaku and Kikkawa[4]. In the transverse gauge of the strings,[5] where the evolution variable is the "null time" $x^+=(x^0+x^1)/\sqrt{2}$ and the independent dynamical variables are the transverse variables $x_\perp(\sigma), p_\perp(\sigma)$ and the conjugate pair x_0^- and p^+, the vertex is simply a continuous δ-function which imposes continuity of both transverse coordinates and momenta of the strings, so that at the vertex two of the strings are joined together at the end points to form the third one[3,4]. The action of the vertex, therefore, manifests itself through a change of boundary conditions: the coordinates and momenta are continuos across the vertex, but are analyzed into different Fourier expansions on the two sides of the vertex to extract the normal mode operators.

A general feature of a change of boundary conditions is that annihilation (creation) operators acting at the left(right) of the vertex are partially transmitted through the vertex and partially reflected as creation(annihilation) operators. This is for instance what happens at the transition between the Reggeon and Pomeron sector of the dual models[6,7] and the transition between the bosonic and fermionic sectors the dual model with spin[8,9,10].

In this note we present a rather simple technique to analyze the transmission and reflection of normal mode operators at the vertex among three strings, in the transverse gauge. We start from the continuity requirements on $x_\perp(\sigma)$ and $p_\perp(\sigma)$ at the vertex [3,4], to derive a relation among weighted averages of the generalized momentum operator $P_\perp(x)$ on the unit circle. By selecting averaging functions $f(x)$ with suitable properties of analyticity, we can then solve the problem of transmission and reflection of the positive and negative frequency parts of $P_\perp(x)$.

The results which we obtain are of course implicit in the works of refs 1÷4 (see also ref. 11), where the matrix elements of the vertex are evaluated. We present our approach here because we believe that it leads to the required commutation properties of the vertex in a very straightforward and concise way and also because it provides another nice proof of the connection between the general ideas of refs. 3 and 4 (the vertex is an overlap function) and the formalism of ref.1.

In the last section of this article we apply our method to a study of the singularities of the string operators at the point of splitting.

2. Transmission and reflection of normal modes at the vertex.

Let us consider the vertex operator V which describes in the transverse gauge the joining of two strings, b and c, at their end points, to form an outgoing string a. We use a parameter range $0 \leqslant \sigma \leqslant \pi$ for the three strings, and orient the strings so that the extremities which join correspond to $\sigma_b = \sigma_c = \pi$ (see Fig.1). In the transverse gauge the density of p^+ momentum is a constant along the string; at the point of joining we have therefore

$$\sigma_a = \frac{p_b^+}{p_a^+} \pi = \left(1 - \frac{p_c^+}{p_a^+} \right) \pi. \qquad 2.1)$$

In the following we shall use the notation

$$\beta = \frac{p_b^+}{p_a^+}, \quad \gamma = \frac{p_c^+}{p_a^+} = 1 - \beta. \qquad 2.2)$$

The vertex V is an operator with matrix elements between the ket states $|\lambda_b, \lambda_c>$, which describe the states of excitation of the incoming strings b and c, and the bra states $<\lambda_a|$, which describe the states of excitation of the outgoing string a. V satisfies the following equations (it is indeed defined by them)[3,4]:

$$x_a(\sigma)\ V = V x_b\left(\frac{\sigma}{\beta}\right) \text{ for } 0 \leqslant \sigma \leqslant \pi\beta, \qquad 2.3a)$$

$$x_a(\sigma)\ V = V x_c\left(\frac{\pi-\sigma}{\gamma}\right) \text{ for } \pi\beta \leqslant \sigma \leqslant \pi, \qquad 2.3b)$$

$$p_a(\sigma)\ V = V\ p_b\left(\frac{\sigma}{\beta}\right)/\beta \text{ for } 0 \leqslant \sigma \leqslant \pi\beta, \qquad 2.4a)$$

$$p_a(\sigma)\ V = V\ p_c(\frac{\pi-\sigma}{\gamma})/\gamma \text{ for } \pi\beta \leqslant \sigma \leqslant \pi, \qquad 2.4b)$$

where the operators $x(\sigma)$ and $p(\sigma)$ are the transverse position and density of momentum operators of the three strings. Notice that as the argument of x_a, p_a varies from 0 to π, the argument of x_b, p_b increases from 0 to π in Eqs. 2.3a) and 2.4a) first, and then the argument of x_c, p_c decreases from π to 0 in Eqs. 2.3b) and 2.4b). - The factors $\frac{1}{\beta}$ and $\frac{1}{\gamma}$ appear in Eqs. 2.4) because the variable p transforms as a density in a change of parametrization.

In the following we shall omit writing explicitly V in equations like 2.3) and 2.4). - It will remain understood that in any relation among the three strings a V must be present at the right of the operators of string a and at the left of the operators of the strings b and c.

The operators x_a and p_a have the following expansion in terms of creation and annihilation operators $a_n^\dagger$ and a_n:

$$x_a(\sigma) = x_{o,a} + \sum_{n>o} \sqrt{\frac{2\alpha'}{n}}\ (\frac{a_n - a_n^\dagger}{i}) \cos n\sigma, \qquad (2.5)$$

$$p_a(\sigma) = p_{o,a} + \sum_{n>o} \sqrt{\frac{n}{2\alpha'}}\ (a_n + a_n^\dagger)\cos n\sigma. \qquad 2.6)$$

(Analogous expansions hold for x_b, p_b, x_c and p_c).

It is very convenient to introduce the generalized momentum operator $P_a(x)$, function of the complex variable $x=e^{i\sigma}$:

$$P_a(x) = p_a(\sigma) + \frac{1}{2\alpha'} \frac{d}{d\sigma} x_a(\sigma) = p_{o,a} + \sum_{n>o} \sqrt{\frac{n}{2\alpha'}} (a_n x^{-n} + a_n^{\dagger} x^n) \qquad 2.7)$$

In P(x) we distinguish a positive frequency part $P^{(+)}(x)$ containing the creation operators, a negative frequency part $P^{(-)}(x)$ containing the destruction operators and a zero mode p_o.

The operators a_n and $a_n^{\dagger}$ can be obtained from $P_a(x)$ by suitable contour integrations, for instance

$$a_n = \frac{1}{2\pi i} \sqrt{\frac{2\alpha'}{n}} \oint_o \frac{dx}{x} x^n P_a(x). \qquad 2.8)$$

In general, the weighted averages

$$P_f = \frac{1}{2\pi i} \oint_o \frac{dx}{x} f(x) P(x) \qquad 2.9)$$

will contain only the negative and zero frequency part of P(x) if f(x) is an analytic function inside the contour of integration, the positive and zero frequency parts of P(x) if f(x) is an analytic function outside of it.

It is immediate to check that Eq's. 2.3) and 2.4) are equivalent to the following very simple relations:

$$P_a(x) = P_b(x^{\frac{1}{\beta}})/\beta \quad \text{for } |x| = 1, |\arg x| \leq \pi\beta \qquad 2.10a)$$

$$P_a(x) = P_c((-x)^{\frac{1}{\gamma}}) / \gamma \text{ for } |x| = 1 \quad \pi\beta \leq |\arg x| \leq \pi, \qquad 2.10b)$$

which in turn can be rewritten as

$$\oint_{|x|=1} \frac{dx}{x} P_a(x) f(x) = \oint_{|x|=1} \frac{dx}{x} P_b(x) f(x^\beta) +$$

$$+ \oint_{|x|=1} \frac{dx}{x} P_c(x) f(-x^\gamma), \qquad 2.11)$$

for an arbitrary function f.

Starting from the fundamental Eq. 2.11), which is a direct consequence of the assumed properties of V, we can analize the transmission and reflection of the positive and negative frequencies of the P operators at the vertex. Consider for instance a linear combination of destruction operators a_n acting on V, which can be obtained by averaging $P_a(x)$ with a function $f^{(+)}(x)$ analytic inside the unit circle. To "commute" these destruction operators through the vertex, we shall look for a function f(x) such that both $f(x^\beta)$ and $f(-x^\gamma)$ are analytic inside the unit circle whereas the function

$$f^{(-)}(x) = f(x) - f^{(+)}(x) \qquad 2.12)$$

is analytic outside the unit circle. Then Eq. 2.11) tells us that the weighted average of annihilation operators $P_{a,f^{(+)}(x)}$, acting on V, is equivalent to the linear combination of the transmitted annihilation operators $P_{b,f(x^{\beta})}$, $P_{c,f(-x^{\gamma})}$ and the reflected creation operators $P_{a,-f^{(-)}(x)}$.

The mathematical problem we face therefore is that of finding a function f(x), having a definite positive frequency part $f^{(+)}(x)$, such that $f(x^{\beta})$ and $f(-x^{\gamma})$ are analytic functions of x inside the unit circle.

To solve this problem let us consider the equation

$$\frac{(b-z)^{\beta}\ (c-z)^{\gamma}}{\rho x z} = 1, \qquad 2.13)$$

where b and c are two positive real numbers, with b<c, and

$\rho = (c-b)\beta^{\beta}\ \gamma^{\gamma}\ b^{\beta-1}\ c^{\gamma-1}$. This equation determines a mapping*

$$x \rightarrow z = z_0(x),$$

where $z_0(x)$ is a root of the equation at a given x.

If we follow the variation of $z_0(x)$ as x varies in the complex plane, we see that, if x describes a circle around the origin with radius R>>1, its image point describes a small contour C around the origin (see Fig.2). If we let R decrease, we see

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* This mapping is very closely related to the mapping $\tilde{z} = \Sigma_i\ p_i^{\dagger} \log (z-a_i)$ used by Mandelstam in ref. 2.

the contour C expand, cross the point at infinity and assume an "eight" shape for R=1. The double point of the contour, z_1, is then the image of the two points $x=e^{\pm i\pi\beta}$ and is a singular point of the mapping. The values of ρ in Eq. 2.3) is chosen so that the singularity occurs precisely for $|x|=1$. If we let now R become smaller than 1, we must distinguish two cases, according to whether x approaches the origin with $|\arg x|<\pi\beta$ or $|\arg x|>\pi\beta$. In the two cases the image point $z_o(x)$ will approach the points b and, respectively, c and $z_o(x)$ will be an analytic function of $x^{\frac{1}{\beta}}$ or of $(-x)^{\frac{1}{\gamma}}$ in the neighbourhoods of the two points.

We use the mapping induced by Eq. 2.13) to define a function f(x) in the following way:

$$f(x) = \frac{1}{2\pi i} \oint_{z_o(x)} \frac{dz\ F(z)}{1 - \frac{(b-z)^{\beta}(c-z)^{\gamma}}{\rho x z}} \frac{\partial}{\partial z} \frac{(b-z)^{\beta}(c-z)^{\gamma}}{\rho x z} . \qquad 2.14)$$

The contour encloses the singularity at $z=z_o(x)$ and no other singularities of the integrand. The derivative has been introduced in Eq. 2.14) so that the residue be precisely $F(z_o(x))$.

If x satisfies Eq. 2.13) for a given z, then x^{β} satisfies the equation

$$\frac{x(\rho z)^{1/\beta}}{(b-z)(c-z)^{\gamma/\beta}} = 1. \qquad 2.15)$$

We see then that

$$f(x^{\beta}) = \frac{1}{2\pi i} \oint_{z_0(x^{\beta})} \frac{dz\ F(z)}{1 - \frac{x(\rho z)^{1/\beta}}{(b-z)(c-z)^{\gamma/\beta}}} \frac{\partial}{\partial z} \frac{x(\rho z)^{1/\beta}}{(b-z)(c-z)^{\gamma/\beta}}. \qquad 2.16)$$

In this equation we can displace the contour, avoiding the possible singularities of F(z), so as to enclose the point $z=b$. We must of course subtract the contribution (constant in x) of the singularity at $z=b$ introduced by the derivative.

Thus we find

$$f(x^{\beta}) = \frac{1}{2\pi i} \oint_{b,z_0(x^{\beta})} \frac{dz\ F(z)}{1 - \frac{x(\rho z)^{1/\beta}}{(b-z)(c-z)^{\gamma/\beta}}} \frac{\partial}{\partial z} \frac{x(\rho z)^{1/\beta}}{(b-z)(c-z)^{\gamma/\beta}} - F(b). \qquad 2.17)$$

Analogously, we can obtain

$$f(-x^{\gamma}) = \frac{1}{2\pi i} \oint_{c,z_0(-x^{\gamma})} \frac{dz\ F(z)}{1 - \frac{x(-\rho z)^{1/\gamma}}{(c-z)(b-z)^{\beta/\gamma}}} \frac{\partial}{\partial z} \frac{x(-\rho z)^{1/\gamma}}{(c-z)(b-z)^{\beta/\gamma}} - F(c). \qquad 2.18)$$

Notice that if we let x approach zero in the integrands of Eq's. 2.17) or 2.18) the singularity z_0 moves inside the contour of integration (approaching b or c). We see then that $f(x^{\beta})$ and $f(-x^{\gamma})$ are regular functions of x inside the unit circle, so that the above construction satisfies the requirement imposed on these two functions.

We must still separate the positive and negative frequency parts of $f(x)$.

We do this by expressing the contour integral around $z_o(x)$ as difference of two contour integrals

$$f(x) = \frac{1}{2\pi i} \oint_{o,z_o(x)} \frac{dz\ F(z)}{1-\frac{(b-z)^{\beta}(c-z)^{\gamma}}{x\rho z}} \frac{\partial}{\partial z} \frac{(b-z)^{\beta}(c-z)^{\gamma}}{x\rho z}$$

$$-\frac{1}{2\pi i} \oint_{o} \frac{dz\ F(z)}{1-\frac{(b-z)^{\beta}(c-z)^{\gamma}}{x\rho z}} \frac{\partial}{\partial z} \frac{(b-z)^{\beta}(c-z)^{\gamma}}{x\rho z}\ . \qquad 2.19)$$

Since $z_o(x)$ moves towards the origin when $x \to \infty$, away from it when $x \to o$, the first contour integral defines a function $f^{(-)}(x)$ analytic outside the unit circle and the second a function $f^{(+)}(x)$ analytic inside the unit circle.

Let us now consider the expression

$$P^{(-)}_{aF} = \frac{1}{2\pi i} \oint_o \frac{dx}{x}\ f^{(+)}(x)\ P_a(x) =$$

$$= \frac{-1}{(2\pi i)^2} \oint_o \frac{dxP_a(x)}{x} \oint_o \frac{dz\ F(z)}{1-\frac{(b-z)^{\beta}(c-z)^{\gamma}}{x\rho z}} \frac{\partial}{\partial z} \frac{(b-z)^{\beta}(c-z)^{\gamma}}{x\rho z}\ . \qquad 2.20)$$

which represents a linear combination of annihilation operators of type a. We shall determine the function $F(z) = F(z,y)$ in such a way that $p_{aF}^{(-)} = p_a^{(-)}(y)$.

Let us take $|y|>1$ so that the contour of the z integration can be taken to enclose the point $z_o(y)$. If we were allowed to set

$$F(z,y) = \frac{1}{1-\dfrac{\rho y z}{(b-z)^{\beta}(c-z)^{\gamma}}} \quad , \qquad 2.21)$$

then, closing the z contour, we would pick up a contribution from the singularity at $z = z_o(y)$ given by (for $z=z_o(y)$ we have

$$\frac{\rho z}{(b-z)^{\beta}(c-z)^{\gamma}} = \frac{1}{y})$$

$$\frac{-1}{2\pi i} \oint_o \frac{dx}{x}\, p_a(x)\, \frac{y}{x-y} = p_{o,a} + p_a^{(-)}(y) \qquad 2.22)$$

and a contribution from the singularity at $z=o$ given by

$$\frac{-1}{2\pi i} \oint \frac{dx}{x}\, p_a(x) = -p_{o,a} \ , \qquad 2.23)$$

so that the sum of the two would produce exatly $p_a^{(-)}(y)$.

Of course, we cannot use that function $F(z,y)$ in our integral representation, because of the singularities at $z=b$ and $z=c$, which would spoil the properties of analyticity of $f(x^{\beta})$ and $f(-x^{\gamma})$.

But we can use a function

$$F(z,y) = \frac{1}{2\pi i} \oint_{o,z_o(y)} \frac{dz'}{z-z'} \frac{1}{1-\dfrac{\rho y z'}{(b-z)^{\beta}(c-z')^{\gamma}}} . \tag{2.24}$$

If we insert this expression into Eq. 2.20) and close the z contour to zero, then the contribution from the singularity at $z=z'$ reproduces exactly $P_a^{(-)}(y)$, whereas the singularity at $z=o$ generates a term containing only the zero mode $p_{a,o}$.

Summarizing, we have the following relation

$$P_a^{(-)}(y) = -\frac{1}{(2\pi i)^2} \oint_o \frac{dx}{x} P_a(x) \times$$

$$\oint_o \frac{dz\, F(z,y)}{1-\dfrac{(b-z)^{\beta}(c-z)^{\gamma}}{x\rho z}} \frac{\partial}{\partial z} \frac{(b-z)^{\beta}(c-z)^{\gamma}}{x\rho z} + p_{a,o} F(o,y) = \tag{2.25}$$

$$= -\frac{1}{(2\pi i)^2} \oint_o \frac{dx}{x} P_a(x) \oint_{o,z_o(x)} \frac{dz\, F(z,y)}{1-\dfrac{(b-z)^{\beta}(c-z)^{\gamma}}{x\rho z}} \frac{\partial}{\partial z} \frac{(b-z)^{\beta}(c-z)^{\gamma}}{x\rho z}$$

$$+ p_{a,o} F(o,y) .$$

$$+ \frac{1}{(2\pi i)^2} \oint_0 \frac{dx}{x} P_b(x) \oint_{b,z_0(x^\beta)} \frac{dz\ F(z,y)}{1-\frac{x(\rho z)^{1/\beta}}{(b-z)(c-z)^{\gamma/\beta}}} \frac{\partial}{\partial z} \frac{x(\rho z)^{1/\beta}}{(b-z)(c-z)^{1/\beta}}$$

$$-P_{b,o}\ F(b,y)$$

$$+ \frac{1}{(2\pi i)^2} \oint_0 \frac{dx}{x} P_c(x) \oint_{c,z_0(x^\gamma)} \frac{dz\ F(z,y)}{1-\frac{x(-\rho z)^{1/\gamma}}{(c-z)(b-z)^{\beta/\gamma}}} \frac{\partial}{\partial z} \frac{x(-\rho z)^{1/\gamma}}{(c-z)(b-z)^{\beta/\gamma}}$$

$$-\ P_{c,o}\ F(c,y),$$

where $F(z,y)$ is given by Eq. 2.24).

This equation follows immediately from Eqs. 2.11), 2.19) 2.17) and 2.18), and it shows that the annihilation part $P_a^{(-)}(y)$ of $P_a(y)$, acting on the three-string vertex, equals a definite linear combination of reflected creation operators(1^{st} term on the r.h.s.), transmitted annihilation operators (3^{rd} and 5^{th} terms on the r.h.s) plus a contribution from the zero modes (2^{nd},4^{th} and 6^{th} terms on the r.h.s).

By using the technique developed in these pages it is also immediate to express $P_b^{(+)}(y)$(or $P_c^{(+)}(y)$) when acting on V, as a linear combination of $P_a^{(+)}$, $P_b^{(+)}$and $P_c^{(-)}$ plus a contribution from the zero modes. We do not reproduce the corresponding equations here, but only remark that, apart from some obvious changes of sign and exchange of creation and annihilation operators, they are nicely crossing symmetric.

3. DISCUSSION

By expanding the various terms on the r.h.s. of Eq. 2.25) into powers of x and y(the position of the singularities determines whether the expansion must be made into positive or negative powers) one recovers the rules given in ref. 1) for the evaluation of the matrix elements of V*.

However, the purpose of this article was not to derive the matrix elements of V, which have been already discussed in refs. 1÷4) and 11), but to illustrate in a simple and concise way the effects of the change of basis inherent in the approach of refs. 3,4) to the three string interaction vertex.

In particular, the role played by the mapping induced by Eq. 3.13) in performing the change of basis emerges clearly from the construction of the previous section. This mapping, introduced by Mandelstam in ref.2), is central to his construction of the dual amplitudes in the transverse gauge, and we may say that, in the approach of Ademollo, Del Giudice, Di Vecchia and Fubini (ref.1), the mapping is realized by the action of the lightlike (+ and -) components of the operators.

We wish to conclude our discussion by applying the results of Sect.2 to the study of the singularities of the P(y) operator

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* The factor ρ introduces a different normalization of the operators, which can be attributed to the effect of propagators.

at the point of splitting. This point corresponds to the two values

$$y = e^{\pm i\pi\beta} \qquad 3.1)$$

of the argument of P.

We recall from section 2 that $|y|=1$ must be reached from $|y| > 1$. In this limit the location of the singularity $z_0(y)$ in the integrand of Eq. 2.24) approaches from the outside the eight shaped contour of Fig. 2. In general it will be possible to displace the contour of the z' integration avoiding the singularity; but for $y \to e^{\pm i\pi\beta}$ a pinching occurs with another singularity $\tilde{z}_0(y)$ emerging from a different sheet, and the function $F(y,z)$ itself becomes singular.

We analyze the behavior of $F(y,z)$ nearby $y=e^{\pm i\pi\beta}$ by setting

$$y= e^{\pm i\pi\beta+i\phi}, \qquad 3.2)$$

$$z_0(y) = z_1+t(y) \qquad 3.3)$$

and expanding for small ϕ,t. Eq. 2.13) then gives us

$$i\phi = A\ t^2 + \text{higher order terms}, \qquad 3.4)$$

where

$$A = \frac{1}{2}\frac{d^2}{dz^2}\log\frac{(b-z)^{\beta}(c-z)^{\gamma}}{z}\Big|_{z=z_1}$$

$$= -\frac{\beta}{2(b-z_1)^2} - \frac{\gamma}{2(c-z_1)^2} + \frac{1}{2z_1^2} \qquad 3.5)$$

and we have used

$$\frac{d}{dz}\log\frac{(b-z)^{\beta}(c-z)^{\gamma}}{z}\Big|_{z=z_1} = 0. \qquad 3.6)$$

It is easy to isolate now the part singular for $y \to e^{\pm i\pi\beta}$ in the contour integral of Eq. 2.24). It is given by

$$F_s(y,z) = \frac{1}{2\pi i}\oint_{z_0(y)} dz'\frac{1}{z-z'}\,\frac{1}{1-\frac{\rho y z'}{(b-z')^{\beta}(c-z')^{\gamma}}} =$$

$$= \frac{-1}{z-z_0(y)}\left(\frac{\beta}{b-z_0(y)} + \frac{\gamma}{c-z_0(y)} + \frac{1}{z_0(y)}\right)^{-1}. \qquad 3.7)$$

For $y \to e^{\pm i\pi\beta}$ we have

$$F_s(y,z) \simeq \frac{1}{At\ (z-z_1)} = \frac{1}{\pm\sqrt{i\phi}\ \sqrt{A}(z-z_1)}\ , \qquad 3.8)$$

where the sign in the last term depends on the direction of approach of $e^{\pm i\pi\beta}$ on the unit circle. We see from this equation that $F(z,y)$ and therefore $P_a^{(-)}(y)$ develop a square root singularity when y approaches the value it takes at the point of splitting. Since the positive and zero frequency parts of $P_a(y)$ are regular there, we conclude that $P(y)$, $p(\sigma)$ and $x'(\sigma) = \frac{dx}{d\sigma}$, are not well defined for $\sigma = \pi\beta$. However, since the singularity is integrable, the position operator $x(\sigma)$ is a well defined operator even at the point of splitting. This is relevant for Mandelstam's analysis of the behavior of V under Lorentz transformations[12]. In ref. 12)it is shown that for d_{tr} (= number of transverse dimensions) = 24 the commutator of the Lorentz generators M^{i-} with V produces a term proportional to $x_i(\pi\beta)V$, which is a well defined quantity. Notice that the terms of the form $\frac{dx_i(\pi\beta)}{dx^+}V$, arising for $d_{tr} = 24$, are instead singular.

The degree of singularity of $p(\sigma)$ and $x'(\sigma)$ becomes worse if one of the strings has $p^+ = o$ (and zero length in parameter space). If we take for instance $\gamma = o$, we have then $z_1 = c$, and $F(c,y)$, coefficient of the zero mode $p_{o,c}$ in Eq. 2.25), diverges linearly for $y = e^{\pm i\pi} = -1$. In this case the position operator $x(\sigma)$ becomes also singular (logarithmically divergent)at the point of splitting.* This singularity however occurs at an isolated value

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* It is useful to recall that, for $p_c^+ = o$ and c in its ground state, $V = :e^{ip_c x(\pi)}:$ (where now $x_a(\sigma) = x_b(\sigma) = x(\sigma)$). The product $x(\pi)V$ is then logarithmically divergent.

of p^+ and is integrable, so that no conflict arises with the results of ref. 12).

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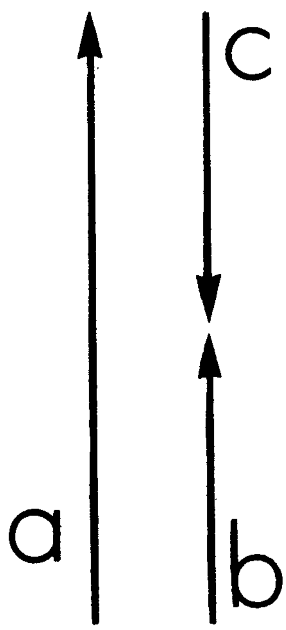

Fig. 1 - RELATIVE ORIENTATION OF THE STRINGS AT THE VERTEX.

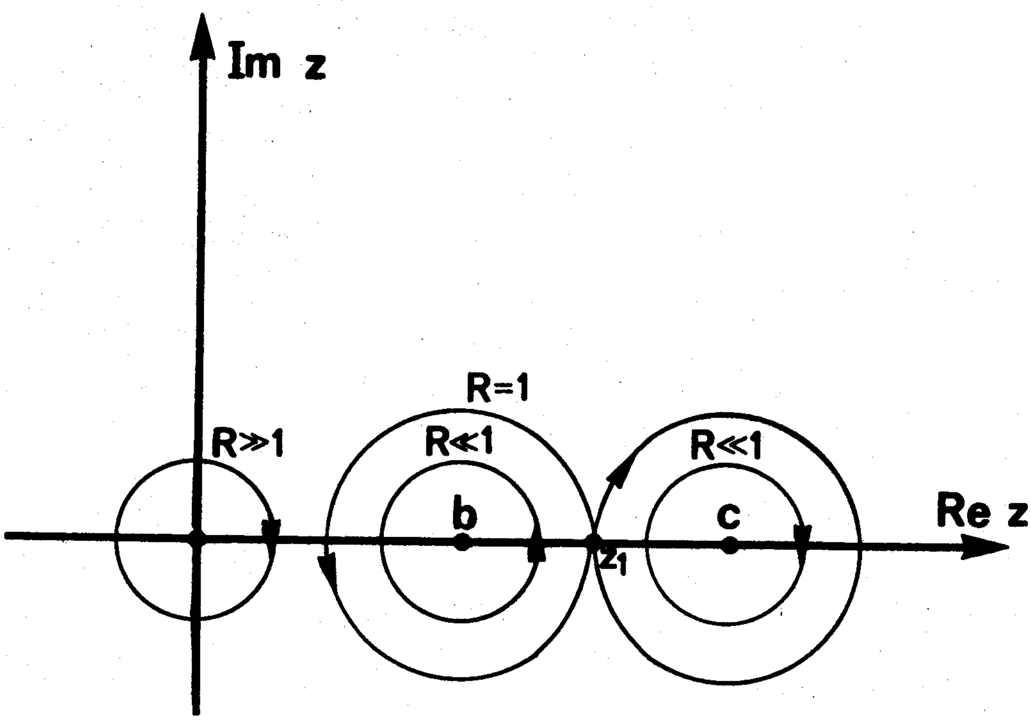

Fig. 2 - IMAGES OF THE CIRCLES $|x|=R$ IN THE MAPPING DETERMINED BY EQ. 2.13).