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REMARK ON A CLASS OF FADDEEV-POPOV LAGRANGIANS IN GAUGE THEORIES

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We will consider a non-abelian gauge theory of the Yang-Mills type [1], consisting of a set of N gauge fields A_μ^a ($a=1,\dots N$), one for each group generator. As is well known, this theory is invariant under a set of gauge transformations of the second kind. Due to this circumstance, in order to quantize the theory, it is necessary to choose gauge determining conditions such as : $F_a(A) = \alpha_a$, where the α's are independent of the fields A_μ^a and of the coupling constant g.

Faddeev and Popov [2] have shown that the vacuum to vacuum amplitude for a gauge theory should be written as :

$$W(\alpha_a) = \int [dA]\, e^{iS(A)} \det \mathcal{M} \prod_a \delta[F_a(A) - \alpha_a] \tag{1}$$

where $S(A)$ is the gauge invariant action and $\mathcal{M}$ is given by :

$$\mathcal{M}_{ab} = \frac{\delta F_a}{\delta A_\mu^c} \left(\delta_{cb} \partial_\mu - g f_{cbe} A_\mu^e \right) \tag{2}$$

In expression (2), f_{cbe} are the antisymmetric structure constants of the group. (In the following, unless otherwise stated, summation and integration over repeated indices will always be understood.) Furthermore, $\det \mathcal{M}$ can be described [2] as loops generated by a set of complex scalar fields G_a and $\bar{G}_a$, obeying Fermi-Dirac statistics :

$$\det \mathcal{M} = \int [dG][d\bar{G}] \exp i \int d^4x\, \bar{G}_a \mathcal{M}_{ab} G_b \tag{3}$$

Here, we wish to investigate whether Lorentz invariant gauge conditions exist, which can generate functions $\mathcal{M}$, so that :

$$S_{FP} = \int d^4x\, \bar{G}_a \mathcal{M}_{ab} G_b = - \int d^4x\, (\mathcal{D}_\mu^2 G)_a^\dagger\, (\mathcal{D}_\mu^1 G)_a \tag{4}$$

with $$(\mathcal{D}_\mu^{1,2})_{ab} = \delta_{ab} - k_{1,2}\, g\, f_{abc} A_\mu^c \doteq (\partial_\mu - k_{1,2} \tilde{A}_\mu)_{ab} \tag{5}$$

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(1) C.N.Yang and R.L.Mills, Phys. Rev. 96, (1954), 191.

(2) L.D.Faddeev and V.Popov, Kiev Report ITP 67-36 (unpublished).

The motivation for considering Faddeev-Popov Lagrangians of this form is that this class contains two important particular cases. First, when $k_1 = k_2 = 0$, one would have a theory where the Faddeev-Popov fields are free [3]. In the second case, when $k_1 = k_2 = 1$, considering that the fields G_a transform according to the adjoint representation of the group, one would have a gauge invariant Faddeev-Popov Lagrangian. From (4) and (5) we obtain the expression for $\mathcal{M}$:

$$\mathcal{M} = \partial^2 - g\,(k_1 \partial . \tilde{A} + k_2 \tilde{A} . \partial) + g^2 k_1 k_2 \tilde{A}^2 \tag{6}$$

with $\partial . \tilde{A} \equiv (\partial . \tilde{A}) + \tilde{A} . \partial$ (7)

Note that $\mathcal{M}$ given by (6) is a Lorentz scalar. From equation (2), since $D_\mu \equiv \partial_\mu - g\tilde{A}_\mu$ is a Lorentz four-vector, it follows that $\frac{\delta F_a}{\delta A_\mu^c} \equiv \left(\frac{\delta F}{\delta A_\mu}\right)_{ac}$ is also a Lorentz four-vector. (This implies that F_a should be a Lorentz scalar.) Therefore, with $\mathcal{M}$ given by (6), the most general solution to equation (2) is given by :

$$\frac{\delta F}{\delta A_\mu} = \mathcal{M}\,(D.D)^{-1} D_\mu + O_\mu \tag{8}$$

Here, O_μ is an arbitrary operator, orthogonal to D_μ:

$$O_\mu D_\mu = 0 \tag{9}$$

Of course, we require that O_μ be well defined, at least in perturbation theory. We can write O_μ as follows :

$$O_\mu = \sum_{n=0}^{\infty} g^n O_\mu^n \tag{10}$$

where O_μ^n are well defined functions, which will be discussed later.

Returning now to equation (8), we see that this represents, in fact, a set of identities which must hold, order by

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(3) There exist non-covariant "axial" gauges, which do not lead to the appearance of ghosts. See, e.g., E.S.Fradkin and I.V.Tyutin, Phys. Rev. D2, (1970), 2841.

order, in perturbation theory. Therefore, with :

$$F^{\alpha}=\sum_{i=0}^{\infty} g^{i} F_i^{\alpha} \tag{11}$$

we must have, in our case :

$$\frac{\delta F_0}{\delta A_\mu} \equiv \partial_\mu + O_\mu^0 \tag{12a}$$

$$\frac{\delta F_1}{\delta A_\mu} \equiv [(1-k_1)\partial.\tilde{A} - k_2\tilde{A}.\partial]\mathcal{D}\partial_\mu + [\tilde{A}.\partial\mathcal{D}\partial_\mu - \tilde{A}_\mu] + O_\mu^1 \tag{12b}$$

etc., where we defined $\mathcal{D}$ by the relation : $\partial^2\mathcal{D}=1$.

Remark that if a gauge function F^{α} exists, which satisfies equation (8), this necessarily requires the existence, for <u>all</u> i, of functions F_i^{α} satisfying the set of identities (12). Therefore, if we can show that for some i, there cannot exist functions F_i^{α} with these properties, we can conclude that there are no gauge determining conditions F^{α}, which can generate Faddeev-Popov Lagrangians of the general form given by equation (4). So it is sufficient to study in detail only the set of equations (12a) and (12b).

We observe that, when $g=0$, $\mathcal{M}=\partial^2$ and $F=F_0$. By inspection of equation (2), we see that in this case :

$$F_0 = (\partial.A) \tag{13}$$

generates the function $\mathcal{M}=\partial^2$. Therefore, in equation (12a), O_μ^0 must be zero.

Now, consider equation (12b). In this equation, O_μ^1 satisfies the relation $O_\mu^1\partial_\mu=0$, which follows from the fact that O_μ^0 vanishes and that equation (9) must hold identically, order by order in g. If a function F_1^{α} can exist, such that functional derivative with respect to A_μ satisfies equation (12b) then, using the property discussed above, we should have :

$$\frac{\delta F_1}{\delta A_\mu}\partial_\mu \equiv (1-k_1)\partial.\tilde{A} - k_2\tilde{A}.\partial \tag{14}$$

From definition (7), we see that relation (14) can hold identically for arbitrary A_μ^{α} only if $k_1=1$. When $k_1\neq 1$, (14) cqn hold only on the manifold $(\partial.A^c)=0$. We will discuss this case later.

So, in general, we must have $k_1 = 1$, and equation (14) becomes :

$$\frac{\delta F_1^a}{\delta A_\mu^b}\partial_\mu = -k_2 f_{abc} A^c \cdot \partial \tag{15}$$

In order to satisfy (15), F_1^a should be quadratic in the gauge fields, if $k_2 \neq 0$. Since it is a Lorentz scalar, the most general form for F_1^a is given by :

$$F_1^a = A_\mu^m O^{mna}_{\mu\nu} A_\nu^n \tag{16}$$

where $O^{mna}_{\mu\nu}$ is a dimensionless operator, independent of the fields A_μ^a. Due to this circumstance, we can write $O^{mna}_{\mu\nu}$ as :

$$O^{mna}_{\mu\nu} = R^{mna} P_{\mu\nu} \tag{17}$$

so that, using relations (15) and (16), we obtain :

$$(R^{bca} P_{\mu\nu} A_\nu^c + A_\nu^c R^{cba} P_{\nu\mu})\partial_\mu = -k_2 f_{abc} A^c \cdot \partial \tag{18}$$

Taking the functional derivative to this relation with respect to A_ν^c and applying the result thus obtained on ∂_ν, using the fact that $\partial_\mu \partial_\nu$ is a symmetric operator, we find :

$$\frac{1}{2}(R^{bca} + R^{cba})(P_{\mu\nu} + P_{\nu\mu})\partial_\mu \partial_\nu = -k_2 f_{abc} \partial^2 \tag{19}$$

However, the only way to satisfy equation (19) is to let $k_2 = 0$, since the left hand side is symmetric in the indices bc, while the right hand side is antisymmetric when $k_2 \neq 0$. So, a solution to equation (14) for arbitrary A_μ^a can exist only when $k_1 = 1$ and $k_2 = 0$, in which case (14) becomes a homogeneous equation.

Identical relations are obtained in higher orders for all F_i. To find a solution in this case, to any order in g, consider equation (8) with :

$$\mathcal{M}(k_1 = 1; k_2 = 0) = \partial^2 - g\,\partial \cdot \tilde{A} \tag{20a}$$

and the orthogonal operator O_μ given by :

$$O_\mu = (\partial^2 - g\,\partial \cdot \check{A})[(\partial \cdot D)^{-1}\partial_\mu - (D \cdot D)^{-1} D_\mu] \tag{20b}$$

Then we obtain identically :

$$\frac{\delta F^a}{\delta A_\mu^b} \equiv \partial_\mu \delta_{ab} \tag{21}$$

so that we get just the Lorentz gauge condition :

$$F^a = \partial . A^a \tag{22}$$

It remains now to consider equation (14) on the manifold $\partial . A \overset{c}{=} 0$. On this manifold, a solution to this equation can readily be found. It is :

$$F_1 = [(1-k_1)\partial . \tilde{A} - k_2 \tilde{A} . \partial] \mathcal{D} \partial . A \tag{23}$$

so that, to first order in g, we can write F as :

$$F = \{\partial^2 + g[(1-k_1)\partial . \tilde{A} - k_2 \tilde{A} . \partial]\} \mathcal{D} \partial . A \tag{24}$$

This relation can easily be generalized to any order in g. We obtain :

$$F = \mathcal{M} \, (\partial^2 - g \partial . \tilde{A})^{-1} \partial . A \tag{25}$$

It can be checked that equation (25) generates, indeed, $\mathcal{M}$ of the form (6) on the manifold $\partial . A \overset{c}{=} 0$. In order to interpret equation (25), we remark that in equation (1) it is necessary to calculate $\mathcal{M}$ only on the manifold $F_a(A) = \alpha_a$. So, by choosing $\alpha_a = 0$, we see that with :

$$\mathcal{M} \, (\partial^2 - g \partial . \tilde{A})^{-1} \partial . A = 0 \tag{26}$$

it follows that $\partial . A \overset{c}{=} 0$. This is due to the fact that equation (26) must hold for arbitrary g, and that, to zero order in g, $\det \mathcal{M} (\partial^2 - g\partial . \tilde{A})^{-1} = 1$, does not vanish. However in this case, using equation (26), we obtain from (1) :

$$W(0) = \int [dA] \, e^{iS(A)} \det \mathcal{M} \prod_a \delta[\mathcal{M}(\partial^2 - g\partial . \tilde{A})^{-1} \partial . A]_a =$$

$$= \int [dA] \, e^{iS(A)} \det(\partial^2 - g\partial . \tilde{A}) \prod_a \delta[\partial . A^a] \tag{27}$$

We see, using (20a) and (22), that expression (27) is just the vacuum to vacuum amplitude for the gauge condition $\partial . A^a = 0$ in the Lorentz gauge $\alpha_a = 0$.

So, excepting the case $k_1 = 1$; $k_2 = 0$, we see that it is not possible to construct Lorentz invariant gauge conditions which can generate Faddeev-Popov Lagrangians of the general form given by equation (4).

It is interesting to understand this result from another point of view. We can think of equation (8) as a set of N (in general distinct) conditions, which the functions F_i^{α} must simultaneously satisfy, for any given α. To illustrate this, consider again relation (14), which is an inhomogeneous equation when $k_2 \neq 0$. The solution corresponding to the inhomogeneous part of this equation should have the form :

$$\frac{\delta F_1^{a}}{\delta A_\mu^{b}} = f_{abc} \left[(1-k_1)\partial . A^c - k_2 A^c . \partial \right] \mathcal{D} \partial_\mu \tag{28}$$

Remark that, when $k_1 = 1$ and $k_2 = 0$, the right hand side vanishes and the constraints that the functions F_i^{α} must satisfy become identical. On the other hand, when $k_1 \neq 1$ and $k_2 \neq 0$, these conditions are, in general, distinct. For example, given any fixed α, consider two indices β and γ, such that $f_{\alpha\beta\gamma} \neq 0$. Then, with $b = \beta$, using the fact that the structure constants are completely antisymmetric, we can solve the above equation, obtaining (now we drop the summation convention) :

$$F_1^{a} = \sum_c f_{a\beta c} \left[(1-k_1)\partial . A^c - k_2 A^c . \partial \right] \mathcal{D} \partial . A^\beta + C^\alpha \tag{29}$$

where the functional C^α is independent of the field A^β. Now, the important point to notice is that F_1^{a}, as given above, at the same time, should also satisfy equation (28) for $b = \gamma$. Then we find :

$$\frac{\delta C^\alpha}{\delta A_\mu^{\gamma}} = f_{a\gamma\beta} \Big\{ \left[(1-k_1)\partial_\mu - k_2 \partial_\mu \right] \mathcal{D} \partial . A^\beta + \\ + \left[(1-k_1)\partial . A^\beta - k_2 A^\beta . \partial \right] \mathcal{D} \partial_\mu \Big\} + \\ + \sum_{s \neq \beta} f_{a\gamma s} \left[(1-k_1)\partial . A^s - k_2 A^s . \partial \right] \mathcal{D} \partial_\mu \tag{30}$$

Since $f_{a\beta\gamma}$ is different from zero, this relation cannot be satisfied unless $k_1 - 1 = k_2 = 0$, because the left hand side is independent of A^β.

We have also considered F^α in higher orders in perturbation theory, (although, as we have seen, this is not strictly necessary for the argument), with the same result. That is, in a non-abelian gauge theory, due to the fact that there are many simultaneous different constraints, it is not possible to construct functions F_i^a which can satisfy equation (8), except in the case $k_1 = 1$ and $k_2 = 0$.

We can then conclude with the statement that, in non-abelian gauge theories, there do not exist Lorentz invariant gauge conditions, which can generate Faddeev-Popov Lagrangians, who are gauge invariant or such that the Faddeev-Popov fields are free fields.

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