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# Solutions of relativistic wave equations in superpositions of Aharonov-Bohm, magnetic, and electric fields.

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## Abstract

We present new exact solutions (in  $3+1$  and  $2+1$  dimensions) of relativistic wave equations (Klein-Gordon and Dirac) in external electromagnetic fields of special form. These fields are combinations of Aharonov-Bohm solenoid field and some additional electric and magnetic fields. In particular, as such additional fields, we consider longitudinal electric and magnetic fields, some crossed fields, and some special non-uniform fields. The solutions obtained can be useful to study Aharonov-Bohm effect in the corresponding electromagnetic fields.

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## I. INTRODUCTION

Aharonov-Bohm (AB) effect [1] plays an important role in quantum theory refining the status of electromagnetic potentials in this theory. First this effect was discussed in relation to a study of interaction between a non-relativistic charged particle and an infinitely long and infinitesimally thin magnetic solenoid field<sup>1</sup> (further AB field). It was discovered that particle wave functions vanish at the solenoid line. In spite of the fact that the magnetic field vanishes out of the solenoid, the phase shift in the wave functions is proportional to the corresponding magnetic flux [3]. A non-trivial particle scattering by the solenoid is interpreted as a possibility for quantum particles to "feel" potentials of the corresponding electromagnetic field. Indeed, potentials of AB field do not vanish out of the solenoid. AB scattering for spinning particles was considered in [4] and [5] using exact solutions of Dirac equation in the AB field. A number of theoretical works and convinced experiments were done to clarify AB effect and to prove its existence (see, for example, [6-9]).

A progress in study of AB effect may be related to revealing new situations, where the effect takes place. For example, one can consider more complicated configurations of electromagnetic fields, different regimes of particle motions, different dimensions, and so on. To study these new possibilities one has to have exact solutions of the corresponding quantum equations in these configurations of electromagnetic fields. In this relation, we ought to mention exact solutions of the Schrödinger equation in a superposition of AB field and a uniform magnetic field [10]. The latter solutions were analyzed in [11-13] from AB effect point of view. The corresponding coherent states were constructed in [12]. Klein-Gordon and Dirac equations for particles moving in a superposition of AB field, Coulomb field, and magnetic monopole field were found and analyzed in [14,15].

In this article we present new exact solutions (in 3+1 and 2+1 dimensions) of relativistic wave equations (Klein-Gordon and Dirac) in external electromagnetic fields of special form.

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<sup>1</sup>A similar effect was discussed earlier by Ehrenberg and Siday [2]

This fields are combinations of AB field and different types of electric and magnetic fields. In Sect.II we consider AB field combined with longitudinal electromagnetic fields. In Sect.III superpositions of AB, longitudinal, and crossed fields are studied. In Sect. IV,V we present solutions in AB field combined with some non-uniform fields. Here we also discuss some relevant solutions in 2 + 1 dimensional QED. Special functions and their properties, which are used in the article, are present in the Appendix.

Most of works, in which AB effect was studied, are based on the use of exact solutions of Schrödinger equation in AB field [1]. Consider the latter a field in 3 + 1 dimensions. If the magnetic solenoid is placed along the axis  $z = x^3$ , then AB field can be given by potentials (we denote these potentials as  $A_\mu^{(0)}(x)$ ,  $x = (x^\mu, \mu = 0, 1, 2, 3)$ ) of the form

$$A_1^{(0)} = \frac{\Phi}{2\pi r^2} x^2, \quad A_2^{(0)} = -\frac{\Phi}{2\pi r^2} x^1, \quad A_0^{(0)} = A_3^{(0)} = 0, \quad r^2 = (x^1)^2 + (x^2)^2. \quad (1.1)$$

AB magnetic field has the form  $\mathbf{H}^{(0)} = (0, 0, H^{(0)})$ , where  $H^{(0)}$  is singular at  $r = 0$ ,

$$H^{(0)} = \Phi \delta(x^1) \delta(x^2). \quad (1.2)$$

AB field creates a finite magnetic flux  $\Phi$  along the axis  $z$ . It is convenient to define a quantity  $\mu$ , which characterizes the magnetic flux  $\Phi$  and is related to the latter as follows

$$\Phi = (l_0 + \mu)\Phi_0, \quad \Phi_0 = 2\pi c\hbar/|e|, \quad 0 \leq \mu < 1, \quad (1.3)$$

where  $l_0$  is integer<sup>2</sup>. In what follows, we call  $\mu$  the mantissa of the magnetic flux  $\Phi$ . By definition  $\mu$  is a positive fractional part of the magnetic flux if the latter is measured in units of quanta  $\Phi_0$ . Cylindrical coordinates  $r, \varphi$  ( $x^1 = r \cos \varphi$ ,  $x^2 = r \sin \varphi$ ) are preferable for AB field consideration. In these coordinates

$$\frac{|e|\hbar}{c} A_1^{(0)} = \frac{l_0 + \mu}{r} \sin \varphi, \quad \frac{|e|\hbar}{c} A_2^{(0)} = -\frac{l_0 + \mu}{r} \cos \varphi. \quad (1.4)$$

In the present article, we are going to consider particle motion in electromagnetic fields  $A_\mu$  that are a combination of AB field and some additional fields with potentials  $A_\mu^{(1)}$ ,

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<sup>2</sup> $e = -|e|$  is the charge of the electron

$$A_\mu = A_\mu^{(0)} + A_\mu^{(1)}. \quad (1.5)$$

Electromagnetic potentials enter in relativistic wave equations only via the operators of momenta  $P_\mu = i\hbar\partial_\mu - \frac{e}{c}A_\mu$ . Doing the transformation  $\Psi(x) = e^{-il_0\varphi}\tilde{\Psi}(x)$  of wave functions, we can eliminate  $l_0$  dependence of AB potentials in equations for  $\tilde{\Psi}(x)$ . Indeed, such equations already contain momentum operators of the form

$$e^{il_0\varphi}P_\mu e^{-il_0\varphi} = i\hbar\partial_\mu - \frac{e}{c}(\tilde{A}_\mu^{(0)} + A_\mu^{(1)}),$$

$$\frac{|e|}{ch}\tilde{A}_1^{(0)} = \frac{\mu}{r}\sin\varphi, \quad \frac{|e|}{ch}\tilde{A}_2^{(0)} = -\frac{\mu}{r}\cos\varphi, \quad \tilde{A}_0^{(0)} = \tilde{A}_3^{(0)} = 0. \quad (1.6)$$

Thus, all the matrix elements of any axial-symmetric operators depend on the mantissa of the magnetic flux only.

## II. AHARONOV-BOHM FIELD COMBINED WITH LONGITUDINAL ELECTROMAGNETIC FIELDS.

Here we consider particle motion in a superposition of AB field and of some longitudinal electromagnetic fields. We call electric  $\mathbf{E}$  and magnetic  $\mathbf{H}$  fields longitudinal ones whenever they are parallel and are directed along the AB solenoid (along the axis  $z$ ).

$$\mathbf{E} = E\mathbf{n}, \quad \mathbf{H} = H\mathbf{n}, \quad \mathbf{n}^2 = 1, \quad \mathbf{n} = (0, 0, 1). \quad (2.1)$$

It follows from Maxwell equations that in this case the functions  $E$  and  $H$  must obey the conditions

$$E = E(x^0, x^3) = \partial_0 A_3 - \partial_3 A_0, \quad A_0 = A_0(x^0, x^3), \quad A_3 = A_3(x^0, x^3);$$

$$H = H(x^1, x^2) = \partial_2 A_1 - \partial_1 A_2, \quad A_1 = A_1(x^1, x^2), \quad A_2 = A_2(x^1, x^2), \quad (2.2)$$

where  $A_0(x^0, x^3)$ ,  $A_1(x^1, x^2)$ ,  $A_2(x^1, x^2)$ ,  $A_3(x^0, x^3)$  are arbitrary functions of the indicated arguments. Exact solutions of the relativistic wave equations in such fields (in the absence of AB field) were studied in [16,17]. As we show below, whenever AB field is present, then exact solutions of the relativistic wave equations can be found only in the

axially symmetric case with the magnetic field having the form  $H = H(r)$ . Thus, potentials of additional fields, which are considered in the present Section, have the form:

$A_0^{(1)} = A_0^{(1)}(x^0, x^3)$ ,  $A_3^{(1)} = A_3^{(1)}(x^0, x^3)$  arbitrary and

$$A_1^{(1)} = \frac{c\hbar A(r)}{|e| r} \sin \varphi, \quad A_2^{(1)} = -\frac{c\hbar A(r)}{|e| r} \cos \varphi, \quad H(r) = \frac{c\hbar A'(r)}{|e| r}, \quad (2.3)$$

where  $A(r)$  is an arbitrary function of  $r$ .

#### A. Classical description of radial motion.

To interpret quantum numbers of wave functions, it is often useful to have classical picture of the problem. That is why we present here a classical analysis of the particle motion in fields under consideration.

Consider classical trajectories that do not intersect the axis  $z$ , thus they do not "feel" the existence of AB field. For such trajectories, the quantity  $P_r^2$  is an integral of motion ( $c^2 P_r^2$  is said to be radial energy),

$$P_r^2 = P_1^2 + P_2^2 = \hbar^2 k_1^2, \quad m_0^2 c^2 + P_r^2 = P_0^2 - P_3^2, \quad (2.4)$$

where  $P_\mu$  is the classical kinetic momentum (a classical analog of the operators  $P_\mu$ ) and  $m_0$  is the rest mass.  $L_z$  is an integral of motion as well ( $L$  is the angular momentum),

$$L_z = \tilde{L}_z - \hbar(l_0 + \mu) = \hbar(l - l_0), \quad \tilde{L}_z = x^1 P^2 - x^2 P^1 - \hbar A(r) = \hbar(l + \mu). \quad (2.5)$$

Here  $l$  is arbitrary ( $l$  will be integer in quantum theory).

As will be seen below, exact solutions of relativistic wave equations can be found whenever the functions  $A(r)$  in (2.3) have the form

$$1. A(r) = 0, \quad (2.6)$$

$$2. A(r) = \frac{\gamma r^2}{2}, \quad \gamma > 0, \quad (2.7)$$

$$3. A(r) = \gamma r, \quad \gamma > 0. \quad (2.8)$$

The first case corresponds to the absence of an additional electromagnetic field, the second one corresponds to the additional constant uniform magnetic field  $H$  along the solenoid ( $\gamma = \frac{|eH|}{c\hbar}$ ), and the third one corresponds to the additional constant magnetic field  $H(r) = b/r$ , ( $\gamma = \frac{|eb|}{c\hbar}$ ). Consider classical motion in these cases.

1. For  $A(r) = 0$ , the momenta  $P_1$  and  $P_2$  are integrals of motion. Then the radial motion (the motion in  $x^1, x^2$  plane) is parametrized by the proper time  $\tau$  and can be presented as

$$x^1 = \frac{P^1}{m_0 c} \tau + x_{(0)}^1, \quad x^2 = \frac{P^2}{m_0 c} \tau + x_{(0)}^2, \quad (2.9)$$

where  $x_{(0)}^1, x_{(0)}^2$  are integration constants. In this case

$$\tilde{L}_z = \hbar(l + \mu) = x_{(0)}^1 P^2 - x_{(0)}^2 P^1. \quad (2.10)$$

Consider the quantity

$$\Delta R = P_r^{-1} (x_{(0)}^1 P^2 - x_{(0)}^2 P^1) = \frac{l + \mu}{k_1}. \quad (2.11)$$

One can show that  $|\Delta R|$  characterizes a minimal distance between the trajectory (2.9) and the axis  $x^3$ . All the classical trajectories are divided in two groups according to the sign of  $l + \mu$ . Trajectories with  $l + \mu > 0$  can be called right ones and those with  $l + \mu < 0$  can be called left ones. The reason is the following: Looking from the positive  $z$ -direction, one can see that a minimal angle rotation from the vector  $\mathbf{r} = (x^1, x^2, x^3)$  to the particle momentum is anticlockwise for the right trajectories and clockwise for left ones.

2. For  $A(r) = \gamma r^2/2$ , the radial motion has the form

$$\begin{aligned} x^1 &= R \cos \kappa + x_{(0)}^1, \quad x^2 = R \sin \kappa + x_{(0)}^2, \\ \kappa &= \omega_0 \tau + \varphi_0, \quad \omega_0 = \frac{\gamma}{m}, \quad m = \frac{m_0 c}{\hbar}. \end{aligned} \quad (2.12)$$

Here  $R, \varphi_0, x_{(0)}^1, x_{(0)}^2$  are integration constants. The trajectories (2.12) are circles of radius  $R$  with centers having coordinates  $x_{(0)}^1, x_{(0)}^2$ ,

$$(x^1 - x_{(0)}^1)^2 + (x^2 - x_{(0)}^2)^2 = R^2. \quad (2.13)$$

One can easily find

$$P_r = \hbar\gamma R, \quad l + \mu = \frac{\gamma}{2} (R^2 - R_0^2), \quad (x_{(0)}^1)^2 + (x_{(0)}^2)^2 = R_0^2,$$

$$l + \mu \leq \frac{\gamma R^2}{2} = \frac{P_r^2}{2\hbar^2\gamma} = \frac{k_1^2}{2\gamma}. \quad (2.14)$$

We can see that classical trajectories with  $l \geq -\mu$  embrace the solenoid, and ones with  $l < -\mu$  do not. In quantum theory these conditions are  $l \geq 0$  and  $l < 0$  respectively. The quantity  $\Delta R$  characterizes a minimal distance between the trajectory (2.12) and the solenoid,

$$\Delta R = |R - R_0| = \frac{2|l + \mu|}{\gamma(R + R_0)}. \quad (2.15)$$

3. Consider finally  $A(r) = \gamma r$ . Here the radial motion depends essentially on values of constants  $a, \varepsilon$ ,

$$a = \frac{P_r}{\hbar\gamma} = \frac{k_1}{\gamma} > 0, \quad \varepsilon = \text{sign}(l + \mu). \quad (2.16)$$

For  $\varepsilon = 1$ , the classical motion is possible only if  $a > 1$ . For  $\varepsilon = -1$ , the classical motion is possible if  $a > 0$ . Whenever  $a \geq 1$  we get unbounded motion for  $r$ . For  $0 < a < 1$ ,  $\varepsilon = -1$ , this motion is bounded. Below we present the radial motion in  $s$  parametrization

$$a > 1: \quad r = \frac{|l + \mu|(a \cosh s + \varepsilon)}{\gamma(a^2 - 1)}, \quad \tau = \frac{|l + \mu|(a \sinh s + \varepsilon s)}{\gamma^2(a^2 - 1)^{3/2}},$$

$$\varphi - \varphi_0 = \frac{s}{\sqrt{a^2 - 1}} + 2\varepsilon \arctan \left( \sqrt{\frac{a - \varepsilon}{a + \varepsilon}} \tanh \frac{s}{2} \right);$$

$$a = 1, \quad \varepsilon = -1: \quad 2\gamma r = |l + \mu|(s^2 + 1), \quad 2\gamma^2 \tau = |l + \mu| m \left( \frac{s^3}{3} + s \right),$$

$$\varphi - \varphi_0 = s - 2 \arctan s;$$

$$a < 1, \quad \varepsilon = -1: \quad r = \frac{|l + \mu|(1 - a \cos s)}{\gamma(1 - a^2)}, \quad \tau = \frac{|l + \mu| m (s - a \sin s)}{\gamma^2(1 - a^2)^{3/2}},$$

$$\varphi - \varphi_0 = s \left( \frac{1}{\sqrt{1 - a^2}} - 1 \right) - 2 \arctan \left( \frac{a \sin s}{1 + \sqrt{1 - a^2} - a \cos s} \right). \quad (2.17)$$

In all the cases under consideration, the minimal distance between a trajectory and the solenoid is defined by the expression



$$\Delta R = \frac{|l + \mu|}{\gamma |a - \varepsilon|}. \quad (2.18)$$

Thus, the quantity  $l$  has a clear classical interpretation.

### B. Klein-Gordon equation in longitudinal fields

Here we consider solutions of Klein-Gordon equation

$$\left( P_\mu P^\mu - m_0^2 c^2 \right) \Psi(x) = 0 \quad (2.19)$$

in the superposition of the external fields (1.1) and (2.3). In this case, the operators (2.4) are integrals of motion. Whenever an additional field is axial-symmetric one (2.3), then the operator (2.5) is an integral of motion as well. Thus, we subject solutions of the equation (2.19) to the following additional conditions

$$\hbar^{-2} \left( P_1^2 + P_2^2 \right) \Psi(x) = k_1^2 \Psi(x), \quad \hbar^{-2} \left( P_0^2 - P_3^2 \right) \Psi(x) = (m^2 + k_1^2) \Psi(x). \quad (2.20)$$

Then such solutions can be presented in the form  $\Psi(x) = \psi(x^1, x^2) \Phi(x^0, x^3)$ , where the functions  $\psi$  and  $\Phi$  obey the equations

$$\hbar^{-2} \left( P_1^2 + P_2^2 \right) \psi(x^1, x^2) = k_1^2 \psi(x^1, x^2), \quad (2.21)$$

$$\hbar^{-2} \left( P_0^2 - P_3^2 \right) \Phi(x^0, x^3) = (m^2 + k_1^2) \Phi(x^0, x^3). \quad (2.22)$$

AB field does not enter in the equation (2.22). This equation can be solved for a large class of electromagnetic fields. All the corresponding solutions of the equation (2.22) are described in detail in [16,17], that is why we do not present them here.

Let us turn to the equation (2.21). As was already mentioned above, exact solutions of this equation can be found only in the superposition of AB field and the fields (2.6)-(2.8). In all these cases, the operator  $L_z$  is an integral of motion, thus we can search for solutions which are eigenvectors for the latter operator. In cylindrical coordinates, we get

$$L_z \psi(r, \varphi) = -i \hbar \partial_\varphi \psi(r, \varphi) = \hbar (l - l_0) \psi(r, \varphi), \quad \psi(r, \varphi) = \frac{\exp[i(l - l_0)\varphi]}{\sqrt{2\pi}} \psi(r). \quad (2.23)$$

where  $K_\nu(r)$  are Macdonald functions ([18], 8.407). These functions have a finite norm ( $\delta = \arg q$ )

$$\int_0^\infty K_\nu^*(qr) K_\nu(qr) r dr = \frac{\pi \sin 2\nu\delta}{2|q|^2 \sin \nu\pi \sin 2\delta}, \quad |\nu| < 1, \quad -\pi/2 < \delta < \pi/2 \quad (\text{Re } q > 0). \quad (2.30)$$

But they are not orthogonal with respect to  $q$ ,

$$\int_0^\infty K_\nu(q'r) K_\nu(qr) r dr = \frac{\pi (q^{2\nu} - q'^{2\nu})}{2(q^2 - q'^2) \sin \nu\pi}, \quad |\nu| < 1. \quad (2.31)$$

In particular, for  $\nu = 0$ , we get

$$\int_0^\infty K_0^*(qr) K_0(qr) r dr = \frac{\delta}{|q|^2 \sin 2\delta}, \quad \int_0^\infty K_0(q'r) K_0(qr) r dr = \frac{\ln q - \ln q'}{q^2 - q'^2}. \quad (2.32)$$

The above peculiarities are related to the loss of hermicity of the operator  $\hat{R}$  for  $l = 0, -1$ ,  $\mu \neq 0$ . Similar problem was discussed in [19].

#### D. Uniform magnetic field

Here we consider a superposition of AB field (1.1) and a uniform magnetic field (2.7). It is useful to introduce dimensionless operators  $a_k, a_k^+$ ,  $k = 1, 2$  by the relations,

$$\begin{aligned} \hbar\sqrt{2\gamma}a_1 &= -iP_1 - P_2, \quad \hbar\sqrt{2\gamma}a_1^+ = iP_1 - P_2, \quad \gamma = \frac{|eH|}{c\hbar}, \\ \hbar\sqrt{2\gamma}a_2 &= -iP_1 + P_2 + \hbar\gamma(x^1 + ix^2), \quad \hbar\sqrt{2\gamma}a_2^+ = iP_1 + P_2 + \hbar\gamma(x^1 - ix^2). \end{aligned} \quad (2.33)$$

Considering coordinates and momenta in these relations as classical quantities, we can get a representation for the classical motion (2.12) in terms of  $a_1$  and  $a_2$

$$a_1 = \sqrt{\frac{\gamma}{2}} R e^{-i\kappa}, \quad a_2 = \sqrt{\frac{\gamma}{2}} (x_0^1 + ix_0^2), \quad (x_0^1 + ix_0^2) = R_0 e^{i\delta}. \quad (2.34)$$

The following operator relations take place

$$P_r^2 = P_1^2 + P_2^2 = \hbar^2\gamma(a_1^+ a_1 + a_1 a_1^+), \quad 2L_z = \hbar(a_1^+ a_1 + a_1 a_1^+ - a_2^+ a_2 - a_2 a_2^+). \quad (2.35)$$

We introduce also a dimensionless coordinate  $\rho$  instead of  $r$ ,

$$\rho = \frac{\gamma r^2}{2}, \quad dx^1 dx^2 = \frac{1}{\gamma} d\rho d\varphi. \quad (2.36)$$

On the classical trajectories (2.12)  $\rho$  evolves as

$$2\rho = \gamma \left[ R^2 + R_0^2 + 2RR_0 \cos(\kappa - \delta) \right]. \quad (2.37)$$

Being written in terms of the variables  $\rho, \varphi$ , the operators  $a_k, a_k^+$  take the form

$$\begin{aligned} a_1 &= \sqrt{\rho} e^{-i\varphi} [(l_0 + \mu + \rho - i\partial_\varphi)/2\rho + \partial_\rho], \\ a_1^+ &= \sqrt{\rho} e^{i\varphi} [(l_0 + \mu + \rho - i\partial_\varphi)/2\rho - \partial_\rho], \\ a_2 &= -\sqrt{\rho} e^{i\varphi} [(l_0 + \mu - \rho - i\partial_\varphi)/2\rho - \partial_\rho], \\ a_2^+ &= -\sqrt{\rho} e^{-i\varphi} [(l_0 + \mu - \rho - i\partial_\varphi)/2\rho + \partial_\rho]. \end{aligned} \quad (2.38)$$

Using the commutation relations for the momentum operators

$$P_\mu P_\nu - P_\nu P_\mu = -i \frac{e\hbar}{c} F_{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (2.39)$$

and definition of the magnetic field (1.2), we arrive to the following commutation relations for the operators  $a_k, a_k^+$

$$\begin{aligned} [a_1, a_1^+] &= 1 + f, \quad [a_2, a_2^+] = 1 - f, \quad [a_1, a_2] = -f, \quad [a_1, a_2^+] = 0, \\ f &= (\Phi/H)\delta(x^1)\delta(x^2) = 2\frac{\Phi}{\Phi_0}\delta(\rho) = 2(l_0 + \mu)\delta(\rho). \end{aligned} \quad (2.40)$$

These commutation relations contain a singular dimensionless function  $f$ . Whenever AB field is absent ( $f = 0$ ), then the operators  $a_k, a_k^+$  form two mutual commuting sets of creation and annihilation operators. It is not true in the presence of AB field. However, as it will be seen further, these operators behave as creation and annihilation ones when acting on functions that tend to zero (sufficiently rapidly) as  $\rho \rightarrow 0$ . Being written in the coordinates  $\rho, \varphi$ , the operators (2.35) have the form

$$P_r^2 = 2\gamma\hbar^2 Q, \quad L_z = -i\hbar\partial_\varphi, \quad Q = \frac{(l_0 + \mu + \rho - i\partial_\varphi)^2}{4\rho} - \partial_\rho - \rho\partial_\rho^2. \quad (2.41)$$

In the case under consideration, the radial equation (2.24) reads

$$\bar{Q}\psi(\rho) = \left(\bar{n} + \frac{1}{2}\right)\psi(\rho), \quad \bar{Q} = \frac{(l + \mu + \rho)^2}{4\rho} - \partial_\rho - \rho\partial_\rho^2, \quad k_1^2 = 2\gamma\left(\bar{n} + \frac{1}{2}\right). \quad (2.42)$$

Bounded and quadratically integrable solutions of this equation are expressed via the Laguerre functions (6.1) (see Appendix). There are two types of solutions the latter equation, we denote them as  $\psi^{(j)}(\rho)$ ,  $j = 1, 2$  (two types of states). The first one  $j = 1$  corresponds to  $l \geq 0$  (classical trajectories embrace the solenoid)

$$\psi^{(1)}(\rho) = I_{n+\mu, n-l}(\rho), \quad 0 \leq l \leq n, \quad n = 0, 1, 2, \dots, \quad \bar{n} = n + \mu. \quad (2.43)$$

The second type of solutions with  $j = 2$  corresponds to  $l < 0$  (classical trajectories do not embrace the solenoid)

$$\psi^{(2)}(\rho) = I_{n-l-\mu, n}(\rho), \quad l < 0, \quad \bar{n} = n. \quad (2.44)$$

In these two cases radial momentum spectra are different,

$$\begin{aligned} (k_1^{(1)})^2 &= 2\gamma\left(n + \mu + \frac{1}{2}\right), \quad 0 \leq l \leq n, \\ (k_1^{(2)})^2 &= 2\gamma\left(n + \frac{1}{2}\right), \quad l < 0, \quad n = 0, 1, 2, \dots \end{aligned} \quad (2.45)$$

The spectrum for  $j = 2$  (which is a part of the total spectrum) corresponds exactly to the spectrum of a spinless particle in a uniform magnetic field (without AB field). The spectrum for  $j = 1$  is shifted by  $2\gamma\mu$  with respect to the one for  $j = 2$ . It is important to note that the presence of AB field lifts partially the degeneracy of the total spectrum in the quantum number  $l$ .

It is convenient to define effective quantum numbers  $\bar{l}$  and  $\bar{n}$  by the relations

$$\bar{n} = n + \mu(2 - j) = \begin{cases} n + \mu, & j = 1, \quad \bar{l} = l + \mu, \quad \bar{l} \leq \bar{n} \\ n, & j = 2, \quad n = 0, 1, 2, \dots \end{cases} \quad (2.46)$$

Using these numbers, we introduce the functions

$$\begin{aligned} \psi_{n,l}^{(1)}(\rho, \varphi) &= (-1)^{n-l} \frac{\exp[i(l-l_0)\varphi]}{\sqrt{2\pi}} I_{\bar{n}, \bar{n}-\bar{l}}(\rho), \\ \psi_{n,l}^{(2)}(\rho, \varphi) &= (-1)^n \frac{\exp[i(l-l_0)\varphi]}{\sqrt{2\pi}} I_{\bar{n}-\bar{l}, \bar{n}}(\rho). \end{aligned} \quad (2.47)$$

According to (6.4), these functions can be expressed via the Laguerre polynomials. Thus, the orthonormality relation can be proved

$$\int_0^\infty d\rho \int_0^{2\pi} d\varphi \psi_{n',l'}^{(j)*}(\rho, \varphi) \psi_{n,l}^{(j)}(\rho, \varphi) = \delta_{l,l'} \delta_{n,n'}. \quad (2.48)$$

The set of the Laguerre functions

$$I_{\alpha+n,n}(x), \quad n = 0, 1, 2, \dots, \quad \alpha > -1 \quad (2.49)$$

is complete in the space of quadratically integrable functions of  $x \geq 0$ ,

$$\sum_{n=0}^{\infty} I_{\alpha+n,n}(x) I_{\alpha+n,n}(y) = \delta(x-y). \quad (2.50)$$

Then the set  $\psi_{n,l}^{(j)}(\rho, \varphi)$  is complete in the space of quadratically integrable functions of  $\rho, \varphi$ , ( $\rho > 0, 0 \leq \varphi \leq 2\pi$ ).

Using the relations (6.6)-(6.11), one can get the action of the operators (2.38) on the functions  $\psi_{n,l}^{(j)}(\rho, \varphi)$ ,

$$\begin{aligned} a_1 \psi_{n,l}^{(j)}(\rho, \varphi) &= \sqrt{\bar{n}} \psi_{n-1,l-1}^{(j)}(\rho, \varphi), \quad a_1^+ \psi_{n,l}^{(j)}(\rho, \varphi) = \sqrt{\bar{n}+1} \psi_{n+1,l+1}^{(j)}(\rho, \varphi), \\ a_2 \psi_{n,l}^{(j)}(\rho, \varphi) &= \sqrt{\bar{n}-\bar{l}} \psi_{n,l+1}^{(j)}(\rho, \varphi), \quad a_2^+ \psi_{n,l}^{(j)}(\rho, \varphi) = \sqrt{\bar{n}-\bar{l}+1} \psi_{n,l-1}^{(j)}(\rho, \varphi). \end{aligned} \quad (2.51)$$

These formulas show that the functions  $\psi_{n,l}^{(1)}$  may be created by an action of the operators  $a_k^+$  on  $\psi_{0,0}^{(1)}$ , and the functions  $\psi_{n,l}^{(2)}$  may be created by an action of the operators  $a_k^+$  on  $\psi_{0,-1}^{(2)}$ .

Namely,

$$\psi_{n,l}^{(1)} = \sqrt{\frac{\Gamma(1+\mu)}{\Gamma(1+\bar{n})\Gamma(1+\bar{n}-\bar{l})}} (a_2^+)^{n-l} (a_1^+)^n \psi_{0,0}^{(1)}, \quad (2.52)$$

$$\psi_{n,l}^{(2)} = \sqrt{\frac{\Gamma(2-\mu)}{\Gamma(1+\bar{n})\Gamma(1+\bar{n}-\bar{l})}} (a_1^+)^n (a_2^+)^{n-l-1} \psi_{0,-1}^{(2)}. \quad (2.53)$$

It is natural to interpret  $\psi_{0,0}^{(1)}$  as a vacuum state for the states  $\psi_{n,l}^{(1)}$ , and to interpret  $\psi_{0,-1}^{(2)}$  as a vacuum state for the states  $\psi_{n,l}^{(2)}$ . Thus, for  $\mu \neq 0$ , we have two vacuum states in the problem. For  $\mu = 0$ , the situations changes. By virtue of (6.18)

$$I_{n,n-l} = (-1)^l I_{n-l,n} \rightarrow \psi_{n,l}^{(1)} = (-1)^l \psi_{n,l}^{(2)}, \quad \mu = 0, \quad (2.54)$$

and for any  $l < n$ , the function  $\psi_{0,0}^{(1)}$  is connected to  $\psi_{0,-1}^{(2)}$  as

$$a_2^+ \psi_{0,0}^{(1)} = \psi_{0,-1}^{(2)}, \quad a_2 \psi_{0,-1}^{(2)} = \psi_{0,0}^{(1)}. \quad (2.55)$$

Thus, we have the only one vacuum in the problem, one energy spectrum (2.45), and all the wave functions are created from the vacuum  $\psi_{0,0}^{(1)}$ .

One ought to stress that all the states obey the property  $\psi_{n,i}^{(j)}(\rho = 0, \varphi) = 0$ , which means that the scalar particle has zero probability to be found in the solenoid area. In fact, the existence of this property allows us to speak about AB effect.

The definitions (2.51) can formally be considered for any values of indices  $n, l$ . In particular, we can consider the following relations

$$\begin{aligned} a_1^+ \psi_{n,-1}^{(2)} &= \sqrt{n+1} \psi_{n+1,0}^{(2)} \\ &= (-1)^{n+1} (1+n) \sqrt{\frac{\Gamma(1+n)}{2\pi\Gamma(2-\mu+n)}} \exp[-il_0\varphi - \frac{\rho}{2}] \rho^{-\frac{\mu}{2}} L_{n+1}^{-\mu}(\rho), \\ a_1 \psi_{n,0}^{(1)} &= \sqrt{n+\mu} \psi_{n-1,-1}^{(1)} \\ &= (-1)^n (n+\mu) \sqrt{\frac{\Gamma(1+n)}{2\pi\Gamma(1+\mu+n)}} \exp[-i(1+l_0)\varphi - \frac{\rho}{2}] \rho^{-\frac{1-\mu}{2}} L_n^{\mu-1}(\rho), \\ a_2^+ \psi_{n,0}^{(1)} &= \sqrt{n+1} \psi_{n,-1}^{(1)} \\ &= (-1)^{n+1} (1+n) \sqrt{\frac{\Gamma(1+n)}{2\pi\Gamma(1+\mu+n)}} \exp[-i(1+l_0)\varphi - \frac{\rho}{2}] \rho^{-\frac{1-\mu}{2}} L_{n+1}^{\mu-1}(\rho), \\ a_2 \psi_{n,-1}^{(2)} &= \sqrt{1-\mu+n} \psi_{n,0}^{(2)} \\ &= (-1)^n (1-\mu+n) \sqrt{\frac{\Gamma(1+n)}{2\pi\Gamma(2-\mu+n)}} \exp[-il_0\varphi - \frac{\rho}{2}] \rho^{-\frac{\mu}{2}} L_n^{-\mu}(\rho). \end{aligned} \quad (2.56)$$

However, the functions  $\psi_{n,-1}^{(1)}$ ,  $\psi_{n,0}^{(2)}$  do not present any physical solutions of the problem, they are not in the set (2.47). Thus, in the general case  $\mu \neq 0$ , the action of the operators  $a_k^+, a_k$  on wave functions may lead them out of a class of physical solutions. The functions (2.56) are singular at  $r = 0$  (for  $\mu \neq 0$ ), however they still remain quadratically integrable.

Thus, we see that  $l = 0, -1$  is a special case. Here there appear unbounded (but quadratically integrable) solution  $\psi_{n,-1}^{(1)}$ ,  $\psi_{n,0}^{(2)}$ . Whenever  $\mu \rightarrow 0$ , these states either coincide with the corresponding states in the pure magnetic field or disappear. The states

$$u^0 = x^0 - x^3, \quad u^3 = x^0 + x^3. \quad (2.63)$$

Then the above mentioned longitudinal running electric fields have the following potentials and strengths

$$A_0^{(1)} = A_3^{(1)} = \frac{1}{2}B(u^0), \quad E = B'(u^0). \quad (2.64)$$

where  $B(u^0)$  is an arbitrary function of  $u^0$ . Consider operators  $\tilde{P}_0, \tilde{P}_3, \tilde{p}_3$ ,

$$\begin{aligned} 2\tilde{P}_0 &= P_0 - P_3, \quad 2\tilde{P}_3 = P_0 + P_3, \quad \tilde{P}_0 = i\hbar \frac{\partial}{\partial u^0}, \\ \tilde{p}_3 &= i\hbar \frac{\partial}{\partial u^3}, \quad \tilde{P}_3 = \tilde{p}_3 + \frac{\hbar g(u^0)}{2}, \quad g = \frac{|e|B(u^0)}{c\hbar}. \end{aligned} \quad (2.65)$$

The operator  $\tilde{p}_3$  commutes with the one  $L_z$  and both are integrals of motion. Thus, we can demand for solutions of Eq. (2.19) to be eigenvectors for these operators,

$$\tilde{p}_3 \Psi(x) = \frac{\hbar \lambda}{2} \Psi(x), \quad L_z \Psi(x) = \hbar(l - l_0) \Psi(x). \quad (2.66)$$

Such solutions have the form

$$\Psi(x) = [\lambda + g(u^0)]^{-1/2} \Phi(r, t) \exp i \left[ (l - l_0) \varphi - m^2 \frac{1}{2} \int \frac{du^0}{\lambda + g(u^0)} - \frac{\lambda u^3}{2} \right], \quad (2.67)$$

where the function  $\Phi(r, t)$  obeys the equation

$$\hat{R}_1 \Phi(r, t) = 0, \quad \hat{R}_1 = i\partial_t + \partial_r^2 + \frac{\partial_r}{r} - \frac{[\bar{l} + A(r)]^2}{r^2}. \quad (2.68)$$

We recall that  $A(r)$  was defined in (2.3).

Consider first the case  $A(r) = 0$ . Here we find a propagation function for the equation (2.68) in the form ( $J_\nu(x)$  are the Bessel functions)

$$\begin{aligned} G_0(r, r', t) &= \frac{1}{2t} J_{|\bar{l}|} \left( \frac{rr'}{2t} \right) e^{iQ_0}, \quad Q_0 = \frac{r^2 + r'^2}{4t} - \frac{(|\bar{l}| + 1)\pi}{2}, \\ \hat{R}_1 \Big|_{A=0} G_0(r, r', t) &= 0, \quad \lim_{t \rightarrow 0} G_0(r, r', t) = \frac{1}{r} \delta(r - r'). \end{aligned} \quad (2.69)$$

The case  $A(r) = \rho = \gamma r^2/2$  can be considered in the same manner. Here the propagation function has the form

$$G(\rho, \rho', t) = \frac{1}{2 \sin \tau} J_{|\bar{l}|} \left( \frac{\sqrt{\rho \rho'}}{\sin \tau} \right) e^{iQ}, \quad Q = \frac{\rho + \rho'}{\sin \tau} - \frac{(|\bar{l}| + 1) \pi}{2} - \bar{l} \tau,$$

$$\hat{R}_1 G(\rho, \rho', t) = 0, \quad \lim_{t \rightarrow 0} G(\rho, \rho', t) = \delta(\rho - \rho'), \quad \tau = \gamma t (u^0). \quad (2.70)$$

The functions  $G_0(r, r', t)$  and  $G(\rho, \rho', t)$  solve the Cauchy problem. For example,

$$\Phi(\rho, t) = \int_0^\infty G(\rho, \rho', t) \Phi(\rho') d\rho', \quad (2.71)$$

where  $\Phi(\rho)$  is an arbitrary functions (an initial data for  $\Phi(\rho, t)$ ).

For the field (2.8), the corresponding propagation function is quite complicated [20].

### G. Exact solutions of Dirac equation

Here we are going to study Dirac equation

$$(\gamma^\mu P_\mu - m_0 c) \Psi(x) = 0 \quad (2.72)$$

in the superposition of AB field and the field (2.1). We use a standard representation (see for example [17]) for  $\gamma$ -matrices. In the case under consideration, we look for solutions with a definite radial momentum. The corresponding bispinors  $\Psi(x)$  can be written in a block form

$$\Psi(x) = Q \begin{pmatrix} \psi_1(x^1, x^2) [m + F - ik_1 \sigma_2] \\ \psi_2(x^1, x^2) [(m - F) \sigma_3 - ik_1 \sigma_1] \end{pmatrix} v \tilde{\Phi}(x^0, x^3), \quad F = \hbar^{-1} (P_0 + P_3), \quad (2.73)$$

where  $v$  is an arbitrary spinor;  $\sigma_k$  ( $k = 1, 2, 3$ ) are Pauli matrices; the function  $\tilde{\Phi}(x^0, x^3)$  obeys the equation

$$\left[ \hbar^{-2} (P_0^2 - P_3^2) + i \frac{|e| E}{c \hbar} \right] \tilde{\Phi}(x^0, x^3) = 0, \quad (2.74)$$

where  $E = E(x^0, x^3)$  is electric field strength (2.2), and functions  $\psi_1, \psi_2$  obey the following equations

$$(P_1 + iP_2) \psi_1(x^1, x^2) = \hbar k_1 \psi_2(x^1, x^2), \quad (P_1 - iP_2) \psi_2(x^1, x^2) = \hbar k_1 \psi_1(x^1, x^2). \quad (2.75)$$



That means that in such states the electron spin has the only one orientation, namely, opposite to the magnetic field.

Consider finally the case of nonuniform magnetic field (2.8). For  $l \neq 0$ ,  $a \neq 1$ , the corresponding bounded solutions (they also vanish at  $r = 0$ ) have the form

$$\begin{aligned}\psi_1(r) &= I_{n-1, n+1-2\bar{l}}(x), \quad \psi_2(r) = -I_{n, n-2\bar{l}}(x), \quad l > 0; \\ \psi_1(r) &= I_{n+1-2\bar{l}, n-1}(x), \quad \psi_2(r) = -I_{n-2\bar{l}, n}(x), \quad l < 0; \\ x &= 2\sqrt{1-a^2}\gamma r, \quad 1-2\bar{l}+2n = \frac{1-2\bar{l}}{\sqrt{1-a^2}}, \quad \bar{l} = l + \mu,\end{aligned}\tag{2.85}$$

where  $I_{n,m}(x)$  are the Laguerre functions (6.1), and we use the notation (2.16). Whenever  $a^2 > 1$ , any  $l \neq 0$  are admissible in complete agreement with classical theory. Whenever  $a^2 < 1$ , the only  $l < 0$  are admissible. In such a case  $n$  is integer and the functions (2.85) are expressed via the Laguerre polynomials according to (6.19). At the same time, the following quantization takes place

$$a^2 = 1 - \frac{(1 + 2|\bar{l}|)^2}{(1 + 2|\bar{l}| + 2n)^2}, \quad n = 0, 1, 2, \dots\tag{2.86}$$

For  $a = 1$ ,  $l \neq 0$ , the only bounded states can be found for  $l < 0$ . They are expressed via the Bessel functions,

$$\psi_1(r) = J_{2|\bar{l}|+2}\left(2\sqrt{(1+2|\bar{l}|)}\gamma r\right), \quad \psi_2(r) = -J_{2|\bar{l}|}\left(2\sqrt{(1+2|\bar{l}|)}\gamma r\right).\tag{2.87}$$

Solutions (2.87) follow from (2.85) as  $a \rightarrow 1$ . That fact can be confirmed by the use of the limit (6.22).

$l = 0$  is a special case. Here there are only unbounded solutions. Some of them are quadratically integrable. Whenever  $a^2 > 1$ , such solutions have the form

$$\begin{aligned}\psi_1(r) &= c_1 I_{n+1-2\mu, n-1}(x) + c_2 I_{n-1, n+1-2\mu}(x), \\ \psi_2(r) &= -c_1 I_{n-2\mu, n}(x) - c_2 I_{n, n-2\mu}(x),\end{aligned}\tag{2.88}$$

where  $c_k$  are arbitrary constants, and for  $a = 1$  these solutions read

$$\begin{aligned}\psi_1(r) &= J_{2-2\mu}(z), \quad \psi_2(r) = -J_{-2\mu}(z), \quad 0 < \mu < \frac{1}{2}; \\ \psi_1(r) &= K_{2-2\mu}(z), \quad \psi_2(r) = K_{2\mu}(z), \quad \frac{1}{2} < \mu < 1; \quad z = 2\sqrt{|1-2\mu|\gamma r}.\end{aligned}\quad (2.89)$$

Quadratically integrable solutions exist for  $a^2 < 1$  as well. For example, for  $0 < \mu < \frac{1}{2}$  they have the form (2.88), where  $c_2 = 0$ . In such a case  $a^2$  is quantized

$$a^2 = 1 - \frac{(1-2\mu)^2}{(1-2\mu+2n)^2}, \quad n = 0, 1, 2, \dots \quad (2.90)$$

Moreover, for any complex  $a^2$  (provided  $\text{Re} \sqrt{1-a^2} > 0$ ) there exist unbounded solutions with a finite norm. They read

$$\psi_1(r) = a\psi_{\lambda, 2(1-\mu)}(x), \quad \psi_2(r) = (1 + \sqrt{1-a^2})\psi_{\lambda, 2\mu}(x), \quad \lambda = \frac{1-2\mu}{2\sqrt{1-a^2}}. \quad (2.91)$$

All the above solutions obey (2.84) for  $n = 0$ .

Finally we present solutions, which do not have an analog in the Klein-Gordon case discussed in Sect. II.F. These solutions are not eigenvectors of the radial momentum operator. With this aim in view we present Dirac wave functions in the following form

$$\Psi(x) = \Psi_{(-)}(x) + \Psi_{(+)}(x), \quad \Psi_{(\pm)}(x) = P_{(\pm)}\Psi(x), \quad 2P_{(\pm)} = 1 \pm (\alpha \mathbf{n}), \quad (2.92)$$

where  $\mathbf{n}$  is a unit vector,  $\alpha = (\alpha_k = \gamma^0 \gamma^k)$ ,  $k = 1, 2, 3$ , and  $P_{(\pm)}$  are projection operators,  $P_{(+)} + P_{(-)} = 1$ ,  $P_{(\pm)}^2 = P_{(\pm)}$ ,  $P_{(+)}P_{(-)} = P_{(-)}P_{(+)} = 0$ . Then we can always present  $\Psi_{(\pm)}(x)$  in the following block form

$$\Psi_{(+)}(x) = \begin{pmatrix} v(x) \\ (\sigma \mathbf{n}) v(x) \end{pmatrix}, \quad \Psi_{(-)}(x) = \begin{pmatrix} u(x) \\ -(\sigma \mathbf{n}) u(x) \end{pmatrix}, \quad (2.93)$$

with  $u(x)$ ,  $v(x)$  being arbitrary spinors. Without loss of generality we can always choose  $\mathbf{n} = (0, 0, 1)$ . The Dirac equation (2.72) demands  $u(x)$ ,  $v(x)$  to obey the equations

$$\begin{aligned}2\tilde{P}_0 u &= [m_0 c - (\sigma \mathbf{B}) \sigma_3] v, \quad 2\tilde{P}_3 v = [m_0 c + (\sigma \mathbf{B}) \sigma_3] u, \\ 2\tilde{P}_0 &= P_0 - P_3, \quad 2\tilde{P}_3 = P_0 + P_3, \quad \mathbf{B} = (P_1, P_2, 0).\end{aligned}\quad (2.94)$$

Suppose we consider external fields, for which the operator  $\tilde{P}_3$  (2.65) is an integral of motion, and suppose we are looking for solutions that are eigenvectors of the latter operator. Then in accordance with (2.66)

$$2\tilde{P}_3 = \hbar(\lambda + g), \quad c\hbar g = |e| \left( A_0^{(1)} + A_3^{(1)} \right). \quad (2.95)$$

It follows from (2.94) that the spinor  $v$  can be restored by the one  $u$ ,

$$\hbar(\lambda + g)v = [m_0c + (\sigma\mathbf{B})\sigma_3]u. \quad (2.96)$$

For external fields under consideration, the operator  $\mathbf{B}$  commutes with  $\lambda + g$ , thus we get a closed equation for  $u$ ,

$$2\hbar(\lambda + g)\tilde{P}_0u = [m_0c - (\sigma\mathbf{B})\sigma_3][m_0c + (\sigma\mathbf{B})\sigma_3]u. \quad (2.97)$$

Considering eigenvectors for the operator  $J_z$  (2.76) in axial-symmetric external fields (2.3), we can write

$$u(x) = \begin{pmatrix} e^{-i\varphi}u_1(r, t) \\ u_{-1}(r, t) \end{pmatrix} \exp i \left[ (l - l_0)\varphi - \frac{m^2}{2} \int \frac{du^0}{\lambda + g(u^0)} - \frac{\lambda u^3}{2} \right], \quad (2.98)$$

where the functions  $u_\zeta(r, t)$ ,  $\zeta = \pm 1$  obey the equations

$$\hat{R}_1^\zeta u_\zeta(r, t) = 0, \quad \hat{R}_1^\zeta = i\partial_t + \partial_r^2 + \frac{\partial_r}{r} - \frac{\left[ \bar{l} - \frac{1+\zeta}{2} + A(r) \right]^2}{r^2} - \zeta \frac{A'(r)}{r}, \quad (2.99)$$

which can be solved similar to the one (2.68).

### III. SUPERPOSITION OF AHARONOV-BOHM, LONGITUDINAL, AND CROSSED FIELDS

We consider here Klein-Gordon and Dirac equations in some superpositions of AB field, longitudinal, and crossed fields. In fact, there are only two types of such fields, which admit exact solutions of these equations.

To define the first type of the fields, we introduce curvilinear coordinates  $u^\mu$  by the relations

$$u^0 = x^0 - x^3, \quad u^1 = q(u^0) r^2, \quad u^2 = \varphi, \quad u^3 = x^0 + x^3 - u^0 u^1, \quad q(u^0) = \left[ (u^0)^2 + a \right]^{-1}, \quad (3.1)$$

where  $a$  is a constant. In these coordinates, covariant components  $A_\mu^{(1)}$  of electromagnetic potentials are given as

$$\begin{aligned} \frac{|e|A_0^{(1)}}{c\hbar} &= q(u^0) \left[ f_1(u^1) + a u^1 g_1(u^0) \right], \quad A_1^{(1)} = 0, \\ \frac{|e|A_2^{(1)}}{c\hbar} &= f_2(u^1) + u^1 g_2(u^0), \quad \frac{|e|A_3^{(1)}}{c\hbar} = \frac{g_1(u^0)}{2}. \end{aligned} \quad (3.2)$$

Here  $g_s(u^0)$ ,  $f_s(u^1)$  ( $s = 1, 2$ ) are arbitrary functions of indicated arguments. The corresponding additional (to AB field) electromagnetic field is given by its components in cylindrical reference frame

$$\begin{aligned} \frac{|e|E_r}{c\hbar} &= \frac{|e|H_\varphi}{c\hbar} = qr \left[ 2q(f_1' + a g_1) + u^0 g_1' \right], \quad \frac{|e|E_z}{c\hbar} = -g_1', \\ \frac{|e|E_\varphi}{c\hbar} &= -\frac{|e|H_r}{c\hbar} = -qr[g_2' - 2qu^0(g_2 + f_2')], \quad \frac{|e|H_z}{c\hbar} = 2q(g_2 + f_2'). \end{aligned} \quad (3.3)$$

Exact solutions in the field (3.3) were studied in [21,17].

In the case under considerations, the operators  $L_z$  (or  $J_z$  in the Dirac equation case) and  $\tilde{P}_3$  (2.65) are integrals of motion. We are going to study solutions that are eigenvectors for such operators. Let us impose the following constraint on the functions  $g_s(u^0)$ ,  $f_s(u^1)$  ( $s = 1, 2$ )

$$u^1 (g_2^2 - a g_1^2 - b) + 2g_2 f_2 - 2g_1 f_1 = 0, \quad b = \text{const}. \quad (3.4)$$

Then we can separate the variables  $u^0$  and  $u^1$  and present Klein-Gordon wave functions in the form

$$\begin{aligned} \Psi &= \sqrt{\frac{q}{P}} e^{-i\Gamma} \psi(u^1), \quad P = \lambda + g_1(u^1), \\ \Gamma &= \frac{\lambda}{2} u^3 - (l - l_0) \varphi + \int \left[ m^2 + 2q(2k_1 + \bar{l}g_2) \right] \frac{du^0}{2P}. \end{aligned} \quad (3.5)$$

The functions  $\psi(u^1)$  obey the equation

$$\psi'' + \frac{1}{u^1} \psi' + R(u^1) \psi = 0, \quad R(u^1) = \frac{2k_1 + \lambda f_1}{2u^1} - \frac{a\lambda^2 + b}{4} - \frac{(\bar{l} + f_2)^2}{4(u^1)^2}. \quad (3.6)$$

In the same case, Dirac wave functions have the form

$$\Psi = \frac{\sqrt{q}}{P} e^{-i\Gamma} KW \left[ (1 + \sigma_3) \psi_1(u^1) + (1 - \sigma_3) \psi_{-1}(u^1) \right] v, \quad (3.7)$$

where  $v$  is an arbitrary constant spinor, and

$$K = \begin{pmatrix} m + P - \sigma_3(\sigma \mathbf{F}) \\ (m - P) \sigma_3 - (\sigma \mathbf{F}) \end{pmatrix}, \quad \mathbf{F} = \mathbf{e}_r q r (2i\partial_{u^1} - u^0 P) \\ + \mathbf{e}_\varphi \left( \frac{\bar{l} + f_2}{r} + q r g_2 \right), \quad W = \cos \delta + i\sigma_3 \sin \delta, \quad \delta = \int \frac{q g_2}{P} du^0. \quad (3.8)$$

The scalar functions  $\psi_\zeta(u^1)$  ( $\zeta = \pm 1$ ) obey a set of independent equations

$$\psi_\zeta'' + \frac{1}{u^1} \psi_\zeta' + \left[ R(u^1) + \zeta \frac{f_2'}{2u^1} \right] \psi_\zeta = 0. \quad (3.9)$$

Explicit solutions of the equations (3.6) and (3.9) can be written for

$$f_1(u^1) = \alpha u^1 + \frac{\beta}{u^1}, \quad f_2(u^1) = \gamma u^1, \quad \alpha, \beta, \gamma = \text{const}. \quad (3.10)$$

In such a case the equations (3.6) and (3.9) are reduced to the one (2.58). Solutions of the latter equation we have studied above.

Let us return to the constraint (3.4). If  $\beta \neq 0$ , then  $g_1 = 0$ ,  $g_2 = \text{const}$ , and  $b$  can be found from (3.4) to be  $b = g_2^2 + 2\gamma g_2$ . If  $\beta = 0$ , then  $g_1, g_2$  are related by an equation

$$(g_2 + \alpha)^2 = a \left( g_1 + \frac{\gamma}{a} \right)^2 + \alpha^2 - \frac{\gamma^2}{a} + b. \quad (3.11)$$

Thus, one of the constants remains arbitrary. We see that there exist a wide class of fields, which admit exact solutions.

To define the second type of fields, which admit exact solutions, we introduce curvilinear coordinates  $u^\mu$  by the relations

$$u^0 = x^0 - x^3, \quad u^1 = \frac{r^2}{u^0}, \quad u^2 = \varphi, \quad u^3 = x^0 + x^3 - \frac{r^2}{2u^0}. \quad (3.12)$$

Covariant components of electromagnetic potentials in the coordinates (3.12) are given by the equations

$$\frac{|e| A_0^{(1)}}{c\hbar} = \frac{f_1(u^1)}{u^0}, \quad A_1^{(1)} = 0, \quad \frac{|e| A_2^{(1)}}{c\hbar} = f_2(u^1), \quad A_3^{(1)} = 0, \quad (3.13)$$

where  $f_s(u^1)$  ( $s = 1, 2$ ) are arbitrary functions of  $u^1$ . The corresponding electromagnetic field is given by its components in the cylindrical reference frame

$$E_r = H_\varphi = \frac{2c\hbar r}{|e|(u^0)^2} f_1', \quad E_\varphi = -H_r = -\frac{2c\hbar r}{|e|(u^0)^2} f_2', \quad H_z = \frac{2c\hbar}{|e|u^0} f_2', \quad E_z = 0. \quad (3.14)$$

In the absence of AB field, exact solutions in such a field were studied in [22,17].

Here integrals of motion are the same as in the previous case. Klein-Gordon wave functions can be written in the form

$$\Psi = \frac{1}{\sqrt{u^0}} \psi(u^1) \exp \left\{ -i \left[ \frac{\lambda}{2} u^3 - i(l - l_0) \varphi + m^2 u^0 + k_1 \ln u^0 \right] \right\}. \quad (3.15)$$

The functions  $\psi(u^1)$  obey the equation

$$\psi'' + \frac{1}{u^1} \psi' + R(u^1) \psi = 0, \quad R(u^1) = \frac{\lambda^2}{16} + \frac{k_1 + 2\lambda f_1}{4u^1} - \frac{(\bar{l} + f_2)^2}{4(u^1)^2}. \quad (3.16)$$

Dirac wave function can be presented in the form (3.7), (3.8) with the following modifications

$$P = \lambda, \quad q = \frac{1}{u^0}, \quad \mathbf{F} = \mathbf{e}_r \frac{r}{u^0} \left( 2i\partial_1 - \frac{\lambda}{2} \right) + \mathbf{e}_\varphi \frac{\bar{l} + f_2}{r}. \quad (3.17)$$

Besides, the functions  $\psi_\zeta(u^1)$  have to obey the equation (3.9) with  $R(u^1)$  defined by (3.16).

Solutions of the latter equations are available for  $f_s(u^1)$  in the form (3.10). Thus, we have again returned to the equation (2.58).

#### IV. SUPERPOSITION OF AHARONOV-BOHM FIELD AND SOME NON-UNIFORM FIELDS

Consider now additional fields, which are given by potentials of the form

$$A_0^{(1)} = \frac{c\hbar}{|e|} f_1(r), \quad A_1^{(1)} = \frac{c\hbar A(r)}{|e| r} \sin \varphi, \quad A_2^{(1)} = -\frac{c\hbar A(r)}{|e| r} \cos \varphi, \quad A_3^{(1)} = \frac{c\hbar}{|e|} f_2(r). \quad (4.1)$$

Here  $f_1(r)$ ,  $f_2(r)$ ,  $A(r)$  are arbitrary functions of  $r$ . The corresponding electromagnetic field components in cylindrical reference frame have the form

$$E_r = -\frac{c\hbar}{|e|} f_1'(r), \quad H_\varphi = \frac{c\hbar}{|e|} f_2'(r), \quad H_z = \frac{c\hbar A'(r)}{|e| r}, \quad E_\varphi = E_z = H_r = 0. \quad (4.2)$$

Exact solutions of relativistic wave equations in such fields were studied in [23,17]. Here we present exact solutions of the equations in the superposition of these fields and AB field.

Stationary solutions of Klein-Gordon equation that are eigenvectors for the operators  $p_0, p_3, L_z$  can be written as

$$\Psi(x) = e^{-i\Gamma} \psi(r), \quad \Gamma = k_0 x^0 + k_3 x^3 - (l - l_0) \varphi. \quad (4.3)$$

Functions  $\psi(r)$  obey the equation

$$\psi''(r) + \frac{1}{r} \psi'(r) + R(r) \psi(r) = 0, \quad R(r) = (k_0 + f_1)^2 - (k_3 + f_2)^2 - \frac{(\bar{l} + A)^2}{r^2} - m^2. \quad (4.4)$$

The corresponding solutions of Dirac equation have the form  $\Psi(x) = e^{-i\Gamma} M \psi$ , where the matrix  $M$  reads  $M = \text{diag}(e^{-i\varphi}, i, e^{-i\varphi}, i)$ , and the bispinor  $\psi = (\psi_k)$  ( $k = 1, 2, 3, 4$ ) obeys the equation

$$\left[ k_0 + f_1 - m\gamma^0 - \frac{1}{r} \left( \bar{l} + A - \frac{1}{2} \right) \alpha_1 - i \left( \frac{d}{dr} + \frac{1}{2r} \right) \alpha_2 + (k_3 + f_2) \alpha \right] \psi = 0. \quad (4.5)$$

In some particular cases the latter equation can be reduced to the one (2.78) and thus solved explicitly. All such cases are described in [23,17].

## V. AHARONOV-BOHM FIELD IN 2+1 QED

Consider Dirac equation in 2 + 1 dimensions (see for example, [24,25]) ( $x = (x^\mu)$ ,  $\mu = 0, 1, 2$ ,  $\gamma^0 = \sigma_3$ ,  $\gamma^1 = i\sigma_2$ ,  $\gamma^3 = -i\sigma_1$ ),

$$(\gamma^\mu P_\mu - m_0 c) \Psi(x) = 0, \quad \Psi(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \end{pmatrix}. \quad (5.1)$$

For components of the spinor  $\Psi(x)$  we get the following equations

$$(P_0 - m)\Psi_1 + (P_1 - iP_2)\Psi_2 = 0, \quad (P_0 + m)\Psi_2 + (P_1 + iP_2)\Psi_1 = 0. \quad (5.2)$$

These equations can be solved exactly for a superposition of AB field and an additional field described below. Potentials (1.6) of the latter field are given as

$$\frac{|e|}{c\hbar}A_0^{(1)} = B(r), \quad \frac{|e|}{c\hbar}A_1^{(1)} = \frac{A(r)}{r}\sin\varphi, \quad \frac{|e|}{c\hbar}A_2^{(1)} = -\frac{A(r)}{r}\cos\varphi, \quad (5.3)$$

where  $A(r), B(r)$  are arbitrary functions of  $r$ . This field is an analog of the field (4.1) in 3 + 1 dimensions. Potentials  $A_0^{(0)}, A_1^{(0)}, A_2^{(0)}$  of AB field in 2 + 1 dimensions are still given by the formulas (1.1). The operators

$$p_0 = i\hbar\frac{\partial}{\partial x^0}, \quad J_3 = -i\hbar\frac{\partial}{\partial\varphi} + \frac{\hbar}{2}\sigma_3 \quad (5.4)$$

are integrals of motions in the external field under consideration. Thus, we can impose the following conditions on the spinor  $\Psi$

$$p_0\Psi = \hbar k_0\Psi, \quad J_3\Psi = \hbar\left(l - l_0 - \frac{1}{2}\right)\Psi. \quad (5.5)$$

A solution of the equations (5.1), (5.5) has the form

$$\Psi(x) = e^{-i\Gamma} \begin{pmatrix} e^{-i\varphi}\psi_1(r) \\ i\psi_2(r) \end{pmatrix}, \quad \Gamma = k_0x^0 - (l - l_0)\varphi, \quad (5.6)$$

where the functions  $\psi_k(r)$  ( $k = 1, 2$ ) obey the equations

$$\begin{aligned} \psi_1'(r) &= \frac{\bar{l} - 1 + A(r)}{r}\psi_1(r) - (k_0 + B(r) + m)\psi_2(r), \\ \psi_2'(r) &= (k_0 + B(r) - m)\psi_1(r) - \frac{\bar{l} + A(r)}{r}\psi_2(r). \end{aligned} \quad (5.7)$$

Explicit solutions of these equations can be found in three particular cases:

$$1. \quad A(r) = B(r) = 0; \quad (5.8)$$

$$2. \quad A(r) = \rho = \frac{\gamma r^2}{2}, \quad \gamma > 0, \quad B(r) = 0; \quad (5.9)$$

$$3. \quad A(r) = \gamma r, \quad B(r) = \frac{b}{r}, \quad \gamma > 0; \quad (5.10)$$



Below we consider each case in detail.

The case (5.8) corresponds to the pure AB field. For  $l \neq 0$ , there exist bounded solutions of the form

$$\begin{aligned}\psi_1(r) &= \sqrt{k_0 + m} J_{|\bar{z}-1|}(kr), \quad k = \sqrt{k_0^2 - m^2}, \\ \psi_2(r) &= \varepsilon \sqrt{k_0 + m} J_{|\bar{z}|}(kr), \quad \varepsilon = \text{sign } l.\end{aligned}\quad (5.11)$$

Here  $J_l(x)$  are Bessel functions. Solutions (5.11) vanish at  $r = 0$ . They are orthogonal and normalized.

For  $l = 0, \mu \neq 0$ , bounded solutions do not exist. However, there are unbounded at  $r = 0$  solutions. Some of them have the form

$$\begin{aligned}\psi_1(r) &= \sqrt{k_0 + m} [c_1 J_{\mu-1}(kr) + c_2 J_{1-\mu}(kr)], \\ \psi_2(r) &= \sqrt{k_0 - m} [c_1 J_{\mu}(kr) - c_2 J_{-\mu}(kr)].\end{aligned}\quad (5.12)$$

Here  $c_1, c_2$  are arbitrary constants. The solutions (5.12) are still orthogonal and normalized. Another set of unbounded solutions (they are expressed via the Macdonald functions) reads

$$\begin{aligned}\psi_1(r) &= \sqrt{m + k_0} K_{1-\mu}(\sqrt{m^2 - k_0^2} r), \quad \text{Re } \sqrt{m^2 - k_0^2} > 0, \\ \psi_2(r) &= -\sqrt{m - k_0} K_{\mu}(\sqrt{m^2 - k_0^2} r).\end{aligned}\quad (5.13)$$

As we see they are defined even for some complex  $k_0$ . It is interesting to remark that the latter solutions have finite norms.

The case (5.9) corresponds a combination of AB and uniform constant magnetic fields. Here we can introduce operators  $a_k^+, a_k$  ( $k = 1, 2$ ) by relations (2.33). Using the substitution

$$\psi_1(x) = e^{-ik_0 x^0} \psi_1(\rho, \varphi), \quad \psi_2(x) = ie^{-ik_0 x^0} \psi_2(\rho, \varphi), \quad (5.14)$$

we present (5.2) in the following form

$$(k_0 - m) \psi_1(\rho, \varphi) - \sqrt{2\gamma} a_1 \psi_2(\rho, \varphi) = 0, \quad (k_0 + m) \psi_2(\rho, \varphi) - \sqrt{2\gamma} a_1^+ \psi_1(\rho, \varphi) = 0. \quad (5.15)$$

Now we can use the functions (2.47) and the relations (2.41).

Consider first states with  $k_0^2 \neq m^2$ ,  $l \neq 0$ . As in 3 + 1 dimensions, these states can be divided in two types ( $j = 1, 2$ ),

$$\psi_1(\rho, \varphi) = \sqrt{k_0 + m} \psi_{n-1, l-1}^{(j)}, \psi_2(\rho, \varphi) = \sqrt{k_0 - m} \psi_{n,l}^{(j)}, k_0^2 = m^2 + 2\gamma\bar{n}, \quad (5.16)$$

where  $\bar{n}$  was defined in (2.46). Solutions (5.16) vanish at  $r = 0$ .

If  $k_0^2 \neq m^2$ ,  $l = 0$ ,  $\mu \neq 0$ , then solutions, which are formally defined by equations (5.16), are unbounded at  $r = 0$ . However, they are still orthogonal and normalized. For  $l = 0$ ,  $\mu \neq 0$ ,  $k_0^2 = m^2 + \gamma(2\lambda + \mu - 1)$  ( $\lambda$  is arbitrary complex), there exist another unbounded solutions. They have the form (5.6), with

$$\psi_1(r) = (k_0 + m) \psi_{\lambda-\frac{1}{2}, 1-\mu}(\rho), \psi_2(r) = \sqrt{2\gamma} \psi_{\lambda, \mu}(\rho), \quad (5.17)$$

where the functions  $\psi_{\lambda, \mu}(x)$  are defined by (6.28), (6.29). These solutions are orthogonal and normalized as well.

Consider now states with  $k_0^2 = m^2$ . Suppose  $k_0 = m$ ; then a general solutions of Eqs. (5.7) reads

$$\begin{aligned} \psi_1(r) &= N f_l(\rho) + c e^{\frac{\rho}{2}} \rho^{\frac{l-1}{2}}, \quad \psi_2(r) = g N e^{-\frac{\rho}{2}} \rho^{-\frac{l}{2}}, \\ f_l(\rho) &= e^{\frac{\rho}{2}} \rho^{\frac{l-1}{2}} \int_{\rho}^{\infty} e^{-x} x^{-l} dx, \quad g = \sqrt{\frac{\gamma}{2m^2}}, \end{aligned} \quad (5.18)$$

where  $N, c$  are arbitrary constants. For  $\mu \neq 0$  the only some states with  $l = 0$  have a finite norm. Namely the states

$$\begin{aligned} \psi_1(r) &= N \rho^{\frac{\mu-1}{2}} e^{\frac{\rho}{2}} \int_{\rho}^{\infty} e^{-x} x^{-\mu} dx, \quad \psi_2(r) = g N e^{-\frac{\rho}{2}} \rho^{-\frac{\mu}{2}}, \\ N &= \sqrt{\frac{\Gamma(1+\mu) \sin \mu\pi}{\pi [1 + \mu\psi(1) - \mu\psi(1+\mu) + \mu g^2]}}, \quad \int_0^{\infty} [\psi_1^2(r) + \psi_2^2] d\rho = 1, \end{aligned} \quad (5.19)$$

where  $\psi(x)$  is the logarithmic derivative of  $\Gamma$ -function ([18], 8.360). All the above solutions are singular at  $r = 0$ . One can see that  $\lim_{\mu \rightarrow 0} N = 0$ . Thus, in 2 + 1 dimensions, in pure magnetic field, Dirac equation does not have solutions with  $k_0 = m$ , in contrast to the corresponding 3+1-dimensional case. For  $\mu \neq 0$  (in the presence of AB field) such solutions appear even in 2 + 1 dimensions.

Suppose  $k_0 = -m$ . Then, the functions

$$\psi_1(r) = 0, \quad \psi_2(r) = [\Gamma(1-\bar{l})]^{-\frac{1}{2}} \rho^{-\frac{\bar{l}}{2}} e^{-\frac{\rho}{2}}, \quad \int_0^{\infty} [\psi_1^2(r) + \psi_2^2] d\rho = 1 \quad (5.20)$$

present solutions for  $l \leq 0$ . One can see that (5.20) is a particular case of (5.16). For  $l < 0$ , the states (5.20) vanish at  $r = 0$ . For  $l = 0, \mu \neq 0$ , they are singular at  $r = 0$ . If  $\mu = 0$ , then this singularity disappears, however the states do not tend to zero as  $r \rightarrow 0$ .

Let us finally turn to the case (5.10). Here it is enough to consider  $b \neq 0$  only. Indeed, suppose  $b = 0$ ; then doing the change of functions ( $\psi \rightarrow f$ )

$$\psi_1(r) = \sqrt{k_0 + m} f_1(r), \quad \psi_2(r) = \sqrt{k_0 - m} f_2(r), \quad (5.21)$$

we can transform Eqs. (5.7) to the form (2.78) with  $k_1 = \sqrt{k_0^2 - m^2}$ . Solutions of Eqs. (2.78) are given by the formulas (2.85)-(2.89). Thus, below we consider the case  $b \neq 0$  only.

We introduce the following notation

$$g_1 = \sqrt{|2\bar{l} - 1| + 2b}, \quad g_{-1} = \sqrt{|2\bar{l} - 1| - 2b}, \quad \varepsilon = \pm 1 = \text{sign } l. \quad (5.22)$$

Thus solutions of Eqs. (5.7) have the form

$$\begin{aligned} \psi_1(r) &= c_1 I_{\alpha+s,s}(y) + c_2 I_{\alpha+s-1,s+1}(y), \quad \psi_2(r) = c_3 I_{\alpha+s,s}(y) + c_4 I_{\alpha+s-1,s+1}(y), \\ y &= 2\sqrt{m^2 + \gamma^2 - k_0^2} r, \quad \alpha = 1 + g_1 g_{-1}, \quad 2 + 2s + g_1 g_{-1} = \frac{2bk_0 - (2\bar{l} - 1)\gamma}{\sqrt{m^2 + \gamma^2 - k_0^2}}, \end{aligned} \quad (5.23)$$

where

$$\begin{aligned} c_1 &= (g_\varepsilon - \varepsilon g_{-\varepsilon}) \sqrt{(2\bar{l} - 1)k_0 - 2b\gamma - mg_1 g_{-1}} = \frac{(g_\varepsilon - \varepsilon g_{-\varepsilon})}{(g_\varepsilon + \varepsilon g_{-\varepsilon})} c_3, \\ c_2 &= -(g_\varepsilon + \varepsilon g_{-\varepsilon}) \sqrt{(2\bar{l} - 1)k_0 - 2b\gamma + mg_1 g_{-1}} = \frac{(g_\varepsilon + \varepsilon g_{-\varepsilon})}{(g_\varepsilon - \varepsilon g_{-\varepsilon})} c_4, \end{aligned} \quad (5.24)$$

and the relations (6.6)-(6.11) for the Laguerre functions  $I_{n,m}$  were used. It follows from (5.22) that both  $g_1$  and  $g_{-1}$  are real and positive for  $(2\bar{l} - 1)^2 > 4b^2$ . If  $(2\bar{l} - 1)^2 < 4b^2$ , then one of these quantities is real and positive and another one is imaginary. Suppose  $(2\bar{l} - 1)^2 > 4b^2$ . Then for the energies  $k_0^2 < m^2 + \gamma^2$  there exist bound states. In the latter case  $s$  is positive integer and  $k_0^2$  is quantized according to the last equation (5.23).

Another observation: For any  $b$  there exist a number  $l$  such that

$$1 > g_1^2 g_{-1}^2 = (2\bar{l} - 1)^2 - 4b^2 > 0. \quad (5.25)$$

For example,  $b = 0$  corresponds to  $l = 0$ . Then one can chose two values for  $\alpha$ ,

$$\alpha_1 = 1 + g_1 g_{-1}, \quad \alpha_2 = 1 - g_1 g_{-1}. \quad (5.26)$$

In both cases the solutions (5.23) have singularity at  $r = 0$  but still have finite norms.

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## VI. APPENDIX

1. The Laguerre functions  $I_{n,m}(x)$  are defined by the relation

$$I_{n,m}(x) = \sqrt{\frac{\Gamma(1+n)}{\Gamma(1+m)}} \frac{\exp(-x/2)}{\Gamma(1+n-m)} x^{\frac{n-m}{2}} \Phi(-m, n-m+1; x). \quad (6.1)$$

Here  $\Phi(\alpha, \gamma; x)$  is the confluent hypergeometric function in a standard definition (see [18], 9.210). For  $\gamma \neq -s$ , where  $s$  is integer and non-negative, the latter function can be presented by a series

$$\Phi(\alpha, \gamma; x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k x^k}{(\gamma)_k k!} = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k) x^k}{\Gamma(\gamma+k) k!}. \quad (6.2)$$

This series is converges for any complex  $x$ . For any complex  $\alpha$ , the Pochhammer symbols  $(\alpha)_k$  are defined as follows

$$(\alpha)_k = \alpha(\alpha+1)\dots(\alpha+k-1) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}. \quad (6.3)$$

2. Let  $m$  be a non-negative integer number; then the Laguerre functions are related to Laguerre polynomials  $L_n^\alpha(x)$  ([18], 8.970, 8.972.1) by the equation

$$I_{n,m}(x) = \sqrt{\frac{\Gamma(1+m)}{\Gamma(1+n)}} \exp(-x/2) x^{\frac{n-m}{2}} L_m^{n-m}(x), \quad m = 0, 1, 2, \dots, \quad (6.4)$$

$$L_n^\alpha(x) = \frac{1}{n!} e^x x^{-\alpha} \frac{d^n}{dx^n} e^{-x} x^{n+\alpha} = \sum_{k=0}^n \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} = \binom{n+\alpha}{n} \Phi(-n, 1+\alpha; x). \quad (6.5)$$

Here  $n$  are non-negative integer numbers such that

$$\binom{\alpha}{n} = \frac{\Gamma(1+\alpha)}{\Gamma(1+n)\Gamma(1+\alpha-n)} = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}.$$

3. Using well-known properties of the confluent hypergeometric function ([18], 9.212; 9.213; 9.216), one can easily get both relations for the Laguerre functions

$$2\sqrt{x(n+1)}I_{n+1,m}(x) = (n-m+x)I_{n,m}(x) - 2xI'_{n,m}(x), \quad (6.6)$$

$$2\sqrt{x(m+1)}I_{n,m+1}(x) = (n-m-x)I_{n,m}(x) + 2xI'_{n,m}(x), \quad (6.7)$$

$$2\sqrt{xn}I_{n-1,m}(x) = (n-m+x)I_{n,m}(x) + 2xI'_{n,m}(x), \quad (6.8)$$

$$2\sqrt{xm}I_{n,m-1}(x) = (n-m-x)I_{n,m}(x) - 2xI'_{n,m}(x), \quad (6.9)$$

$$2\sqrt{nm}I_{n-1,m-1}(x) = (n+m-x)I_{n,m}(x) - 2xI'_{n,m}(x), \quad (6.10)$$

$$2\sqrt{(n+1)(m+1)}I_{n+1,m+1}(x) = (n+m+2-x)I_{n,m}(x) + 2xI'_{n,m}(x), \quad (6.11)$$

and a differential equation for these functions

$$4x^2 I''_{n,m}(x) + 4x I'_{n,m}(x) - [x^2 - 2x(1+n+m) + (n-m)^2] I_{n,m}(x) = 0. \quad (6.12)$$

Suppose  $I_{n,m}(x)$  and  $I_{m,n}(x)$  are linearly independent. Then, a general solution  $I$  of this equation has the form  $I = AI_{n,m}(x) + BI_{m,n}(x)$ . However, whenever the condition (6.18) holds,  $I_{n,m}(x)$  and  $I_{m,n}(x)$  are dependent. The formulas (6.6)-(6.11) and the equation (6.12) are valid for any complex  $n, m, x$ . One has to be careful applying the formulas (6.8)-(6.10) for  $n, m = 0$ . A straightforward calculation, which uses (6.1) and (6.2), gives

$$\lim_{n \rightarrow 0} \sqrt{n} I_{n-1,m}(x) = -\frac{\sin m\pi}{\pi} \sqrt{\Gamma(1+m)} x^{-\frac{1+m}{2}} \exp(x/2), \quad \lim_{m \rightarrow 0} \sqrt{m} I_{n,m-1}(x) = 0. \quad (6.13)$$

A combination of Eqs. (6.6)-(6.9) results in the following relations

$$\begin{aligned} 2\sqrt{x}I'_{n,m}(x) &= \sqrt{n}I_{n-1,m}(x) - \sqrt{n+1}I_{n+1,m}(x) \\ &= \sqrt{m+1}I_{n,m+1}(x) - \sqrt{m}I_{n,m-1}(x), \end{aligned} \quad (6.14)$$

$$\sqrt{x(n+1)}I_{n+1,m}(x) - (n-m+x)I_{n,m}(x) + \sqrt{xn}I_{n-1,m}(x) = 0, \quad (6.15)$$

$$\sqrt{x(m+1)}I_{n,m+1}(x) - (n-m-x)I_{n,m}(x) + \sqrt{xm}I_{n,m-1}(x) = 0. \quad (6.16)$$

4. Using properties of the confluent hypergeometric function, one can get a representation

$$I_{n,m}(x) = \sqrt{\frac{\Gamma(1+n)}{\Gamma(1+m)\Gamma(1+n-m)}} \frac{\exp(x/2)}{x^{\frac{n-m}{2}}} \Phi(1+n, 1+n-m; -x), \quad (6.17)$$

and a relation ([18], 9.214)

$$I_{n,m}(x) = (-1)^{n-m} I_{m,n}(x), \quad n-m \text{ integer}. \quad (6.18)$$

5. An asymptotic formula takes place

$$\Phi(a, c; x) \approx \frac{\Gamma(c)}{\Gamma(a)} e^x x^{a-c}, \quad \operatorname{Re} x \rightarrow \infty. \quad (6.19)$$

Thus we obtain the following asymptotic behavior of  $I_{n,m}(x)$  whenever  $m$  is not integer

$$I_{n,m}(x) = -\frac{\sin m\pi}{\pi} \sqrt{\Gamma(1+n)\Gamma(1+m)} x^{-\frac{n+m+2}{2}} \exp(x/2), \quad \operatorname{Re} x \rightarrow \infty, \quad (6.20)$$

and

$$I_{n,m}(x) = (-1)^m \frac{x^{\frac{n+m}{2}} \exp(-x/2)}{\sqrt{\Gamma(1+n)\Gamma(1+m)}}, \quad \operatorname{Re} x \rightarrow \infty, \quad (6.21)$$

whenever  $m$  is integer.

6. One can prove the following asymptotic formula

$$\lim_{p \rightarrow \infty} I_{p+\alpha, p+\beta} \left( \frac{x^2}{4p} \right) = J_{\alpha-\beta}(x), \quad (6.22)$$

where  $J_\nu(x)$  are Bessel functions.

7. Taking into account (6.20) and (6.21), one can see that only the functions  $I_{\alpha+n,n}(x)$  with non-negative integer  $n$  and  $\alpha > -1$  are quadratically integrable on the interval  $(0, \infty)$ .

Such functions obey the orthonormality relation

$$\int_0^{\infty} I_{\alpha+n,n}(x) I_{\alpha+m,m}(x) dx = \delta_{m,n}, \quad (6.23)$$

which follows from the corresponding properties of the Laguerre polynomials ([18], 7.414.3).

In such a case, the relation

$$I_{\alpha+n,n}(x) = \sqrt{\frac{n!}{\Gamma(n+\alpha+1)}} e^{-\frac{x}{2}} x^{\frac{\alpha}{2}} L_n^{\alpha}(x) \quad (6.24)$$

follows from (6.4)

8. Consider a class of functions, which are closely related to Laguerre functions, and which appear often in various problems of mathematical physics.

As it follows from Eq. (6.12), the Laguerre functions are solutions of the following eigenvalue problem

$$R_{\alpha}\psi = \lambda\psi, \quad R_{\alpha} = \frac{\alpha^2}{4x} + \frac{x}{4} - \frac{d}{dx} - x \frac{d^2}{dx^2}, \quad 0 < x < \infty, \quad \alpha = \text{const.} \quad (6.25)$$

A general solution of this problem has the form

$$\psi(x) = aI_{n,m}(x) + bI_{m,n}(x), \quad \alpha = n - m, \quad 2\lambda = n + m + 1, \quad (6.26)$$

where  $a, b$  are arbitrary constants. In the general case functions  $\psi(x)$  vanish as  $x \rightarrow \infty$  only if one of the numbers  $n$  or  $m$  is positive and integer. However, one can provide such a behavior for any  $n, m$ , choosing some special values of  $a, b$ . Consider the functions

$$\psi_{\lambda,\alpha}(x) = x^{-\frac{1}{2}} W_{\lambda, \frac{\alpha}{2}}(x), \quad \psi_{\lambda,\alpha}(x) = \psi_{\lambda,-\alpha}(x), \quad (6.27)$$

where  $W_{\lambda,\mu}(x)$  are Whittaker functions ([18], 9.220.4). The functions  $\psi_{\lambda,\alpha}(x)$  can be expressed via the confluent hypergeometric functions

$$\begin{aligned} \psi_{\lambda,\alpha}(x) = e^{-\frac{x}{2}} & \left[ \frac{\Gamma(-\alpha) x^{\frac{\alpha}{2}}}{\Gamma\left(\frac{1-\alpha}{2} - \lambda\right)} \Phi\left(\frac{1+\alpha}{2} - \lambda, 1 + \alpha; x\right) \right. \\ & \left. + \frac{\Gamma(\alpha) x^{-\frac{\alpha}{2}}}{\Gamma\left(\frac{1+\alpha}{2} - \lambda\right)} \Phi\left(\frac{1-\alpha}{2} - \lambda, 1 - \alpha; x\right) \right], \end{aligned} \quad (6.28)$$

or, using (6.1), via the Laguerre functions

$$\psi_{\lambda,\alpha}(x) = \frac{\sqrt{\Gamma(1+n)\Gamma(1+m)}}{\sin(n-m)\pi} (\sin n\pi I_{n,m}(x) - \sin m\pi I_{m,n}(x)),$$

$$\alpha = n - m, 2\lambda = 1 + n + m, n = \lambda - \frac{1-\alpha}{2}, m = \lambda - \frac{1+\alpha}{2}. \quad (6.29)$$

By the help of (6.6)-(4.16), the following properties of the functions  $\psi_{\lambda,\alpha}(x)$  can be established,

$$\psi_{\lambda,\alpha}(x) = \sqrt{x}\psi_{\lambda-\frac{1}{2},\alpha-1}(x) + \frac{1+\alpha-2\lambda}{2}\psi_{\lambda-1,\alpha}(x), \quad (6.30)$$

$$\psi_{\lambda,\alpha}(x) = \sqrt{x}\psi_{\lambda-\frac{1}{2},\alpha+1}(x) + \frac{1-\alpha-2\lambda}{2}\psi_{\lambda-1,\alpha}(x), \quad (6.31)$$

$$2x\psi'_{\lambda,\alpha}(x) = (2\lambda-1-x)\psi_{\lambda,\alpha}(x) + \frac{1}{2}(2\lambda-1-\alpha)(2\lambda-1+\alpha)\psi_{\lambda-1,\alpha}(x), \quad (6.32)$$

$$2x\psi'_{\lambda,\alpha}(x) = (\alpha-x)\psi_{\lambda,\alpha}(x) + (2\lambda-1-\alpha)\sqrt{x}\psi_{\lambda-\frac{1}{2},\alpha+1}(x) \quad (6.33)$$

$$= (x-2\lambda-1)\psi_{\lambda,\alpha} - 2\psi_{\lambda+1,\alpha}. \quad (6.34)$$

As a consequence of these properties we get

$$A_\alpha\psi_{\lambda,\alpha}(x) = \frac{2\lambda-1+\alpha}{2}\psi_{\lambda-\frac{1}{2},\alpha-1}(x), \quad A_\alpha^+\psi_{\lambda-\frac{1}{2},\alpha-1}(x) = \psi_{\lambda,\alpha}(x),$$

$$A_\alpha = \frac{x+\alpha}{2\sqrt{x}} + \sqrt{x}\frac{d}{dx}, \quad A_\alpha^+ = \frac{x+\alpha-1}{2\sqrt{x}} - \sqrt{x}\frac{d}{dx}. \quad (6.35)$$

The operator  $R_\alpha$  can be expressed via the operators  $A_\alpha, A_\alpha^+$ ,

$$R_\alpha = A_\alpha^+A_\alpha + \frac{1-\alpha}{2}, \quad R_{\alpha-1} = A_\alpha A_\alpha^+ - \frac{\alpha}{2}. \quad (6.36)$$

Since (6.29) is a particular case of (6.26), then  $\psi_{\lambda,\alpha}(x)$  are also an eigenfunctions for the operator  $R_\alpha$ .

Using well-known asymptotics of the Whittaker function ([18], 9.227), we get

$$\psi_{\lambda,\alpha}(x) \sim x^{\lambda-\frac{1}{2}}e^{-\frac{x}{2}}, \quad x \rightarrow \infty; \quad \psi_{\lambda,\alpha}(x) \sim \frac{\Gamma(|\alpha|)}{\Gamma(\frac{1+|\alpha|}{2}-\lambda)}x^{-\frac{|\alpha|}{2}}, \quad \alpha \neq 0, \quad x \sim 0. \quad (6.37)$$

The functions  $\psi_{\lambda,0}(x)$  have a logarithmic singularity at  $x \sim 0$ . It is important to stress that the functions  $\psi_{\lambda,\alpha}(x)$  are correctly defined and infinitely differentiable for  $0 < x < \infty$  and for any complex  $\lambda, \alpha$ . In this respect one can mention that the Laguerre functions are not defined for negative integer  $n, m$ . In particular cases, when one of the numbers  $n, m$



is non-negative and integer, the functions  $\psi_{\lambda,\alpha}(x)$  coincides (up to a constant factor) with Laguerre functions. Thus,  $\psi_{\lambda,\alpha}(x)$  are eigenfunctions (of the operator  $R_\alpha$ ), which vanish at  $x \rightarrow \infty$ .

According to (6.37), the functions  $\psi_{\lambda,\alpha}(x)$  are quadratically integrable on the interval  $0 < x < \infty$  whenever  $|\alpha| < 1$ . It is not true for  $|\alpha| \geq 1$ . The corresponding integrals can be calculated ([18], 7.611),

$$\int_0^\infty \psi_{\lambda,\alpha}(x) \psi_{\lambda',\alpha}(x) dx = \frac{\pi}{(\lambda' - \lambda) \sin \alpha\pi} \left\{ \left[ \Gamma\left(\frac{1 + \alpha - 2\lambda'}{2}\right) \Gamma\left(\frac{1 - \alpha - 2\lambda}{2}\right) \right]^{-1} - \left[ \Gamma\left(\frac{1 - \alpha - 2\lambda'}{2}\right) \Gamma\left(\frac{1 + \alpha - 2\lambda}{2}\right) \right]^{-1} \right\}, \quad |\alpha| < 1, \quad (6.38)$$

$$\int_0^\infty |\psi_{\lambda,\alpha}(x)|^2 dx = \frac{\pi}{\sin \alpha\pi} \frac{\psi\left(\frac{1+\alpha-2\lambda}{2}\right) - \psi\left(\frac{1-\alpha-2\lambda}{2}\right)}{\Gamma\left(\frac{1+\alpha-2\lambda}{2}\right) \Gamma\left(\frac{1-\alpha-2\lambda}{2}\right)}, \quad |\alpha| < 1, \quad (6.39)$$

$$\int_0^\infty |\psi_{\lambda,0}(x)|^2 dx = \frac{\psi'\left(\frac{1}{2} - \lambda\right)}{\Gamma^2\left(\frac{1}{2} - \lambda\right)}, \quad \int_0^\infty |\psi_{n+\frac{1}{2},0}(x)|^2 dx = \Gamma^2(1+n). \quad (6.40)$$

Here  $\psi(x)$  is the logarithmic derivative of the  $\Gamma$ -function ([18], 8.360).

For  $|\alpha| \geq 1$ , the situation is the following: the only quadratically integrable eigenfunctions of the operator  $R_\alpha$  are Laguerre functions, they also form a complete set. The functions  $\psi_{\lambda,\alpha}(x)$  are orthogonal whenever arguments of the  $\Gamma$ -function in (6.38) are integer and negative. That corresponds to  $n, m$  integer and non-negative. Thus, that is again the case of Laguerre functions according to (6.29). If  $|\alpha| < 1$ , then, in the general case, the functions  $\psi_{\lambda,\alpha}(x)$  and  $\psi_{\lambda',\alpha}(x)$ ,  $\lambda' \neq \lambda$ , are not orthogonal, as it follows from (6.38). That is a reflection of the fact that  $R_\alpha$  is not anymore self-conjugate operator for such values of  $\alpha$ .

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