Converging Perturbative Solutions of the Schrödinger Equation for a Two-Level System with a Hamiltonian Depending Periodically on Time

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Abstract. *We study the Schrödinger equation of a class of two-level systems under the action of a periodic time-dependent external field in the situation where the energy difference* 2ϵ *between the free energy levels is sufficiently small with respect to the strength of the external interaction. Under suitable conditions we show that this equation has a solution in terms of converging power series expansions in* ϵ*. In contrast to other expansion methods, like in the Dyson expansion, the method we present is not plagued by the presence of "secular terms". Due to this feature we were able to prove absolute and uniform convergence of the Fourier series involved in the computation of the wave functions and to prove absolute convergence of the* ϵ*-expansions leading to the "secular frequency" and to the coefficients of the Fourier expansion of the wave function.*

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Contents

1 Introduction

Let us consider the following Hamiltonian for a two-level system under the action of an external time-dependent field

$$H_1(t) \;=\; H_0 + H_I(t) \;=\; \epsilon\sigma_3 - f(t)\sigma_1 \tag{1.1}$$

and the corresponding Schrödinger equation[2]

$$i\partial_t\Psi(t) \;=\; H_1(t)\Psi(t), \tag{1.2}$$

with $\Psi : \mathbb{R} \to \mathbb{C}^2$. Here $f(t)$ is a function of time t and $\epsilon \in \mathbb{R}$ is a parameter representing half of the energy difference between the "free" (i.e., for $f \equiv 0$) energy levels. The symbols σ_1, σ_2 and σ_3 denote the Pauli matrices in their usual representations:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

satisfying the commutation relations $[\sigma_1,\ \sigma_2] = 2i\sigma_3$, plus cyclic permutations.

The "interaction Hamiltonian" $H_I(t) := -f(t)\sigma_1$ represents a time-dependent external interaction coupled to the system inducing transitions between the two eigen-states of the free Hamiltonian $H_0 := \epsilon\sigma_3$. The situation where ϵ is "small" characterizes the "large coupling domain" [2]-[3].

The system described above is certainly one of the simplest non-trivial time-depending quantum systems and the study of the solutions of (1.2) is of basic importance for many physical applications as, e.g., in quantum optics or in problems of quantum tunnelling.

Equation (1.2) has been analysed by many authors in various approximations. In the wide literature on this subject we mention the pioneering work of Autler and Townes [4], where these authors studied the solutions of (1.2) for the case where, in our notation, $f(t) = -2\beta\cos(\omega t)$, $\beta \in \mathbb{R}$. Their work is exact but non-rigorous and involved a combination of the method of continued fractions, for relating the coefficients the Fourier decomposition of the wave functions, with numerical analysis. No proof has been obtained that the continued fractions converge and further unjustified restrictions have been made in order to transform some transcendental equations into low order algebraic equations, which are then solved either exactly or, specially for strong fields, numerically.

For a recent review on the mathematical theory of quantum systems submited to time-depending periodic and quasi-periodic perturbations see [2]. For an introduction to the subject of "quantum chaos" and quantum stability, see [5]. See also [3] for a spectral analysis of the quasi-energy operator for two-level atoms in the quasi-periodic case.

In [1] we studied the system described by (1.2) in the situation where f is a quasi-periodic function of time and a special sort of perturbative expansion (power series expansion in ϵ) has been developed. Its main virtue is to be free of the so-called "secular terms", i.e., polynomials in t that appear order by order in perturbation theory and that spoil the analysis of convergence of the series and the proofs of quasi-periodicity of the perturbative terms. Although we have not been able to prove convergence of our power series expansion in the general case where f is quasi-periodic it has been established that the coefficients of the expansion are indeed quasi-periodic functions of time.

[2] For simplicity we shall adopt here a system of units with $\hbar = 1$.

One of the obstacles found in the attempt to prove convergence of the series is the presence of “small denominators”. This typical feature of perturbative approximations for solutions of differential equations with quasi-periodic coefficients is well known as one of the main sources of problems in the mathematically precise treatment of such equations.

On what concerns proofs of convergence it should, therefore, be expected that better results could be obtained if the function f were restricted to be periodic since, in this case, no problems with small denominators should afflict our expansions.

However, the problem with small denominators is not the only problem to be faced in the perturbative expansion of [1]. In this paper we show how to circumvent the additional sources of difficulties and to finally establish convergence of our perturbative expansion for periodic f.

By a time-independent unitary transformation, representing a rotation of $\pi/2$ around the 2-axis, we may replace $H_1(t)$ by

$$H_2(t) \;:=\; \left(e^{-i\pi\sigma_2/4}\right)\, H_1(t)\, \left(e^{i\pi\sigma_2/4}\right) \;=\; \epsilon\sigma_1 + f(t)\sigma_3 \tag{1.3}$$

and the Schrödinger equation becomes

$$i\partial_t\Phi(t) \;=\; H_2(t)\Phi(t), \tag{1.4}$$

with

$$\Phi(t) \;:=\; e^{-i\pi\sigma_2/4}\Psi(t). \tag{1.5}$$

The theorem below, proven in [1], presents the solution of the Schrödinger equation (1.4) in terms of particular solutions of a generalized Riccati equation.

1.1 Theorem. *Let $f : \mathbb{R} \to \mathbb{R}$, $f \in C^1(\mathbb{R})$ and $\epsilon \in \mathbb{R}$ and let $g : \mathbb{R} \to \mathbb{C}$, $g \in C^1(\mathbb{R})$, be a particular solution of the generalized Riccati equation*

$$G' - iG^2 - 2ifG + i\epsilon^2 = 0. \tag{1.6}$$

Then, the function $\Phi : \mathbb{R} \to \mathbb{C}^2$ given by

$$\Phi(t) \;=\; \begin{pmatrix} \phi_+(t) \\ \phi_-(t) \end{pmatrix} \;=\; U(t)\Phi(0) \;=\; U(t,\ 0)\Phi(0), \tag{1.7}$$

where

$$U(t) \;:=\; \begin{pmatrix} R(t)\left(1 + ig(0)S(t)\right) & -i\epsilon R(t)S(t) \\ \\ -i\epsilon\overline{R(t)}\;\overline{S(t)} & \overline{R(t)}\left(1 - i\,\overline{g(0)}\;\overline{S(t)}\right) \end{pmatrix}, \tag{1.8}$$

with

$$R(t) \;:=\; \exp\left(-i\int_0^t \left(f(\tau) + g(\tau)\right) d\tau\right) \tag{1.9}$$

and

$$S(t) \;:=\; \int_0^t R(\tau)^{-2}\, d\tau \tag{1.10}$$

is a solution of the Schrödinger equation (1.4) with initial value $\Phi(0) \;=\; \begin{pmatrix} \phi_+(0) \\ \phi_-(0) \end{pmatrix} \in \mathbb{C}^2$. □

For a proof of Theorem 1.1, see [1]. Let us briefly describe some of the ideas leading to Theorem 1.1 and to other results of [1]. As we saw in [1], the solutions of the Schrödinger equation (1.4) can be studied in terms of the solutions of a particular complex version of Hill's equation:

$$\phi''(t) + \left(if'(t) + \epsilon^2 + f(t)^2\right)\phi(t) = 0. \tag{1.11}$$

In fact, a simple computation (see [1]) shows that the components $\phi_\pm$ of $\Phi(t)$ satisfy precisely

$$\begin{aligned} \phi_+'' + (+if' + \epsilon^2 + f^2)\,\phi_+ &= 0 \\ \phi_-'' + (-if' + \epsilon^2 + f^2)\,\phi_- &= 0 \end{aligned} \quad . \tag{1.12}$$

As a side remark we note that equations (1.12) are simpler and more convenient than the equations obtained by separating ψ_+ and ψ_- from (1.2):

$$\begin{aligned} \psi_+'' - \left(\frac{f'}{f}\right)\psi_+' + \left(\epsilon^2 + f^2 - i\epsilon\left(\frac{f'}{f}\right)\right)\psi_+ &= 0 \\ \psi_-'' - \left(\frac{f'}{f}\right)\psi_-' + \left(\epsilon^2 + f^2 + i\epsilon\left(\frac{f'}{f}\right)\right)\psi_- &= 0 \end{aligned} \quad . \tag{1.13}$$

This last pair of equations, mentioned (but not used) in [4], is mathematically less convenient because the coefficient f'/f can be discontinuous and unbounded in typical cases as, for instance when $f(t) = -2\beta\cos(\omega t)$, the case analysed in [4].

In [1] we attempted to solve (1.11) using the Ansatz

$$\phi(t) = \exp\left(-i\int_0^t (f(\tau) + g(\tau))d\tau\right). \tag{1.14}$$

It follows that g has to satisfy the generalized Riccati equation (1.6) and we tried to find solutions for g in terms of a power expansion in ϵ like

$$g(t) \;=\; q(t)\sum_{n=1}^{\infty}\epsilon^n\, c_n(t), \tag{1.15}$$

where

$$q(t) \;:=\; \exp\left(i\int_0^t f(\tau)d\tau\right). \tag{1.16}$$

The heuristic idea behind the Ansätze (1.14) and (1.15) is the following. For $\epsilon \equiv 0$ a solution for (1.11) is given by $\exp\left(-i\int_0^t f(\tau)d\tau\right)$. Thus, in (1.14) and (1.15) we are searching for solutions in terms of an "effective external field" of the form $f + g$, with g vanishing for $\epsilon = 0$.

Notice that a solution of the form (1.14) leads to only one of the two independent solutions of the second order Hill's equation (1.11). The complete solution of the Schrödinger equation (1.4) in terms of solutions of the generalized Riccati equation (1.6) is that described in Theorem 1.1.

As mentioned above, perturbative solutions of quasi-periodically time-dependent systems are usually plagued by small denominators and by the presence of the so-called "secular terms". In

[1] we discovered a particular way to eliminate completely the secular terms from the perturbative expansion of g (see Appendix A) and we were able to show, under some special assumptions, that the coefficients $c_n(t)$ are all quasi-periodic functions. In [1] we proved convergence of our perturbative solution in the somewhat trivial case where $f(t)$ is a non-zero constant function. Unfortunately no conclusion could be drawn about the convergence of the perturbative expansion for g in the general case of quasi-periodic f. We conjectured, however, that our expansion is uniformly convergent at least in the situation where $f(t) - M(f)$ is uniformly small. Here $M(h)$ is the so-called "mean value" of an *almost periodic* function h, defined as (see, e.g. [6])

$$M(h) \ := \ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} h(t)\, dt. \tag{1.17}$$

The technically central result of the present paper is the proof that, under suitable assumptions, the series (1.15) converges absolutely and uniformly on $\mathbb{R}$ as a function of time for $|\epsilon|$ small enough and f periodic. This is the content of Theorem 3.1. Moreover, we show that the functions c_n and, hence, g, have uniformly converging Fourier series representations. We use this fact together with the solution (1.8) to find the Floquet representation of the components $\phi_\pm$ of the wave function in terms of uniformly converging Fourier series representations. This is the content of Theorem 1.2. Absolutely converging power series in ϵ for the Fourier coefficients and for the secular frequency are also presented.

We believe that the methods employed in this paper are also of importance for the general theory of Hill's equation. It would be of great interest to know whether the ideas described in [1] and here can be generalized and applied to a larger class of Hill's equations than those we studied so far.

1.1 The Main Result

On what concerns the solutions of the Schrödinger equation (1.4) the next theorem summarises our main results.

1.2 Theorem. *Let f be a real T_ω-periodic function of time ($T_\omega := 2\pi/\omega$) whose Fourier decomposition*

$$f(t) \ = \ \sum_{n\in\mathbb{Z}} F_n e^{in\omega t}, \tag{1.18}$$

with $\omega > 0$, contains only a finite number of terms, i.e., the set of integers $\{n \in \mathbb{Z}|\ F_n \neq 0\}$ is a finite set. Moreover, assume that $F_0 = 0$.

Consider the two following mutually exclusive conditions on f:

I) $M(q^2) \neq 0$.

II) $M(q^2) = 0$ but $M(\mathcal{Q}_1) \neq 0$, where

$$\mathcal{Q}_1(t) \ := \ q(t)^2 \int_0^t q^{-2}(\tau) d\tau. \tag{1.19}$$

Then, for each f as above, satisfying condition I or II, there exists a constant $K > 0$ (depending on the Fourier coefficients $\{F_n,\ n \in \mathbb{Z}\ , n \neq 0\}$ and on $\omega > 0$) such that, for each ϵ with $|\epsilon| < K$, *there exist $\Omega \in \mathbb{R}$ and T_ω-periodic functions $u_{11}^{\pm}$ and $u_{12}^{\pm}$ such that the propagator $U(t)$ of (1.7) can be written as*

$$U(t) = \begin{pmatrix} U_{11}(t) & U_{12}(t) \\ U_{21}(t) & U_{22}(t) \end{pmatrix} = \begin{pmatrix} U_{11}(t) & U_{12}(t) \\ -\overline{U_{12}(t)} & \overline{U_{11}(t)} \end{pmatrix}, \tag{1.20}$$

with

$$U_{11}(t) = e^{-i\Omega t}\, u_{11}^{-}(t) + e^{i\Omega t}\, u_{11}^{+}(t), \tag{1.21}$$

$$U_{12}(t) = e^{-i\Omega t}\, u_{12}^{-}(t) + e^{i\Omega t}\, u_{12}^{+}(t). \tag{1.22}$$

The functions $u_{11}^{\pm}$ and $u_{12}^{\pm}$ have absolutely and uniformly converging Fourier expansions

$$u_{11}^{\pm}(t) = \sum_{n\in\mathbb{Z}} \mathcal{U}_{11}^{\pm}(n) e^{in\omega t},$$

$$u_{12}^{\pm}(t) = \sum_{n\in\mathbb{Z}} \mathcal{U}_{12}^{\pm}(n) e^{in\omega t}.$$

Moreover, under the same assumptions, Ω and the Fourier coefficients $\mathcal{U}_{11}^{\pm}(n)$ and $\mathcal{U}_{12}^{\pm}(n)$ can be expressed in terms of absolutely converging power series on ϵ. □

Remarks on Theorem 1.2

1. Expressions (1.21) and (1.22) represent the so-called "Floquet form" of the matrix elements $U_{11}(t)$ and $U_{12}(t)$. The frequency Ω is called the "secular frequency".

2. In this paper we will assume that $F_0 = 0$. Results on the almost resonant case $F_0 \neq 0$, with F_0/ω satisfying some appropriated Diophantine conditions, will appear in a forthcoming publication [12].

3. The physically realistic condition that the Fourier decomposition of f contains only a finite number of terms can be weakened. The only condition we use is the fast decay for $|m| \to \infty$ of the Fourier coefficients Q_m of the function $q(t)$ (defined in (1.16)), as found in Proposition 4.1.

4. The second equality in (1.20) is due to (1.8).

5. It is important to stress that conditions I and II are restrictions on the function f and not on the parameter ϵ.

6. Possibly there are other conditions beyond *I* and *II* which could be considered, but they have not been explored so far. They are relevant in some cases. Theorem 1.2 still does not provide a complete solution of (1.4) for all possible periodic functions f, but examples and some qualitative arguments show that the remaining cases are rather exceptional. For instance, for $f(t) = \varphi_1 \cos(\omega t) + \varphi_2 \sin(\omega t)$ condition I covers all pairs $(\varphi_1,\ \varphi_2) \in \mathbb{R}^2$, except only the countable family of circles centered at the origin with radius $x_a\omega/2$, $a = 1, 2, \ldots$, where x_a if the a-th zero of J_0 in $\mathbb{R}_+$ (J_0 is the Bessel function of order zero). However, in these circles condition II is nowhere fulfilled. See the discussion in Section 6.

7. From the computational point of view the solution given by our method can be easily implemented in numerical programs and has been successfully tested, providing ways to study our two-level system for large times with controllable errors (due to the uniform convergence). Results on these numerical studies will be published elsewhere.

8. Unitarity of $U(t)$ for all $t \in \mathbb{R}$ is a consequence of Dyson's expansion (see f.i. [7]).

9. Conditions I and II define, in principle, distinct solutions of the generalized Riccati equation (1.6) and, hence, of the Schrödinger equation (1.4). To fix a name we will call these solutions "classes" of solutions.

1.2 Remarks on the Notation

Let us make some remarks on the notation we use here and recall the notation used in [1]. Given the Fourier representation[3]

$$f(t) \;=\; \sum_{\underset{\sim}{m} \in \mathbb{Z}^B} F_{\underset{\sim}{m}} \, e^{i \underset{\sim}{m} \cdot \underset{\sim}{\omega}_f t} \tag{1.23}$$

of the quasi-periodic function f, we denote (as in [1]) by $\underline{\omega}$ the vector of frequencies defined by

$$\underline{\omega} \;:=\; \begin{cases} \underset{\sim}{\omega}_f \in \mathbb{R}^B, & \text{if } F_{\underset{\sim}{0}} = 0 \\ \\ (\underset{\sim}{\omega}_f, \; F_{\underset{\sim}{0}}) \in \mathbb{R}^{B+1}, & \text{if } F_{\underset{\sim}{0}} \neq 0, \end{cases} \quad . \tag{1.24}$$

Since we assume that $\underset{\sim}{\omega}_f \in \mathbb{R}^B_+$, the definition above says that all components of $\underline{\omega}$ are always non-zero. Moreover, we denote

$$A \;:=\; \begin{cases} B, & \text{if } F_{\underset{\sim}{0}} = 0 \\ \\ B+1, & \text{if } F_{\underset{\sim}{0}} \neq 0 \end{cases} \quad . \tag{1.25}$$

We denote vectors in $\mathbb{Z}^B$ (or $\mathbb{R}^B$) by $\underset{\sim}{v}$ and vectors in $\mathbb{Z}^A$ (or $\mathbb{R}^A$) by $\underline{v}$. The symbol $|\underline{n}|$ denotes the $l^1(\mathbb{Z}^A)$ norm of a vector $\underline{n} = (n_1, \ldots, n_A) \in \mathbb{Z}^A$: $|\underline{n}| := |n_1| + \cdots + |n_A|$. We use the symbol $\mathbb{1}$ for the identity matrix. $\mathrm{Mat}(n, \, \mathbb{C})$ is the set of all $n \times n$ matrices with complex entries.

We denote by $\lfloor x \rfloor$ the largest integer lower or equal to $x \in \mathbb{R}$.

For $m \in \mathbb{Z}$ we denote by $\ll m \gg$ the following function:

$$\ll m \gg := \begin{cases} |m|, & \text{for } m \neq 0 \\ 1, & \text{for } m = 0 \end{cases} \quad . \tag{1.26}$$

For $m \in \mathbb{Z}$ we denote by J_m the Bessel function of first kind and order m.

The symbol □ denotes end of statement and the symbol ■ denotes end of proof.

[3]For convenience we adopt here a different notation as that found in [1], where the Fourier decomposition of f was written as $f(t) \;=\; \sum_{\underset{\sim}{m} \in \mathbb{Z}^B} f_{\underset{\sim}{m}} \, e^{i \underset{\sim}{m} \cdot \underset{\sim}{\omega}_f t}$.

2 Some Previous Results

In [1] some results could be proven about the nature of some particular solutions of (1.6) for the case where f is a quasi-periodic function subjected to some additional restrictions. These results are described in Theorem 2.1.

2.1 Theorem. *Let $f : \mathbb{R} \to \mathbb{R}$ be quasi-periodic with*

$$f(t) \;=\; \sum_{\underset{\sim}{n} \in \mathbb{Z}^B} F_{\underset{\sim}{n}}\, e^{i \underset{\sim}{\omega}_f \cdot \underset{\sim}{n} t},$$

and such that the sum above contains only a finite number of terms. Assume that the vector $\underline{\omega}$ (defined in (1.24)) satisfies Diophantine conditions, i.e., assume the existence of constants $\Delta > 0$ and $\sigma > 0$ such that, for all $\underline{n} \in \mathbb{Z}^A$, $\underline{n} \neq \underline{0}$,

$$|\underline{n} \cdot \underline{\omega}| \geq \Delta^{-1} |\underline{n}|^{-\sigma}.$$

I. *Assume that f satisfies the condition $M(q^2) \neq 0$. Then, there exists a formal power series*

$$g(t) \;=\; q(t) \sum_{n=1}^{\infty} c_n(t) \epsilon^n, \tag{2.1}$$

representing a particular solution of the generalized Riccati equation (1.6) such that all coefficients c_n can be chosen to be quasi-periodic and can be represented as

$$c_n(t) \;=\; \sum_{\underline{m} \in \mathbb{Z}^A} C_{\underline{m}}^{(n)}\, e^{i \underline{m} \cdot \underline{\omega} t}, \tag{2.2}$$

where, for the Fourier coefficients $C_{\underline{m}}^{(n)}$, we have

$$|C_{\underline{m}}^{(n)}| \;\leq\; \mathcal{K}_n e^{-\chi_0 |\underline{m}|},$$

where $\chi_0 > 0$ is a constant and $\mathcal{K}_n \geq 0$.

II. *Assume that f satisfies the conditions $M(q^2) = 0$ and $M(\mathcal{Q}_1) \neq 0$, where $\mathcal{Q}_1$ is defined in (1.19). Then, there exists a formal power series*

$$g(t) \;=\; q(t) \sum_{n=1}^{\infty} e_n(t) \epsilon^{2n}, \tag{2.3}$$

representing a particular solution of the generalized Riccati equation (1.6) such that all coefficients e_n can be chosen to be quasi-periodic and can be represented as

$$e_n(t) \;=\; \sum_{\underline{m} \in \mathbb{Z}^A} E_{\underline{m}}^{(n)}\, e^{i \underline{m} \cdot \underline{\omega} t}, \tag{2.4}$$

where, for the Fourier coefficients $E_{\underline{m}}^{(n)}$, we have

$$|E_{\underline{m}}^{(n)}| \;\leq\; \mathcal{L}_n e^{-\chi_0 |\underline{m}|},$$

where $\chi_0 > 0$ is a constant and $\mathcal{L}_n \geq 0$. □

There are other conditions beyond *I* and *II* which could be considered, but they have not been explored so far. See the discussion in Section 6.

The statements of this last theorem are not sufficient for proving convergence of the power series expansions in ϵ for g. Unfortunately, as discussed in [1], the behavior for large n of the constants $\mathcal{K}_n$ and $\mathcal{L}_n$ is apparently too bad to guarantee absolute convergence of the formal power series above.

For the restricted case were f is periodic we will prove in the present paper stronger results (Theorem 3.1 below) than that implied by Theorem 2.1. As we will see, these stronger results, in contrast, imply convergence of the ϵ-power series for g (Theorem 3.3 below).

Some of the more technical results of [1] have been obtained through the analysis of the Fourier coefficients of the functions c_n and e_n defined in Theorem 2.1 above. Specially important for us are the recursion relations found in [1] for the Fourier coefficients $C^{(n)}_{\underline{m}}$ and $E^{(n)}_{\underline{m}}$ defined in (2.2) and (2.4), respectively. Those recursion relations follow by imposing the generalized Riccati equation (1.6) to the power expansions (2.1) and (2.3). In Appendix A we reproduce some of the main ideas of [1] leading to a power series expansion for g free of secular terms and leading to the recursion relations below.

It is important for our present purposes to reproduce those recursive relations here, what we shall do now.

As in [1], let us denote by $Q_{\underline{m}}$ the Fourier coefficients of the function q (defined in (1.16))

$$q(t) = \sum_{m\in\mathbb{Z}} Q_m e^{im\omega t} \tag{2.5}$$

and by $Q^{(2)}_{\underline{m}}$ the Fourier coefficients of the function q^2. For the Fourier coefficients of the functions c_n we have found the following relations:

$$C^{(1)}_{\underline{m}} = \alpha_1 Q_{\underline{m}}, \tag{2.6}$$

$$C^{(2)}_{\underline{m}} = \sum_{\substack{\underline{n}\in\mathbb{Z}^A \\ \underline{n}\neq\underline{0}}} \frac{\left(\alpha_1^2 Q^{(2)}_{\underline{n}} - \overline{Q^{(2)}_{-\underline{n}}}\right)}{\underline{n}\cdot\underline{\omega}} \left[Q_{\underline{m}-\underline{n}} - \frac{Q_{\underline{m}} Q^{(2)}_{-\underline{n}}}{Q^{(2)}_{\underline{0}}} \right], \tag{2.7}$$

$$C^{(n)}_{\underline{m}} = \sum_{\substack{\underline{n}_1,\,\underline{n}_2\in\mathbb{Z}^A \\ \underline{n}_1+\underline{n}_2\neq\underline{0}}} \frac{1}{(\underline{n}_1+\underline{n}_2)\cdot\underline{\omega}} \left(\sum_{p=1}^{n-1} C^{(p)}_{\underline{n}_1} C^{(n-p)}_{\underline{n}_2} \right) \left[Q_{\underline{m}-(\underline{n}_1+\underline{n}_2)} - \frac{Q_{\underline{m}} Q^{(2)}_{-\underline{n}_1-\underline{n}_2}}{Q^{(2)}_{\underline{0}}} \right]$$

$$- \frac{Q_{\underline{m}}}{2\alpha_1 Q^{(2)}_{\underline{0}}} \sum_{\underline{n}\in\mathbb{Z}^A} \sum_{p=2}^{n-1} C^{(p)}_{\underline{n}} C^{(n+1-p)}_{-\underline{n}}, \qquad \text{for } n\geq 3. \tag{2.8}$$

Above $\underline{m}\in\mathbb{Z}^A$, $\alpha_1^2 = \dfrac{\overline{M(q^2)}}{M(q^2)}$. For the Fourier coefficients of the functions e_n we have found the following relations.

$$E^{(1)}_{\underline{m}} = \sum_{\substack{\underline{n}\in\mathbb{Z}^A \\ \underline{n}\neq\underline{0}}} \frac{Q_{\underline{m}+\underline{n}} \overline{Q^{(2)}_{\underline{n}}}}{\underline{n}\cdot\underline{\omega}} + \frac{Q_{\underline{m}}}{2iM(\mathcal{Q}_1)} \sum_{\substack{\underline{n}_1,\,\underline{n}_2\in\mathbb{Z}^A \\ \underline{n}_1\neq\underline{0},\ \underline{n}_2\neq\underline{0}}} \frac{Q^{(2)}_{\underline{n}_1+\underline{n}_2} \overline{Q^{(2)}_{\underline{n}_1}}\ \overline{Q^{(2)}_{\underline{n}_2}}}{(\underline{n}_1\cdot\underline{\omega})(\underline{n}_2\cdot\underline{\omega})} \tag{2.9}$$

$$E^{(n)}_{\underline{m}} \;=\; \sum_{\substack{\underline{n}_1,\,\underline{n}_2\in\mathbb{Z}^A\\ \underline{n}_1+\underline{n}_2\neq\underline{0}}} \left[Q_{\underline{m}-\underline{n}_1-\underline{n}_2} + \frac{Q_{\underline{m}}}{iM(\mathcal{Q}_1)} \left(Q^{(2)}_{-\underline{n}_1-\underline{n}_2}\mathcal{R} - \sum_{\substack{\underline{n}\in\mathbb{Z}^A\\ \underline{n}\neq\underline{0}}} \frac{Q^{(2)}_{\underline{n}+\underline{n}_1+\underline{n}_2}\overline{Q^{(2)}_{\underline{n}}}}{\underline{n}\cdot\underline{\omega}} \right)\right] \frac{\sum_{p=1}^{n-1} E^{(p)}_{\underline{n}_1}E^{(n-p)}_{\underline{n}_2}}{(\underline{n}_1+\underline{n}_2)\cdot\underline{\omega}}$$

$$+\frac{Q_{\underline{m}}}{2iM(\mathcal{Q}_1)} \sum_{\underline{n}\in\mathbb{Z}^A}\sum_{p=2}^{n-1} E^{(p)}_{\underline{n}}E^{(n+1-p)}_{-\underline{n}}, \qquad \text{for } n\geq 2. \tag{2.10}$$

Above $\underline{m}\in\mathbb{Z}^A$, $\mathcal{Q}_1$ is defined in (1.19) and

$$\mathcal{R} \;:=\; \frac{1}{2iM(\mathcal{Q}_1)} \sum_{\substack{\underline{n}_1,\,\underline{n}_2\in\mathbb{Z}^A\\ \underline{n}_1\neq\underline{0},\;\underline{n}_2\neq\underline{0}}} \frac{Q^{(2)}_{\underline{n}_1+\underline{n}_2}\,\overline{Q^{(2)}_{\underline{n}_1}}\;\overline{Q^{(2)}_{\underline{n}_2}}}{(\underline{n}_1\cdot\underline{\omega})(\underline{n}_2\cdot\underline{\omega})}. \tag{2.11}$$

The above expressions for the Fourier coefficients are somewhat complex but two important features can be distinguished. The first is the inevitable presence of "small denominators", represented by the various factors of the form $(\underline{n}\cdot\underline{\omega})^{-1}$ (with $\underline{n}\neq\underline{0}$) appearing above. The second is the presence of convolution products (a consequence, lately, of the quadratic character of the generalized Riccati equation). The presence of the later is the additional source of complications mentioned before, for they also, together with the small denominators, contribute to spoil the decay of the Fourier coefficients needed to prove convergence of the ϵ-expansions.

3 The Recursive Relations in the Periodic Case

In [1] the recursion relations presented above have been used to prove inductively exponential bounds for the Fourier coefficients. As mentioned before two main difficulties have to be faced in this enterprise: the presence of "small denominators" and of convolution products in the recursion relations. Both are independently responsible for reducing the rate of decay of the Fourier coefficients at each induction step.

Let us consider the origin of the "small denominators problem" in our recursion relations. It comes from the many factors of the form $(\underline{n}\cdot\underline{\omega})^{-1}$ (with $\underline{n}\neq\underline{0}$) appearing in the recursion relations. In the case where f is a periodic function with frequency ω with $F_0\neq 0$, we have $A=2$, $\underline{n}=(n_1,\ n_2)\in\mathbb{Z}^2$ and $\underline{n}\cdot\underline{\omega}=n_1\omega+n_2F_0$. On the other hand, in the case where f is a periodic function with frequency ω and with $F_0=0$, we have $A=1$, $\underline{n}=n\in\mathbb{Z}$ and $\underline{n}\cdot\underline{\omega}=n\omega$. To avoid the quasi-resonant situation where $n_1\omega+n_2F_0$ is small we will, as mentioned, consider in this paper the case where $F_0=0$.

For the Fourier coefficients of the functions c_n, the recursive relations become

$$C^{(1)}_m \;=\; \alpha_1 Q_m, \tag{3.1}$$

$$C^{(2)}_m \;=\; \sum_{\substack{n_1\in\mathbb{Z}\\ n_1\neq 0}} \frac{\left(\alpha_1^2 Q^{(2)}_{n_1} - \overline{Q^{(2)}_{-n_1}}\right)}{n_1\omega}\left[Q_{m-n_1} - \frac{Q_m Q^{(2)}_{-n_1}}{Q^{(2)}_0}\right], \tag{3.2}$$

$$C_m^{(n)} = \sum_{\substack{n_1,\, n_2 \in \mathbb{Z} \\ n_1+n_2 \neq 0}} \frac{1}{(n_1+n_2)\cdot\omega} \left(\sum_{p=1}^{n-1} C_{n_1}^{(p)} C_{n_2}^{(n-p)} \right) \left[Q_{m-(n_1+n_2)} - \frac{Q_m Q_{-n_1-n_2}^{(2)}}{Q_0^{(2)}} \right]$$

$$- \frac{Q_m}{2\alpha_1 Q_0^{(2)}} \sum_{n_1 \in \mathbb{Z}} \sum_{p=2}^{n-1} C_{n_1}^{(p)} C_{-n_1}^{(n+1-p)}, \qquad \text{for } n \geq 3. \tag{3.3}$$

Above $m \in \mathbb{Z}$.

For the Fourier coefficients of the functions e_n we have:

$$E_m^{(1)} = \sum_{\substack{n_1 \in \mathbb{Z} \\ n_1 \neq 0}} \frac{Q_{m+n_1} \overline{Q_{n_1}^{(2)}}}{n_1 \omega} + \frac{Q_m}{2iM(\mathcal{Q}_1)} \sum_{\substack{n_1,\, n_2 \in \mathbb{Z} \\ n_1 \neq 0,\, n_2 \neq 0}} \frac{Q_{n_1+n_2}^{(2)} \overline{Q_{n_1}^{(2)}}\, \overline{Q_{n_2}^{(2)}}}{(n_1\omega)(n_2\omega)} \tag{3.4}$$

$$E_m^{(n)} = \sum_{\substack{n_1,\, n_2 \in \mathbb{Z} \\ n_1+n_2 \neq 0}} \left[Q_{m-n_1-n_2} + \frac{Q_m}{iM(\mathcal{Q}_1)} \left(Q_{-n_1-n_2}^{(2)} \mathcal{R} - \sum_{\substack{n_3 \in \mathbb{Z} \\ n_3 \neq 0}} \frac{Q_{n_3+n_1+n_2}^{(2)} \overline{Q_{n_3}^{(2)}}}{n_3 \omega} \right) \right] \frac{\sum_{p=1}^{n-1} E_{n_1}^{(p)} E_{n_2}^{(n-p)}}{(n_1+n_2)\omega}$$

$$+ \frac{Q_m}{2iM(\mathcal{Q}_1)} \sum_{p=2}^{n-1} \sum_{n_1 \in \mathbb{Z}} E_{n_1}^{(p)} E_{-n_1}^{(n+1-p)}, \qquad \text{for } n \geq 2. \tag{3.5}$$

It is clear here that no "small divisors" appear in this case, since now $|(\underline{n}\cdot\underline{\omega})^{-1}| \geq \omega^{-1}$ for $\underline{n} \neq \underline{0}$. Hence, the convolution products are the only remaining factors eventually forcing the reduction of the decay rate of the Fourier coefficients at the successive induction steps.

In the Section 4 we will show how the effect of the convolution products can be taken under control. The result is expressed in the following three theorems.

3.1 Theorem. *Let $f : \mathbb{R} \to \mathbb{R}$ be periodic with a finite Fourier decomposition as in (1.18) and with $F_0 = 0$.*

Case I. *Consider the Fourier coefficients $C_m^{(n)}$ satisfying the recursion relations (3.1), (3.2) and (3.3). Under the hypothesis that $M(q^2) \neq 0$ we have*

$$|C_m^{(n)}| \leq K_n \frac{e^{-\chi|m|}}{\ll m \gg^2} \tag{3.6}$$

for all $n \in \mathbb{N}$, and all $m \in \mathbb{Z}$, where $\chi > 0$ is a constant and $\ll m \gg$ is defined in (1.26). Above, the coefficients K_n do not depend on m and satisfy the recursion relation

$$K_n = \mathcal{C}_2 \left[\left(\sum_{p=1}^{n-1} K_p K_{n-p} \right) + \left(\sum_{p=2}^{n-1} K_p K_{n+1-p} \right) \right], \tag{3.7}$$

with $K_1 = K_2 = \mathcal{C}_1$, where $\mathcal{C}_1$ and $\mathcal{C}_2$ are positive constants which can be chosen larger than or equal to 1.

Case II. *Consider the Fourier coefficients $E_m^{(n)}$ satisfying the recursion relations (3.4) and (3.5). Under the hypothesis that $M(q^2) = 0$ and $M(\mathcal{Q}_1) \neq 0$ we have*

$$|E_m^{(n)}| \ \leq \ K'_n \, \frac{e^{-\chi|m|}}{\ll m \gg^2} \tag{3.8}$$

for all $n \in \mathbb{N}$, and all $m \in \mathbb{Z}$, where $\chi > 0$ is a constant. Above, the coefficients K'_n do not depend on m and satisfy the recursion relation

$$K'_n \ = \ \mathcal{E}_2 \left[\left(\sum_{p=1}^{n-1} K'_p K'_{n-p} \right) + \left(\sum_{p=2}^{n-1} K'_p K'_{n+1-p} \right) \right] , \tag{3.9}$$

with $K'_1 = K'_2 = \mathcal{E}_1$, where $\mathcal{E}_1$ and $\mathcal{E}_2$ are positive constants which can be chosen larger than or equal to 1. □

Theorem 3.1 will be proven in Section 4. The importance of the recursive definition of the constants K_n given in (3.7) or (3.9) is expressed in the following theorem, which says that the constants K_n grow at most exponentially with n.

3.2 Theorem. *Let the constants K_n be defined through the recurrence relations (3.7) or (3.9). Then there exist constants $K > 0$ and $K_0 > 0$ (depending eventually on f) such that $K_n \leq K_0 K^n$ for all $n \in \mathbb{N}$.* □

The proof of Theorem 3.2 is found in Appendix D and makes interesting use of properties of the Catalan sequence. Theorems 3.1 and 3.2 have the following immediate corollary:

3.3 Theorem. *The power series expansions in (2.1) and (2.3) are absolutely convergent for all $\epsilon \in \mathbb{C}$ with $|\epsilon| < K$ for all $t \in \mathbb{R}$ and, hence, (2.1) and (2.3) define particular solutions of the generalized Riccati equation (1.6) in cases I and II, respectively, of Theorem 3.1. The function g can be expressed in terms of an absolutely and uniformly converging Fourier series whose coefficients can be expressed in terms of absolutely converging power series in ϵ for all $\epsilon \in \mathbb{C}$ with $|\epsilon| < K$.* □

Proof of Theorem 3.3 We prove the statement for case I. Case II is analogous. The first step is to determine the Fourier expansion of the function g, as given in (1.15), and to study some of their properties. One clearly has

$$g(t) \ = \ \sum_{m \in \mathbb{Z}} G_m e^{im\omega t}, \tag{3.10}$$

with

$$G_m \ := \ \sum_{n=1}^{\infty} \epsilon^n G_m^{(n)}, \tag{3.11}$$

where

$$G_m^{(n)} \ := \ \sum_{l \in \mathbb{Z}} Q_{m-l} C_l^{(n)}. \tag{3.12}$$

Now and in future proofs we will make use of the following important lemma, whose proof is given in Appendix C.

3.4 Lemma. *For $\chi > 0$ and $m \in \mathbb{Z}$ define*

$$\mathcal{B}(m) \equiv \mathcal{B}(m,\, \chi) \;:=\; \sum_{n\in\mathbb{Z}} \frac{e^{-\chi(|m-n|+|n|)}}{\ll m-n \gg^2 \, \ll n \gg^2}. \tag{3.13}$$

Then one has

$$\mathcal{B}(m) \;\le\; B_0 \frac{e^{-\chi|m|}}{\ll m \gg^2} \tag{3.14}$$

for some constant $B_0 \equiv B_0(\chi) > 0$ and for all $m \in \mathbb{Z}$. □

We have the following proposition:

3.5 Proposition. *For all $\chi > 0$ there exists a constant $\mathcal{C}_g \equiv \mathcal{C}_g(\chi) > 0$ such that*

$$|G_m^{(n)}| \;\le\; \mathcal{C}_g K_n \frac{e^{-\chi|m|}}{\ll m \gg^2} \tag{3.15}$$

for all $m \in \mathbb{Z}$ and all $n \in \mathbb{N}$. Consequently, for $|\epsilon| < K$ one has

$$|G_m| \;\le\; \mathcal{C}_g' \frac{e^{-\chi|m|}}{\ll m \gg^2} \tag{3.16}$$

for some constant $\mathcal{C}_g'(\chi, \epsilon) > 0$ and for all $m \in \mathbb{Z}$. □

Proof of Proposition 3.5. Inserting (3.6) and (4.1) into (3.12) we have, for any $\chi > 0$

$$\left|G_m^{(n)}\right| \;\le\; K_n \mathcal{Q}\, \mathcal{B}(m,\, \chi), \tag{3.17}$$

where $\mathcal{B}(m,\, \chi)$ is defined in (3.13). Relation (3.15) follows now from Lemma 3.4. ■

From this the rest of the proof of Theorem 3.3 follows immediately. ■

The solutions for the generalized Riccati equation (1.6) mentioned in Theorem 3.3 are, through (1.8), the main ingredient for the solution of the Schrödinger equation (1.4). This will be further discussed in Section 5. Now we have to prove Theorem 3.1.

4 Inductive Bounds for the Fourier Coefficients

In this section we will prove Theorem 3.1 in cases I and II. We will make use of the following proposition on the decay of the Fourier coefficients Q_m and $Q_m^{(2)}$ of the functions q and q^2, respectively. The proof of this proposition appears in Appendix B.

4.1 Proposition. *Let $f : \mathbb{R} \to \mathbb{R}$ be periodic and be represented by a finite Fourier series as in (1.18). Then, for any constant $\chi > 0$ there is a positive constant $\mathcal{Q} \equiv \mathcal{Q}(\chi)$ such that*

$$|Q_m| \ \leq \ \mathcal{Q}\frac{e^{-\chi|m|}}{\ll m \gg^2} \tag{4.1}$$

and

$$|Q_m^{(2)}| \ \leq \ \mathcal{Q}\frac{e^{-\chi|m|}}{\ll m \gg^2} \tag{4.2}$$

for all $m \in \mathbb{Z}$, where $\ll m \gg$ is defined in (1.26). □

4.1 Case I

In this section we will prove Theorem 3.1 in case I. Making use of Proposition 4.1 and of relations (3.1), (3.2) and (3.3) we easily derive the following estimates:

$$|C_m^{(1)}| \ \leq \ \mathcal{Q}\frac{e^{-\chi|m|}}{\ll m \gg^2}, \tag{4.3}$$

$$|C_m^{(2)}| \ \leq \ 2\omega^{-1}\mathcal{Q}\sum_{n_1\in\mathbb{Z}}\frac{e^{-\chi|n_1|}}{\ll n_1 \gg^2}\left[\frac{e^{-\chi|m-n_1|}}{\ll m-n_1 \gg^2} + \frac{\mathcal{Q}}{|Q_0^{(2)}|}\frac{e^{-\chi(|m|+|n_1|)}}{\ll m \gg^2 \ll n_1 \gg^2}\right], \tag{4.4}$$

$$\begin{aligned} |C_m^{(n)}| \ &\leq \ \omega^{-1}\mathcal{Q}\sum_{n_1,\,n_2\in\mathbb{Z}}\left(\sum_{p=1}^{n-1}|C_{n_1}^{(p)}|\,|C_{n_2}^{(n-p)}|\right)\left[\frac{e^{-\chi|m-(n_1+n_2)|}}{\ll m-(n_1+n_2) \gg^2} + \frac{\mathcal{Q}}{|Q_0^{(2)}|}\frac{e^{-\chi(|m|+|n_1+n_2|)}}{\ll m \gg^2 \ll n_1+n_2 \gg^2}\right] \\ &+ \frac{\mathcal{Q}}{2|Q_0^{(2)}|}\frac{e^{-\chi|m|}}{\ll m \gg^2}\sum_{n_1\in\mathbb{Z}}\sum_{p=2}^{n-1}|C_{n_1}^{(p)}|\,|C_{-n_1}^{(n+1-p)}|, \qquad \text{for } n \geq 3. \end{aligned} \tag{4.5}$$

It follows from (4.4), from the definition of $\mathcal{B}(m)$ in (3.13) and from Lemma 3.4 that

$$|C_m^{(2)}| \leq 2\omega^{-1}\mathcal{Q}\left(\mathcal{B}(m) + \frac{\mathcal{Q}}{|Q_0^{(2)}|}\frac{e^{-\chi|m|}}{\ll m \gg^2}\sum_{n_1\in\mathbb{Z}}\frac{e^{-2\chi|n_1|}}{\ll n_1 \gg^4}\right) \leq K_2\frac{e^{-\chi|m|}}{\ll m \gg^2} \tag{4.6}$$

for some convenient choice of the constant K_2.

Now, we will use an induction argument to establish (3.6) for all $n \geq 3$. Let us assume that, for a given $n \in \mathbb{N}$, $n \geq 3$, one has

$$|C_m^{(p)}| \ \leq \ K_p\,\frac{e^{-\chi|m|}}{\ll m \gg^2}, \qquad \forall m \in \mathbb{Z}, \tag{4.7}$$

for all p such that $1 \leq p \leq n-1$, for some convenient constants K_p. We will establish that this implies the same sort of bound for $p = n$. Notice, by taking $K_1 \geq \mathcal{Q}$, that relation (4.3) guarantees (4.7) for $p = 1$ and that relation (4.6) guarantees the case $p = 2$.

From (4.5) and from the induction hypothesis,

$$
\begin{aligned}
|C_m^{(n)}| &\leq \omega^{-1}\mathcal{Q}\left(\sum_{p=1}^{n-1} K_p K_{n-p}\right)\Bigg[\sum_{n_1,\,n_2\in\mathbb{Z}} \frac{e^{-\chi(|m-(n_1+n_2)|+|n_1|+|n_2|)}}{\ll m-(n_1+n_2)\gg^2 \ll n_1\gg^2 \ll n_2\gg^2} \\
&+ \frac{\mathcal{Q}}{|Q_0^{(2)}|}\frac{e^{-\chi|m|}}{\ll m\gg^2}\sum_{n_1,\,n_2\in\mathbb{Z}} \frac{e^{-\chi(|n_1+n_2|+|n_1|+|n_2|)}}{\ll n_1+n_2\gg^2 \ll n_1\gg^2 \ll n_2\gg^2}\Bigg] \\
&+ \frac{\mathcal{Q}}{2|Q_0^{(2)}|}\frac{e^{-\chi|m|}}{\ll m\gg^2}\left(\sum_{p=2}^{n-1} K_p K_{n+1-p}\right)\sum_{n_1\in\mathbb{Z}} \frac{e^{-2\chi|n_1|}}{\ll n_1\gg^4}. \qquad (4.8)
\end{aligned}
$$

Now,

$$
\sum_{n_1,\,n_2\in\mathbb{Z}} \frac{e^{-\chi(|n_1+n_2|+|n_1|+|n_2|)}}{\ll n_1+n_2\gg^2 \ll n_1\gg^2 \ll n_2\gg^2} \qquad \text{and} \qquad \sum_{n_1\in\mathbb{Z}} \frac{e^{-2\chi|n_1|}}{\ll n_1\gg^4}
$$

are just finite constants and

$$
\begin{aligned}
\sum_{n_1,\,n_2\in\mathbb{Z}} \frac{e^{-\chi(|m-(n_1+n_2)|+|n_1|+|n_2|)}}{\ll m-(n_1+n_2)\gg^2 \ll n_1\gg^2 \ll n_2\gg^2} &= \sum_{n_1\in\mathbb{Z}} \frac{e^{-\chi|n_1|}}{\ll n_1\gg^2}\sum_{n_2\in\mathbb{Z}} \frac{e^{-\chi(|(m-n_1)-n_2)|+|n_2|)}}{\ll (m-n_1)-n_2)\gg^2 \ll n_2\gg^2} \\
&= \sum_{n_1\in\mathbb{Z}} \frac{e^{-\chi|n_1|}}{\ll n_1\gg^2}\mathcal{B}(m-n_1) \\
&\leq B_0 \sum_{n_1\in\mathbb{Z}} \frac{e^{-\chi(|n_1|+|m-n_1|)}}{\ll n_1\gg^2 \ll m-n_1\gg^2} \\
&= B_0\mathcal{B}(m) \\
&\leq (B_0)^2\frac{e^{-\chi|m|}}{\ll m\gg^2}, \qquad (4.9)
\end{aligned}
$$

where we again used Lemma 3.4.

Therefore, we conclude

$$
|C_m^{(n)}| \leq \left[\mathcal{C}_a\left(\sum_{p=1}^{n-1} K_p K_{n-p}\right) + \mathcal{C}_b\left(\sum_{p=2}^{n-1} K_p K_{n+1-p}\right)\right]\frac{e^{-\chi|m|}}{\ll m\gg^2}, \qquad (4.10)
$$

for two positive constants $\mathcal{C}_a$ and $\mathcal{C}_b$. Taking $\mathcal{C}_2 := \max\{\mathcal{C}_a,\ \mathcal{C}_b,\ 1\}$ relation (3.7) is proven with $\mathcal{C}_2 \geq 1$.

Notice that, without loss, we are allowed to choose $K_1 = K_2 \geq 1$ by choosing both equal to $\max\{K_1,\ K_2,\ 1\}$. ■

4.2 Case II

In this section we will prove Theorem 3.1 in case II. From (3.4) and (3.5), from Proposition 4.1 and from the assumption (3.8) we have

$$\begin{aligned}
\left|E_m^{(1)}\right| &\leq \frac{\mathcal{Q}^2}{\omega}\sum_{n_1\in\mathbb{Z}} \frac{e^{-\chi(|m+n_1|+|n_1|)}}{\ll m+n_1\gg^2 \ll n_1\gg^2} \\
&+ \frac{\mathcal{Q}^4 e^{-\chi|m|}}{2\ll m\gg^2\,\omega^2|M(\mathcal{Q}_1)|}\sum_{n_1,\,n_2\in\mathbb{Z}} \frac{e^{-\chi(|n_1+n2|+|n_1|+|n_2|)}}{\ll n_1+n_2\gg^2\ll n_1\gg^2\ll n_2\gg^2}\quad, \qquad (4.11)
\end{aligned}$$

$$\begin{aligned}
E_m^{(n)} &= \frac{1}{\omega}\sum_{n_1,\,n_2\in\mathbb{Z}}\Bigg[\mathcal{Q}\frac{e^{-\chi(|m-n_1-n_2|+|n_1|+|n_2|)}}{\ll m-n_1-n_2\gg^2\ll n_1\gg^2\ll n_2\gg^2} \\
&+\frac{\mathcal{Q}^2 e^{-\chi|m|}}{|M(\mathcal{Q}_1)|\ll m\gg^2}\Bigg(\frac{e^{-\chi(|n_1+n_2|+|n_1|+|n_2|)}|\mathcal{R}|}{\ll n_1+n_2\gg^2\ll n_1\gg^2\ll n_2\gg^2} \\
&+\frac{\mathcal{Q}}{\omega}\sum_{n_3\in\mathbb{Z}}\frac{e^{-\chi(|n_1+n_2+n_3|+|n_1|+|n_2|+|n_3|)}}{\ll n_1+n_2+n_3\gg^2\ll n_1\gg^2\ll n_2\gg^2\ll n_3\gg^2}\Bigg)\Bigg]\left(\sum_{p=1}^{n-1}K'_pK'_{n-p}\right) \\
&+\frac{\mathcal{Q}e^{-\chi|m|}}{2|M(\mathcal{Q}_1)|\ll m\gg^2}\left(\sum_{n_1\in\mathbb{Z}}\frac{e^{-2\chi|n_1|}}{\ll n_1\gg^4}\right)\left(\sum_{p=2}^{n-1}K'_pK'_{n+1-p}\right),\qquad \text{for } n\geq 2. \qquad (4.12)
\end{aligned}$$

Sums like

$$\sum_{n_1,\,n_2\in\mathbb{Z}}\frac{e^{-\chi(|n_1+n2|+|n_1|+|n_2|)}}{\ll n_1+n_2\gg^2\ll n_1\gg^2\ll n_2\gg^2}\quad\text{and}\quad\sum_{n_1,\,n_2,\,n_3\in\mathbb{Z}}\frac{e^{-\chi(|n_1+n_2+n_3|+|n_1|+|n_2|+|n_3|)}}{\ll n_1+n_2+n_3\gg^2\ll n_1\gg^2\ll n_2\gg^2\ll n_3\gg^2}$$

are just finite constants. By applying Lemma 3.4 we get

$$|E_m^{(1)}| \leq \mathcal{E}_a\frac{e^{-\chi|m|}}{\ll m\gg^2} \qquad (4.13)$$

$$|E_m^{(n)}| \leq \frac{e^{-\chi|m|}}{\ll m\gg^2}\left[\mathcal{E}_b\left(\sum_{p=1}^{n-1}K'_pK'_{n-p}\right)+\mathcal{E}_c\left(\sum_{p=2}^{n-1}K'_pK'_{n+1-p}\right)\right],\qquad\text{for } n\geq 2, \qquad (4.14)$$

where $\mathcal{E}_a$, $\mathcal{E}_b$ and $\mathcal{E}_c$ are constants. The rest of the proof follows the same steps of the proof of Theorem 3.1 in case I. ■

5 The Fourier Expansion for the Wave Function

Now we return to the discussion of the solution (1.8) of the Schrödinger equation (1.4). Our intention is to find the Fourier expansion of the wave function $\Phi(t)$.

5.1 The Floquet Form of the Wave Function. The Fourier Decomposition and the Secular Frequency

As explained in [1] and in Section 1, the components $\phi_\pm$ of the wave function $\Phi(t)$ are solutions of Hill's equation (1.12). For periodic f the classical theorem of Floquet (see e.g. [10] and [11]) claims that there are particular solutions of equations like (1.12) with the general form $e^{i\Omega t}u(t)$, where $u(t)$ is periodic with the same period of f. In order to preserve unitarity we must have $\Omega \in \mathbb{R}$. This form of the particular solutions is called the "Floquet form" and the frequencies Ω are called "secular frequencies".

In this section we will recover the Floquet form of the wave function in terms of Fourier expansions and we will find out expansions for the secular frequencies as converging power series expansions in ϵ.

According to the solution expressed in relation (1.7) and (1.8), we have first to find out the Fourier expansion for the functions R and S defined in (1.9) and (1.10), respectively.

We begin with the function R. The Fourier expansion of the function $f + g$ is

$$f(t) + g(t) \;=\; \Omega + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} (F_n + G_n(\epsilon))\, e^{in\omega t}, \tag{5.1}$$

where

$$\Omega \;\equiv\; \Omega(\epsilon) \;:=\; G_0(\epsilon). \tag{5.2}$$

One has,

$$R(t) \;=\; e^{-i\gamma_f(\epsilon)}\; e^{-i\Omega t}\; \exp\left(-\sum_{n\in\mathbb{Z}} H_n e^{in\omega t}\right) \tag{5.3}$$

with

$$H_n \;\equiv\; H_n(\epsilon) \;:=\; \begin{cases} \dfrac{F_n + G_n(\epsilon)}{n\omega}, & \text{for } n \neq 0 \\[2ex] 0, & \text{for } n = 0 \end{cases}, \tag{5.4}$$

and

$$\gamma_f(\epsilon) \;:=\; i \sum_{m\in\mathbb{Z}} H_m. \tag{5.5}$$

Notice that $\gamma_f(0) = \gamma_f$, where γ_f is defined in (B.4).

Since we are assuming that there are only finitely many non-vanishing coefficients F_n, we have the following proposition as an obvious corollary of Proposition 3.5:

5.1 Proposition. *For all $\chi > 0$ and $|\epsilon|$ small enough, there exists a constant $\mathcal{C}_H \equiv \mathcal{C}_H(\chi,\, \epsilon) > 0$ such that*

$$|H_m| \;\leq\; \mathcal{C}_H \frac{e^{-\chi|m|}}{\ll m \gg^2} \tag{5.6}$$

for all $m \in \mathbb{Z}$. □

Writing now the Fourier expansion of $R(t)$ in the form

$$R(t) \; = \; e^{-i\Omega t}\sum_{n\in\mathbb{Z}} R_n e^{in\omega t} \tag{5.7}$$

we find from (5.3)

$$R_n \equiv R_n(\epsilon) = \begin{cases} e^{-i\gamma_f(\epsilon)}\left(1+\displaystyle\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{(p+1)!}\sum_{n_1,\ldots,\,n_p\in\mathbb{Z}} H_{n_1}\cdots H_{n_p}H_{-N_p}\right), & \text{for } n=0, \\ e^{-i\gamma_f(\epsilon)}\left(-H_n+\displaystyle\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{(p+1)!}\sum_{n_1,\ldots,\,n_p\in\mathbb{Z}} H_{n_1}\cdots H_{n_p}H_{n-N_p}\right), & \text{for } n\neq 0. \end{cases} \tag{5.8}$$

with

$$N_p \; := \; \sum_{a=1}^{p} n_a, \tag{5.9}$$

for $p \geq 1$.

In order to compute the Fourier expansion of S we have to compute first the Fourier expansion of R^{-2}. This is now an easy task, since the replacement $R(t) \to R(t)^{-2}$ corresponds to the replacement $(f+g) \to -2(f+g)$ and, hence, to $H_n \to -2H_n$. We get

$$R(t)^{-2} \; = \; e^{2i\Omega t}\sum_{n\in\mathbb{Z}} R_n^{(-2)} e^{in\omega t}, \tag{5.10}$$

with

$$R_n^{(-2)} \; \equiv \; R_n^{(-2)}(\epsilon) \; := \begin{cases} e^{2i\gamma_f(\epsilon)}\left(1+\displaystyle\sum_{p=1}^{\infty}\frac{2^{p+1}}{(p+1)!}\sum_{n_1,\ldots,\,n_p\in\mathbb{Z}} H_{n_1}\cdots H_{n_p}H_{-N_p}\right), & \text{for } n=0, \\ e^{2i\gamma_f(\epsilon)}\left(2H_n+\displaystyle\sum_{p=1}^{\infty}\frac{2^{p+1}}{(p+1)!}\sum_{n_1,\ldots,\,n_p\in\mathbb{Z}} H_{n_1}\cdots H_{n_p}H_{n-N_p}\right), & \text{for } n\neq 0. \end{cases} \tag{5.11}$$

The following proposition will be used below.

5.2 Proposition. *For all $\chi > 0$ and $|\epsilon|$ small enough, there exist constants $\mathcal{C}_R \equiv \mathcal{C}_R(\chi,\,\epsilon) > 0$ and $\mathcal{C}_{R^{(-2)}} \equiv \mathcal{C}_{R^{(-2)}}(\chi,\,\epsilon) > 0$ such that*

$$|R_m| \; \leq \; \mathcal{C}_R \frac{e^{-\chi|m|}}{\langle\!\langle m\rangle\!\rangle^2} \tag{5.12}$$

$$|R_m^{(-2)}| \; \leq \; \mathcal{C}_{R^{(-2)}} \frac{e^{-\chi|m|}}{\langle\!\langle m\rangle\!\rangle^2} \tag{5.13}$$

for all $m \in \mathbb{Z}$. □

Proof of Proposition 5.2. Using Proposition 5.1 we have, for any $p \geq 1$,

$$\left| \sum_{n_1,\ldots,\, n_p \in \mathbb{Z}} H_{n_1} \cdots H_{n_p} \, H_{n-N_p} \right| \leq (\mathcal{C}_H)^{p+1} \sum_{n_1,\ldots,\, n_p \in \mathbb{Z}} \frac{\exp\left(-\chi(|n_1| + \cdots + |n_p| + |n - n_1 - \cdots - n_p|)\right)}{\left(\ll n_1 \gg \cdots \ll n_p \gg \ll n - n_1 - \cdots - n_p \gg\right)^2}. \tag{5.14}$$

Making repeated use of Lemma 3.4 on the right hand side of (5.14) we get

$$\left| \sum_{n_1,\ldots,\, n_p \in \mathbb{Z}} H_{n_1} \cdots H_{n_p} \, H_{n-N_p} \right| \leq \frac{(\mathcal{C}_H B_0)^{p+1}}{B_0} \frac{e^{-\chi|n|}}{\ll n \gg^2}. \tag{5.15}$$

Inserting this into (5.8) gives (since $B_0 > 1$)

$$|R_n| \quad \leq \quad \left(\frac{e^{\left|\mathrm{Im}(\gamma_f(\epsilon))\right| + \mathcal{C}_H B_0}}{B_0} \right) \frac{e^{-\chi|n|}}{\ll n \gg^2} \tag{5.16}$$

for all $n \in \mathbb{Z}$, as desired. The proof for $R_n^{(-2)}$ is analogous. ■

Assuming for a while

$$n\omega + 2\Omega \neq 0 \qquad \text{for all } n \in \mathbb{Z}, \tag{5.17}$$

we have[4]

$$S(t) \; = \; \sigma_0 + e^{2i\Omega t} \sum_{n \in \mathbb{Z}} S_n e^{in\omega t} \tag{5.18}$$

with

$$S_n \; := \; -i \frac{R_n^{(-2)}}{n\omega + 2\Omega} \qquad \text{and} \qquad \sigma_0 \; := \; -\sum_{n \in \mathbb{Z}} S_n. \tag{5.19}$$

Assumption (5.17) is actually a consequence of unitarity, as will be discussed in Section 5.2.

The following proposition is an elementary corollary of Proposition 5.2:

5.3 Proposition. *For all $\chi > 0$ and $|\epsilon|$ small enough, there exists a constant $\mathcal{C}_S \equiv \mathcal{C}_S(\chi, \epsilon) > 0$ such that*

$$|S_m| \; \leq \; \mathcal{C}_S \frac{e^{-\chi|m|}}{\ll m \gg^2} \tag{5.20}$$

for all $m \in \mathbb{Z}$. □

Writing

$$U(t) \; = \; \begin{pmatrix} U_{11}(t) & U_{12}(t) \\ U_{21}(t) & U_{22}(t) \end{pmatrix} \; = \; \begin{pmatrix} U_{11}(t) & U_{12}(t) \\ -\overline{U_{12}(t)} & \overline{U_{11}(t)} \end{pmatrix}, \tag{5.21}$$

[4] For the case $n = 0$, (5.17) says that $\Omega \neq 0$. This must hold except for $\epsilon = 0$ when $\Omega = 0$.

we have for U_{11} and U_{12}:

$$U_{11}(t) \;=\; e^{-i\Omega t}\, u_{11}^{-}(t) + e^{i\Omega t}\, u_{11}^{+}(t) \tag{5.22}$$

$$U_{12}(t) \;=\; e^{-i\Omega t}\, u_{12}^{-}(t) + e^{i\Omega t}\, u_{12}^{+}(t) \tag{5.23}$$

with

$$\begin{aligned} u_{11}^{-}(t) &:= (1 + ig(0)\sigma_0)\, r(t), \qquad & u_{11}^{+}(t) &:= ig(0)\, v(t), \\ u_{12}^{-}(t) &:= -i\epsilon\sigma_0\, r(t), \qquad & u_{12}^{+}(t) &:= -i\epsilon\, v(t), \end{aligned} \tag{5.24}$$

for

$$r(t) \;:=\; \sum_{n\in\mathbb{Z}} R_n\, e^{in\omega t} \qquad \text{and} \qquad v(t) \;:=\; \sum_{n\in\mathbb{Z}} V_n\, e^{in\omega t}, \tag{5.25}$$

with

$$V_n \;:=\; \sum_{m\in\mathbb{Z}} S_{n-m} R_m. \tag{5.26}$$

This provides the desired Floquet form for the components of the wave function $\Phi(t)$. We notice from the expressions above that the secular frequencies are $\pm\Omega$. For Ω we have the ϵ-expansion

$$\Omega = \sum_{n=1}^{\infty} \epsilon^n G_0^{(n)}, \tag{5.27}$$

and for $g(0)$,

$$g(0) \;=\; \sum_{m\in\mathbb{Z}} G_m \;=\; \sum_{n=1}^{\infty} \epsilon^n \sum_{m\in\mathbb{Z}} G_m^{(n)}. \tag{5.28}$$

Both converge absolutely for $|\epsilon| < K$, where K is mentioned in Theorem 3.2.

As before, we have the following corollary of Propositions 5.2, 5.3 and Lemma 3.4:

5.4 Proposition. *For all $\chi > 0$ and $|\epsilon|$ small enough, there exists a constant $\mathcal{C}_V \equiv \mathcal{C}_V(\chi,\, \epsilon) > 0$ such that*

$$|V_m| \;\leq\; \mathcal{C}_V \frac{e^{-\chi|m|}}{\ll m \gg^2} \tag{5.29}$$

for all $m \in \mathbb{Z}$. □

This last proposition closed the proof of Theorem 1.2.

5.2 Remarks on the Unitarity of the Propagator

The unitarity of the propagator $U(t)$ means $U(t)^* U(t) = \mathbb{1}$. After (5.21), this means

$$|U_{11}(t)|^2 + |U_{12}(t)|^2 \;=\; 1. \tag{5.30}$$

Looking at relations (5.22) and (5.23) two conclusions can be drawn from (5.30). The first is the following proposition:

5.5 Proposition. *For $\epsilon \in \mathbb{R}$ and under the hypothesis leading to (5.22) and (5.23) one has $\Omega \in \mathbb{R}$.* □

The proof follows from the obvious observation that (5.30) would be violated for $|t|$ large enough if Ω had a non-vanishing imaginary part. Unfortunately a proof of this fact using directly the ϵ-expansion of Ω (5.27) is difficult and has not been found yet.

The second conclusion is that (5.17) indeed holds. For, without this assumption there would be a term linear in t in (5.18), violating (5.30) for large $|t|$.

As in the case of Proposition 5.5, no direct proof of this fact out of the ϵ-expansion for Ω (5.27) has been found yet. The proof will probably follow the idea that $|\Omega|$ is always smaller than 2ω because Ω is of order ϵ and $|\epsilon|$ has to be chosen small in order to provide convergence for the expansions. Analogously $\Omega \neq 0$ because Ω is analytic in ϵ and, hence, has isolated zeros. If the analyticity domain must be small enough no zeros occur, except at $\epsilon = 0$.

6 Discussion on the Classes of Solutions

Let us now discuss some aspects of conditions I and II of Theorem 1.2. It is important to stress that these conditions are restrictions on the function f and not on the parameter ϵ.

As in (B.1), let us write the Fourier decomposition of f as

$$f(t) = \sum_{a=1}^{2J} f_a e^{i n_a \omega t}, \tag{6.1}$$

with $n_a = -n_{2J-a+1}$ and $\overline{f_a} = f_{2J-a+1}$ for all a with $1 \leq a \leq J$. Comparing with (1.18) one has $f_a \equiv F_{n_a}$, $1 \leq a \leq J$.

Hence, for $F_0 = 0$ and for fixed J and ω, there are J independent complex coefficients f_a and we can identify the parameter space $\mathbb{R}^{2J}$ with the set $\mathfrak{F}_{J,\omega}$ of all possible functions f with a given J and ω.

Condition $M(q^2) = 0$ determines a $(2J-1)$ or $(2J-2)$-dimensional subset of $\mathfrak{F}_{J,\omega}$ and there condition II applies. It is also on this subset that the more restrictive condition $M(q^2) = M(\mathcal{Q}_1) = 0$ should hold, restricting the parameter space of f to a $(2J-2)$, $(2J-3)$ or $(2J-4)$-dimensional subset. Hence, successive conditions like I and II would eventually exhaust completely the parameter space $\mathfrak{F}_{J,\omega}$.

Conditions beyond I and II have not been yet analysed and many questions concerning the classes of solutions are still open. For instance, will further conditions like I and II really exhaust the parameter space of the functions f? Will the subtraction method of [1] and the convergence proofs of the present paper also work under these further conditions? What are the physically qualitative distinctions between the classes? Are these classes of solutions in some sense analytic continuations of each other?

A distinction between class I and II may be pointed with the observation that in class I we have power expansions in ϵ while in II we have power expansions in ϵ^2. Compare relations (2.1) and (2.3) of Theorem 2.1.

6.1 An Explicit Example

To illustrate these ideas and point to some problems let us consider the important example where f is given by

$$f(t) \;=\; \varphi_1 \cos(\omega t) + \varphi_2 \sin(\omega t), \tag{6.2}$$

$\varphi_1,\ \varphi_2 \in \mathbb{R}$. We have $f(t) = f_1 e^{-i\omega t} + f_2 e^{i\omega t}$ with $f_1 = (\varphi_1 + i\varphi_2)/2,\ f_2 = \overline{f_1},\ J = 1,\ n_1 = -1,$ $n_2 = 1$. Applying now (B.5) for this case with $m = 0$ we get

$$M(q^2) \;=\; Q_0^{(2)} \;=\; e^{2i\gamma_f} \sum_{p=0}^{\infty} \frac{(-1)^p}{(p!)^2} \left(\frac{4|f_1|}{2\omega}\right)^{2p} \;=\; e^{2i\gamma_f} J_0\left(\frac{2\varphi_0}{\omega}\right), \tag{6.3}$$

where $\varphi_0 := \sqrt{\varphi_1^2 + \varphi_2^2}$ and where J_0 is the Bessel function of first kind and order zero. In this case $\gamma_f = \varphi_2/\omega$.

Relation (6.3) shows that condition I is not empty and that the locus in the $(\varphi_1,\ \varphi_2)$-space of the condition $M(q^2) = 0$ (necessary for condition II) is the countable family of circles centered at the origin with radius $x_a \omega/2$, $a = 1, 2, \ldots$, where x_a if the a-th zero of J_0 in $\mathbb{R}_+$.

One shows analogously that

$$Q_m \;=\; e^{i\gamma_f} \left(\frac{\overline{f_1}}{|f_1|}\right)^m J_m\left(\frac{2|f_1|}{\omega}\right) \tag{6.4}$$

and

$$Q_m^{(2)} \;=\; e^{2i\gamma_f} \left(\frac{\overline{f_1}}{|f_1|}\right)^m J_m\left(\frac{4|f_1|}{\omega}\right), \tag{6.5}$$

for all $m \in \mathbb{Z}$.

For $Q_0^{(2)} = 0$ the function $\mathcal{Q}_1$ is periodic and we have in general

$$M(\mathcal{Q}_1) \;=\; \frac{i}{\omega} \sum_{\substack{m\in\mathbb{Z} \\ m\neq 0}} \frac{\left|Q_m^{(2)}\right|^2}{m} \;=\; \frac{i}{\omega} \sum_{m=1}^{\infty} \left(\frac{\left|Q_m^{(2)}\right|^2 - \left|Q_{-m}^{(2)}\right|^2}{m} \right) \tag{6.6}$$

Since $|J_m(x)| = |J_{-m}(x)|$ for all $x \in \mathbb{R}$, $\forall m \in \mathbb{Z}$, it follows that $|Q_m^{(2)}| = |Q_{-m}^{(2)}|$, $\forall m \in \mathbb{Z}$. Hence, for functions f like (6.2)

$$M(\mathcal{Q}_1) \;=\; 0. \tag{6.7}$$

Therefore, condition II is nowhere fulfilled. For a complete solution of the problem for functions like (6.2), including the circles mentioned above, higher restrictions than that implied by condition II are necessary.

6.2 A Second Example

For functions f with $J > 1$ the situation leading to (6.7) is not expected in general and condition II, and eventually others, may hold in non-empty regions of the parameter space of f. This can be seen in the following example with $J = 2$. Let us take

$$f(t) \;=\; f_1(t) + f_2(t) \tag{6.8}$$

with

$$f_1(t) = f_1 e^{-i\omega t} + \overline{f_1} e^{i\omega t} \tag{6.9}$$

$$f_2(t) = f_2 e^{-i2\omega t} + \overline{f_2} e^{i2\omega t} \tag{6.10}$$

$f_i \in \mathbb{C}$, $i = 1, 2$. We have $q(t) = q_1(t)q_2(t)$, where

$$q_1(t) := e^{i\gamma_{f_1}} \sum_{n\in\mathbb{Z}} e^{in\zeta_1} J_n\left(\frac{2|f_1|}{\omega}\right) e^{in\omega t}, \tag{6.11}$$

$$q_2(t) := e^{i\gamma_{f_2}} \sum_{n\in\mathbb{Z}} e^{in\zeta_2} J_n\left(\frac{|f_2|}{\omega}\right) e^{in2\omega t}, \tag{6.12}$$

with

$$e^{i\zeta_i} = \frac{\overline{f_i}}{|f_i|}, \qquad i = 1, 2.$$

It follows that

$$Q_m = e^{i(\gamma_{f_1}+\gamma_{f_2})} \sum_{k\in\mathbb{Z}} e^{i((m-2k)\zeta_1 + k\zeta_2)} J_{m-2k}\left(\frac{2|f_1|}{\omega}\right) J_k\left(\frac{|f_2|}{\omega}\right), \tag{6.13}$$

$$Q_m^{(2)} = e^{2i(\gamma_{f_1}+\gamma_{f_2})} \sum_{k\in\mathbb{Z}} e^{i((m-2k)\zeta_1 + k\zeta_2)} J_{m-2k}\left(\frac{4|f_1|}{\omega}\right) J_k\left(\frac{2|f_2|}{\omega}\right). \tag{6.14}$$

From this we see (using $J_{-n}(x) = (-1)^n J_n(x)$) that

$$\overline{Q_{-m}^{(2)}} = (-1)^m e^{-4i(\gamma_{f_1}+\gamma_{f_2})} \left\{ e^{2i(\gamma_{f_1}+\gamma_{f_2})} \sum_{k\in\mathbb{Z}} (-1)^k e^{i((m-2k)\zeta_1 + k\zeta_2)} J_{m-2k}\left(\frac{4|f_1|}{\omega}\right) J_k\left(\frac{2|f_2|}{\omega}\right) \right\}. \tag{6.15}$$

The factor between brackets differs from $Q_m^{(2)}$ due to the presence of the factor $(-1)^k$ in the sum over $k \in \mathbb{Z}$. Hence, we should rather expect $|Q_m^{(2)}| \neq |Q_{-m}^{(2)}|$ in this case, what most likely implies $M(\mathcal{Q}_1) \neq 0$ for $M(q^2) = 0$, leading to a non-empty condition II.

Appendices

A Short Description of the Strategy Followed in [1]

For convenience of the reader we reproduce the main steps of the strategy developed in [1] for finding a power series solution of the generalized Riccati equation (1.6) without secular terms.

As discussed in Section 1, a natural proposal is to express g, a particular solution of (1.6), as a formal power expansion on ϵ which vanishes at $\epsilon = 0$. For convenience, we write this expansion as in (1.15) where $q(t)$ is defined in (1.16). This would give the desired solution, provided the infinite sum converges. Inserting (1.15) into (1.6) leads to

$$\sum_{n=1}^{\infty} \left((qc_n)' - i \sum_{p=1}^{n-1} q^2 c_p c_{n-p} - 2ifqc_n \right) \epsilon^n + i\epsilon^2 = 0. \tag{A.1}$$

Assuming that the coefficients vanish order by order we conclude

$$(qc_1)' - 2ifqc_1 = 0, \tag{A.2}$$

$$(qc_2)' - iq^2c_1^2 - 2ifqc_2 + i = 0, \tag{A.3}$$

$$(qc_n)' - i \sum_{p=1}^{n-1} q^2 c_p c_{n-p} - 2ifqc_n = 0, \quad n \geq 3. \tag{A.4}$$

The solutions of (A.2)-(A.3) are

$$c_1(t) \;=\; \alpha_1 \, q(t), \tag{A.5}$$

$$c_2(t) \;=\; q(t) \left[i \int_0^t \left(\alpha_1^2 q(t')^2 - q(t')^{-2} \right) dt' + \alpha_2 \right], \tag{A.6}$$

$$c_n(t) \;=\; q(t) \left[i \left(\sum_{p=1}^{n-1} \int_0^t c_p(t') c_{n-p}(t') \, dt' \right) + \alpha_n \right], \quad \text{for } n \geq 3, \tag{A.7}$$

where the α_n's above, $n = 1, 2, \ldots$, are arbitrary integration constants.

The key idea is to fix the integration constants α_i in such a way as to eliminate the constant terms from the integrands in (A.6) and (A.7). The remaining terms involve sums of exponentials like $e^{in\omega t}$, $n \neq 0$, which do not develop secular terms when integrated, in contrast to the constant terms. For instance, fixing α_1 such that $M(\alpha_1^2 q^2 - q^{-2}) = 0$, that means, $\alpha_1^2 = M(q^{-2})/M(q^2)$, prevents secular terms in (A.6).

As shown in [1] this procedure can be implemented in all orders, fixing all constants α_i and preventing secular terms in all functions $c_n(t)$. In case I, relations (2.6)-(2.8) represent precisely relations (A.5)-(A.7) in Fourier space with the integration constants fixed as explained above. Case II is analogous.

B The Decay of the Fourier Coefficients of q and q^2

To prove our main results on the Fourier coefficients of the functions c_n and e_n we have to establish some results on the decay of the Fourier coefficients of q and q^2.

We write the Fourier series (1.18) of f in the form[5]

$$f(t) \ = \ \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} F_n e^{in\omega t},$$

with $\overline{F_n} = F_{-n}$, since f is real. In order to simplify our analysis we will consider here the case where the sum above is a finite sum. This situation is physically more realistic anyway.

By assumption, the set of integers $\{n \in \mathbb{Z}|\ F_n \neq 0\}$ is a finite set and, by the condition that f is real and $F_0 = 0$, it contains an even number of elements, say $2J$ with $J \geq 1$. Let us write this set of integers as $\{n_1, \ldots, n_{2J}\}$ and write

$$f(t) = \sum_{a=1}^{2J} f_a e^{in_a \omega t}, \tag{B.1}$$

with the convention that $n_a = -n_{2J-a+1}$, for all $1 \leq a \leq J$, with $f_a \equiv F_{n_a}$. Clearly $\overline{f_a} = f_{2J-a+1}$, $1 \leq a \leq J$.

A simple computation (see [1]) now shows that q has a Fourier decomposition as in (2.5) with

$$Q_m \ = \ e^{i\gamma_f} \sum_{p_1,\,\ldots,\,p_{2J}=0}^{\infty} \delta\,(P,\ m) \prod_{a=1}^{2J} \left[\frac{1}{p_a!}\left(\frac{f_a}{n_a\omega}\right)^{p_a}\right], \tag{B.2}$$

where

$$P \ \equiv \ P(p_1, \ldots, p_{2J}, n_1, \ldots, n_{2J}) \ := \ \sum_{b=1}^{2J} p_b n_b \in \mathbb{Z}, \tag{B.3}$$

and where

$$\gamma_f \ := \ i\sum_{a=1}^{2J} \frac{f_a}{n_a \omega}. \tag{B.4}$$

As one easily sees, $\gamma_f \in \mathbb{R}$. Above $\delta\,(P,\ m)$ is the Krönecker delta:

$$\delta\,(P,\ m) \ := \ \begin{cases} 1, & \text{if } P = m, \\ 0, & \text{else.} \end{cases}$$

[5] As above, here we adopt $F_0 = 0$.

Since the function q^2 is obtained from q by replacing $f \to 2f$ we have from (B.2)

$$Q_m^{(2)} \; = \; e^{2i\gamma_f} \sum_{p_1,\, \ldots,\, p_{2J}=0}^{\infty} \delta\left(P,\; m\right) \prod_{a=1}^{2J} \left[\frac{1}{p_a!} \left(\frac{2f_a}{n_a \omega} \right)^{p_a} \right], \tag{B.5}$$

where $Q_m^{(2)}$ are the Fourier coefficients of q^2. The coefficients Q_m and $Q_m^{(2)}$ can also be expressed in terms of Bessel functions of the first kind and integer order. See Section 6 for some examples.

As in [1], define

$$\varphi \; := \; \max_{1 \leq a \leq 2J} \left| \frac{f_a}{n_a \omega} \right| .$$

and

$$\mathcal{N} \; := \; \sum_{b=1}^{2J} |n_b| .$$

Notice that, since the n_b's are fixed by the choice of f, $\mathcal{N}$ is non-zero.

The following important bounds have been proven in [1], Appendix D:

$$|Q_m| \; \leq \; \left(2J e^{(2J-1)\varphi} \right) \; \frac{\varphi^{\lceil \mathcal{N}^{-1}|m| \rceil}}{\lceil \mathcal{N}^{-1}|m| \rceil !} \; \left(1 - \frac{\varphi}{\lceil \mathcal{N}^{-1}|m| \rceil + 1} \right)^{-1} , \tag{B.6}$$

and

$$|Q_m^{(2)}| \; \leq \; \left(2J e^{(2J-1)2\varphi} \right) \; \frac{(2\varphi)^{\lceil \mathcal{N}^{-1}|m| \rceil}}{\lceil \mathcal{N}^{-1}|m| \rceil !} \; \left(1 - \frac{2\varphi}{\lceil \mathcal{N}^{-1}|m| \rceil + 1} \right)^{-1} , \tag{B.7}$$

for all m with $\lceil \mathcal{N}^{-1}|m| \rceil + 1 > 2\varphi$. Above $\lceil x \rceil$ is the lowest integer larger than or equal to x.

In [1] we derived from (B.6) a simple exponential bound for $|Q_m|$, namely,

$$|Q_m| \; \leq \; \mathcal{Q}\, e^{-\chi |m|}, \tag{B.8}$$

where $\mathcal{Q}$ and χ are some positive constants. For the purposes of this paper a sharper bound than (B.8) is needed and we have to study relation (B.6) more carefully. The result is expressed in Proposition 4.1 whose proof we present now.

Proof of Proposition 4.1. Let us consider first the coefficients Q_m. Due to the dominating factor $\lceil \mathcal{N}^{-1}|m| \rceil !$, one has

$$\lim_{|m| \to \infty} \frac{\ll m \gg^2}{e^{-\chi |m|}} \frac{\varphi^{\lceil \mathcal{N}^{-1}|m| \rceil}}{\lceil \mathcal{N}^{-1}|m| \rceil !} \; = \; 0.$$

for any constant $\chi > 0$. Hence, one can choose a constant $M_1 > 0$ depending on χ such that

$$\frac{\varphi^{\lceil \mathcal{N}^{-1}|m| \rceil}}{\lceil \mathcal{N}^{-1}|m| \rceil !} \leq M_1 \, \frac{e^{-\chi |m|}}{\ll m \gg^2}$$

for all $m \in \mathbb{Z}$. Therefore, there exists a positive constant $\mathcal{Q}_1 > 0$ (depending on χ) such that $|Q_m| \leq \mathcal{Q}_1 \ll m \gg^{-2} e^{-\chi |m|}$ for all $m \in \mathbb{Z}$. For $Q_m^{(2)}$ we proceed in the same way and get the bound $|Q_m^{(2)}| \leq \mathcal{Q}_2 \ll m \gg^{-2} e^{-\chi |m|}$ for all $m \in \mathbb{Z}$. In (4.1) and (4.2) we adopt $\mathcal{Q} = \max\{\mathcal{Q}_1,\; \mathcal{Q}_2\}$. ■

Remark. The proof of Proposition 4.1 shows that we have also sharper bounds like

$$|Q_m| \; \leq \; \mathcal{Q}_k \, \frac{e^{-\chi |m|}}{\ll m \gg^k}$$

for any $k \in \mathbb{N}$. For the purposes of the present paper it was enough to consider $k = 2$.

C Bounds on Convolutions

Here we will prove Lemma 3.4. Consider for $\chi > 0$ and $m \in \mathbb{Z}$

$$\mathcal{B}(m) \equiv \mathcal{B}(m, \chi) := \sum_{n\in\mathbb{Z}} \frac{e^{-\chi(|m-n|+|n|)}}{\langle\!\langle m-n \rangle\!\rangle^2 \, \langle\!\langle n \rangle\!\rangle^2}. \tag{C.1}$$

First notice that $\mathcal{B}(m) = \mathcal{B}(-m)$ for all $m \in \mathbb{Z}$. Choosing B_0 to be such that

$$B_0 \geq \sum_{n\in\mathbb{Z}} \frac{e^{-2\chi|n|}}{\langle\!\langle n \rangle\!\rangle^4}$$

the statement of the lemma becomes trivially correct for $m = 0$. Hence, it is enough to consider the case where $m > 0$.

In (C.1), the sum over all $n \in \mathbb{N}$ can be split into three sums:

$$\mathcal{B}(m) \;=\; e^{-\chi m} \sum_{n=-\infty}^{-1} \frac{e^{2\chi n}}{(m-n)^2 n^2} + e^{-\chi m} \sum_{n=0}^{m} \frac{1}{\langle\!\langle m-n \rangle\!\rangle^2 \, \langle\!\langle n \rangle\!\rangle^2} + e^{\chi m} \sum_{n=m+1}^{\infty} \frac{e^{-2\chi n}}{(m-n)^2 n^2} \tag{C.2}$$

In the first sum above we perform the change of variables $n \to -n$ and in the third sum we perform the change of variables $n \to n+m$. The result is

$$\mathcal{B}(m) \;=\; e^{-\chi m} \left(2 \sum_{n=1}^{\infty} \frac{e^{-2\chi n}}{(m+n)^2 n^2} + \sum_{n=0}^{m} \frac{1}{\langle\!\langle m-n \rangle\!\rangle^2 \, \langle\!\langle n \rangle\!\rangle^2} \right) \tag{C.3}$$

Now we will study separately each of the sums in (C.3). Since for $n \geq 1$ one has $m+n \geq \langle\!\langle m \rangle\!\rangle$ one has for the first sum

$$\sum_{n=1}^{\infty} \frac{e^{-2\chi n}}{(m+n)^2 n^2} \leq \frac{B_1}{\langle\!\langle m \rangle\!\rangle^2} \tag{C.4}$$

where $B_1 := \displaystyle\sum_{n=1}^{\infty} \frac{e^{-2\chi n}}{n^2}$.

The second sum in (C.3) is a little more involving. We have

$$\sum_{n=0}^{m} \frac{1}{\langle\!\langle m-n \rangle\!\rangle^2 \, \langle\!\langle n \rangle\!\rangle^2} = \sum_{n=0}^{\lfloor m/2 \rfloor} \frac{1}{\langle\!\langle m-n \rangle\!\rangle^2 \, \langle\!\langle n \rangle\!\rangle^2} + \sum_{n=\lfloor m/2 \rfloor+1}^{m} \frac{1}{\langle\!\langle m-n \rangle\!\rangle^2 \, \langle\!\langle n \rangle\!\rangle^2} \tag{C.5}$$

For the first sum in the right hand side of (C.5) we have $\langle\!\langle m-n \rangle\!\rangle \geq m-n \geq m - \lfloor m/2 \rfloor \geq m/2$. For the second sum in the right hand side of (C.5) we have $n \geq \lfloor m/2 \rfloor + 1 \geq m/2$. Hence, for $m > 0$,

$$\begin{aligned} \sum_{n=0}^{m} \frac{1}{\langle\!\langle m-n \rangle\!\rangle^2 \, \langle\!\langle n \rangle\!\rangle^2} \;&\leq\; \left(\frac{2}{m}\right)^2 \left[\sum_{n=0}^{\lfloor m/2 \rfloor} \frac{1}{\langle\!\langle n \rangle\!\rangle^2} + \sum_{n=\lfloor m/2 \rfloor+1}^{m} \frac{1}{\langle\!\langle m-n \rangle\!\rangle^2} \right] \\ &\leq\; 2 \left(\frac{2}{\langle\!\langle m \rangle\!\rangle} \right)^2 \sum_{n=0}^{\infty} \frac{1}{\langle\!\langle n \rangle\!\rangle^2} \end{aligned} \tag{C.6}$$

Therefore, choosing

$$B_0 = 2B_1 + 8\sum_{n=0}^{\infty} \frac{1}{\ll n \gg^2} \tag{C.7}$$

the lemma is proven. ■

The proof of this lemma has the following proposition as corollary, generalizing Lemma 3.4:

C.1 Proposition. *For $\chi > 0$, $k \in \mathbb{N}$, $k \geq 2$, let*

$$\mathcal{B}_k(m) \;:=\; \sum_{n\in\mathbb{Z}} \frac{e^{-\chi(|m-n|+|n|)}}{\ll m-n \gg^k \ll n \gg^k}. \tag{C.8}$$

Then, there exists a constant $B_{0,\,k}$, depending eventually on k, such that

$$\mathcal{B}_k(m) \;\leq\; B_{0,\,k}\; \frac{e^{-\chi|m|}}{\ll m \gg^k} \tag{C.9}$$

for all $m \in \mathbb{Z}$. □

D Catalan Numbers. Bounds on the Constants K_n

Here we will prove Theorem 3.2. Let us start recalling that we have chosen $K_1 = K_2 = \mathcal{C}_1$ for some constant $\mathcal{C}_1$ which, in turn, can be chosen without loss to be larger than or equal to 1. The proof of Theorem 3.2 will be presented on four steps.

Step 1. In this step we show that the sequence K_n, defined in (3.7), is an increasing sequence.

First notice that $K_3 = \mathcal{C}_2(2K_1K_2 + (K_2)^2) = 3\mathcal{C}_2(K_2)^2$. Since $K_1 = K_2 \geq 1$ and $\mathcal{C}_2 \geq 1$, we have $K_1 = K_2 < K_3$.

Let us now suppose that

$$K_1 = K_2 < K_3 < \cdots < K_n \tag{D.1}$$

for some $n \geq 3$. We will show that $K_{n+1} > K_n$. We have

$$\begin{aligned}
K_{n+1} - K_n \;&=\; \mathcal{C}_2\left[\sum_{p=1}^{n} K_pK_{n-p+1} + \sum_{p=2}^{n} K_pK_{n-p+2} - \sum_{p=1}^{n-1} K_pK_{n-p} - \sum_{p=2}^{n-1} K_pK_{n-p+1}\right] \\
&=\; \mathcal{C}_2\left[2K_1K_n + \sum_{p=2}^{n} K_pK_{n-p+2} - \sum_{p=1}^{n-1} K_pK_{n-p}\right] \\
&=\; \mathcal{C}_2\left[2K_1K_n + (K_2K_n - K_{n-2}K_1) + (K_3 - K_1)K_{n-1} + \cdots + (K_n - K_{n-2})K_2\right] \\
&=\; \mathcal{C}_2\left[2K_1K_n + (K_n - K_{n-2})K_1 + (K_3 - K_1)K_{n-1} + \cdots + (K_n - K_{n-2})K_2\right],
\end{aligned}$$

where in the last equality we used $K_1 = K_2$. Now, from hypothesis (D.1) we conclude that $K_{n+1} > K_n$, thus proving that K_n is an increasing sequence.

Step 2. Here we show that the sequence K_n defined in (3.7) satisfies

$$K_n \;\leq\; 3\mathcal{C}_2 \sum_{p=2}^{n-1} K_p K_{n-p+1} \tag{D.2}$$

for all $n \geq 3$.

We have already shown that $K_3 = 3\mathcal{C}_2 (K_2)^2$. Hence, (D.2) is obeyed for $n = 3$.

Assume now that (D.2) is satisfied for all K_p with $p \in \{1, \ldots, n-1\}$, for some $n \geq 4$. We will show that it is also satisfied for K_n. In fact, we have from (3.7)

$$K_n \;=\; \mathcal{C}_2 \left[K_1 K_{n-1} + K_2(K_{n-2} + K_{n-1}) + K_3(K_{n-3} + K_{n-2}) + \cdots + K_{n-1}(K_1 + K_2)\right]. \tag{D.3}$$

From this and from the fact proven in step 1 that the sequence K_n is increasing, it follows that

$$K_n \;\leq\; \mathcal{C}_2 \left[K_1 K_{n-1} + 2\left(K_2 K_{n-1} + K_3 K_{n-2} + \cdots + K_{n-1} K_2\right)\right] \tag{D.4}$$

Now, using the obvious relation

$$K_1 K_{n-1} = K_2 K_{n-1} \leq \left(K_2 K_{n-1} + K_3 K_{n-2} + \cdots + K_{n-1} K_2\right)$$

we get finally from (D.4)

$$K_n \;\leq\; 3\mathcal{C}_2 \left[K_2 K_{n-1} + K_3 K_{n-2} + \cdots + K_{n-1} K_2\right] \;=\; 3\mathcal{C}_2 \sum_{p=2}^{n-1} K_p K_{n-p+1}, \tag{D.5}$$

thus proving (D.2).

Step 3. Here we will prove the following statement. Let L_n be defined as the sequence such that $L_1 = L_2 = K_1 = K_2 = \mathcal{C}_1$ and

$$L_n \;=\; 3\mathcal{C}_2 \sum_{p=2}^{n-1} L_p L_{n-p+1}. \tag{D.6}$$

Then, one has

$$K_n \leq L_n, \qquad \forall n \in \mathbb{N}. \tag{D.7}$$

First notice that $K_3 = 3\mathcal{C}_2 (K_1)^2 = 3\mathcal{C}_2 (L_1)^2 = L_3$. Hence, (D.7) is valid for $n \in \{1,\ 2,\ 3\}$. Now suppose $K_p \leq L_p$ for all $p \in \{1, \ldots, n-1\}$ for some $n \geq 4$. One has from (D.2)

$$K_n \;\leq\; 3\mathcal{C}_2 \sum_{p=2}^{n-1} K_p K_{n-p+1} \;\leq\; 3\mathcal{C}_2 \sum_{p=2}^{n-1} L_p L_{n-p+1} \;=\; L_n, \tag{D.8}$$

thus proving (D.7).

Step 4. Consider the sequence $\mathbf{c}_n$ defined as follows: $\mathbf{c}_1 = \mathbf{c}_2 = 1$ and

$$\mathbf{c}_n \;=\; \sum_{p=2}^{n-1} \mathbf{c}_p \mathbf{c}_{n-p+1} \tag{D.9}$$

for $n \geq 3$. The so defined numbers $\mathbf{c}_n$ are called "Catalan numbers", after the mathematician Eugène C. Catalan. The Catalan numbers arise in several combinatorial problems (for a historical account with proofs, see [8]) and can be expressed in a closed form as

$$\mathbf{c}_n = \frac{(2n-4)!}{(n-1)!(n-2)!}, \qquad n \geq 2. \tag{D.10}$$

(see, f.i, [8] or [9]). Using Stirling's formula we get the following asymptotic behaviour for the Catalan numbers:

$$\mathbf{c}_n \approx \frac{1}{16\sqrt{\pi}} \frac{4^n}{n^{3/2}}, \qquad n \text{ large.} \tag{D.11}$$

The existence of a connection between the Catalan numbers and the sequence L_n defined above is evident. Two distinctions are the factor $3\mathcal{C}_2$ appearing in (D.6) and the fact that $L_1 = L_2 = \mathcal{C}_1$ is not necessarily equal to 1. Nevertheless, using the definition of the Catalan numbers in (D.9), it is easy to prove the following closed expression for the numbers L_n:

$$L_n = (\mathcal{C}_1)^{n-1} (3\mathcal{C}_2)^{n-2} \frac{(2n-4)!}{(n-1)!(n-2)!}, \qquad n \geq 2. \tag{D.12}$$

We omit the proof here. Hence, the following asymptotic behaviour can be established:

$$L_n \approx \frac{1}{144\mathcal{C}_1\mathcal{C}_2^2\sqrt{\pi}} \frac{(12\mathcal{C}_1\mathcal{C}_2)^n}{n^{3/2}}, \qquad n \text{ large.} \tag{D.13}$$

From the inequality $K_n \leq L_n$, proven in step 3, it follows that $K_n \leq K_0(12\mathcal{C}_1\mathcal{C}_2)^n$ for some constant $K_0 > 0$, for all $n \in \mathbb{N}$. Theorem 3.2 is now proven. ■

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