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# On the $q$ -deformation of the NJL model

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Using a  $q$ -deformed fermionic algebra we perform explicitly a deformation of the Nambu–Jona-Lasinio (NJL) Hamiltonian. In the Bogoliubov–Valatin approach we obtain the deformed version of the functional for the total energy, which is minimized to obtain the corresponding gap equation. The breaking of chiral symmetry and its restoration in the limit  $q \rightarrow 0$  are then discussed.

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One of the most beautiful aspects in physics is the appearance of concepts which are universal in the sense that they are common to many different branches in physics. A very important phenomenon in this context is the dynamical symmetry breaking, which appears in areas as different as statistical mechanics or nuclear and particle physics. In the last few years, results obtained in the treatment of many body systems when the underlying algebra is deformed suggests that the appearance of symmetry breaking in this new framework might be an universal aspect as well.

The chiral symmetry breaking in Quantum Chromodynamics (QCD) due to the appearance of the quark condensates, and its restoration at high temperature is of fundamental importance in medium and high energy physics. These mechanisms are responsible for the mass generation of the more fundamental constituents of matter.

On the other hand, the  $q$ -deformed algebras have been an alternative and elegant way to investigate the symmetry breaking process in different areas of physics [1,2]. It has been shown that the  $q$ -deformation of the fermionic algebra produces changes in the creation and annihilation operators [3]. Frequently, physical quantities depend on the action of these operators in some physical state and therefore they must be sensitive to changes in the operators' definition.

The aim of this work is to apply this prescription to hadronic systems by performing directly the  $q$ -deformation in the Hamiltonian that describes a particular system. For this purpose we have chosen the NJL model [4], which has become a popular effective model for QCD, due to its simplicity and at the same time the richness in describing some features which are very difficult to reproduce directly from the more fundamental quantum chromodynamics. For instance, the dynamical breaking of chiral symmetry and its restoration at large temperature and density are very well described using such an effective model [5,6].

In a recent work [2], the effect of a  $q$ -deformation in the NJL gap equation was studied through a  $q$ -deformed calculation of the quark condensates leading to an enhancement of the dynamical mass. This can be understood as a result of a larger effective coupling between the quarks in the deformed case. In this case, the deformed gap equation is given by

$$m = -2G \langle \bar{\psi}\psi \rangle_q, \quad (1)$$

where  $\langle \bar{\psi}\psi \rangle_q$  stands for the  $q$ -deformed calculation of the quark condensates.

In this work we use the same deformation procedure as in the previous paper but, instead of deforming the gap equation, we apply the  $q$ -deformation in a more fundamental way by deforming directly the NJL Hamiltonian, which is the starting point to obtain the gap equation in the variational Bogoliubov–Valatin approach [7]. Here it is important to note that this approach is completely different from simply deforming the condensate in the gap equation, as will be seen in our results.

The Hamiltonian of the Nambu–Jona-Lasinio model is given by

$$\mathcal{H}_{NJL} = -i\bar{\psi}\gamma \cdot \nabla\psi - G \left[ (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\tau\psi)^2 \right], \quad (2)$$

and corresponds to the following Lagrangian

$$\mathcal{L}_{NJL} = \bar{\psi}i\gamma^\mu\partial_\mu\psi + G \left[ (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\tau\psi)^2 \right]. \quad (3)$$

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This Lagrangian is constructed from contact interactions in such way that it has the main symmetries of QCD. As it is well known, one of the most important features of the QCD Lagrangian is that it has chiral symmetry, which is the most important symmetry concerning the dynamics of the lightest hadrons.

The  $q$ -deformed fermionic algebra that we shall use is based on the work of Ubricco [3], where the thermodynamic properties of a many fermion system were studied. An extension of this procedure was used in the construction of a  $q$ -covariant form of the BCS approximation [8], and further applied to the NJL gap equation [2]. As a consequence, this deformation procedure only modifies negative helicity quarks (anti-quarks) operators.

Making use of the  $q$ -deformed creation and annihilation operators we write the modified quark fields as

$$\psi_q(x, 0) = \sum_s \int \frac{d^3p}{(2\pi)^3} [B(\mathbf{p}, s)u(\mathbf{p}, s)e^{i\mathbf{p}\cdot\mathbf{x}} + D^\dagger(\mathbf{p}, s)v(\mathbf{p}, s)e^{-i\mathbf{p}\cdot\mathbf{x}}]. \quad (4)$$

The  $q$ -deformed quark and anti-quark creation and annihilation operators  $B$ ,  $B^\dagger$ ,  $D$ , and  $D^\dagger$ , are expressed in terms of the non-deformed ones

$$B_- = b_- (1 + Qb_+^\dagger b_+) \quad , \quad B_-^\dagger = b_-^\dagger (1 + Qb_+^\dagger b_+) \quad , \quad (5)$$

$$D_- = d_- (1 + Qd_+^\dagger d_+) \quad , \quad D_-^\dagger = d_-^\dagger (1 + Qd_+^\dagger d_+) \quad , \quad (6)$$

where  $Q = q^{-1} - 1$ . The positive helicity operators are not modified [2].

The variational approach to obtain the gap equation consists on the following procedure: a) to define a variational vacuum, b) to calculate the vacuum expectation value of the Hamiltonian, obtaining the functional for the total energy, and c) to minimize the functional, obtaining the variational parameters and the gap equation.

Following the above steps, we now define our variational BCS-like vacuum

$$|NJL\rangle = \prod_{\mathbf{p}, s=\pm 1} [\cos \theta(p) + s \sin \theta(p) b^\dagger(\mathbf{p}, s) d^\dagger(-\mathbf{p}, s)] |0\rangle, \quad (7)$$

which, for a given momentum  $\mathbf{p}$ , is expanded as

$$\begin{aligned} |NJL\rangle = & \cos^2 \theta(p) |0\rangle \\ & + \sin \theta(p) \cos \theta(p) b^\dagger(\mathbf{p}, +) d^\dagger(-\mathbf{p}, +) |0\rangle \\ & - \sin \theta(p) \cos \theta(p) b^\dagger(\mathbf{p}, -) d^\dagger(-\mathbf{p}, -) |0\rangle \\ & - \sin^2 \theta(p) b^\dagger(\mathbf{p}, -) d^\dagger(-\mathbf{p}, -) b^\dagger(\mathbf{p}, +) d^\dagger(-\mathbf{p}, +) |0\rangle. \end{aligned} \quad (8)$$

Here it is important to note that the deformed version of this vacuum differs from the non-deformed one only by a phase and, therefore, the effects of the deformation comes solely from the deformed component of the Hamiltonian.

The deformed functional for the total energy will be obtained from the vacuum expectation value of the  $q$ -deformed NJL Hamiltonian

$$\mathcal{W}^q[\theta(p)] = \langle NJL | \mathcal{H}_{NJL}^q | NJL \rangle, \quad (9)$$

where

$$\mathcal{H}_{NJL}^q = -i\bar{\psi}_q \gamma \cdot \nabla \psi_q - G (\bar{\psi}_q \psi_q)^2, \quad (10)$$

and  $\psi_q$  is given by Eq. (4). Due to the additive structure of the  $q$ -deformation in Eq. (5) and Eq. (6), the deformed Hamiltonian can be written as

$$\mathcal{H}_{NJL}^q = \mathcal{H}_{NJL} + H(Q), \quad (11)$$

and the functional reads

$$\mathcal{W}^q[\theta(p)] = \mathcal{W}[\theta(p)] + W[Q, \theta(p)]. \quad (12)$$

The last terms of Eqs. (11) and (12), namely  $H(Q)$  and  $W[Q, \theta(p)]$ , stand for the new terms of first order in  $Q$  generated when the algebra is deformed, and therefore, they must vanish for  $q = 1$  ( $Q = 0$ ). Table I shows the increase in the number of operators in the NJL Hamiltonian as also in its matrix elements, due to the deformation of the fermionic algebra. The hard task is then to find the non-vanishing matrix elements of the  $q$ -deformed Hamiltonian.

In the non-deformed case, which corresponds to  $q = 1$  ( $Q = 0$ ), the total energy is given by

$$\mathcal{W}[\theta(p)] = -2N_c N_f \int \frac{d^3 p}{(2\pi)^3} p \cos 2\theta(p) - 4G (N_c N_f)^2 \left[ \int \frac{d^3 p}{(2\pi)^3} \sin 2\theta(p) \right]^2. \quad (13)$$

The minimization of this functional

$$\frac{\delta \mathcal{W}[\theta(p)]}{\delta [2\theta(p)]} = 0, \quad (14)$$

leads to the NJL gap equation

$$p \tan 2\theta(p) = 4GN_c N_f \int \frac{d^3 p'}{(4\pi)^3} \sin 2\theta(p'), \quad (15)$$

which takes its more familiar form

$$m = 4GN_c N_f \int \frac{d^3 p}{(2\pi)^3} \frac{m}{\sqrt{\mathbf{p}^2 + m^2}}, \quad (16)$$

provided the variational angles acquire the following structure

$$\tan 2\theta(p) = \frac{m}{p}, \quad \sin 2\theta(p) = \frac{m}{\sqrt{\mathbf{p}^2 + m^2}}, \quad \cos 2\theta(p) = \frac{p}{\sqrt{\mathbf{p}^2 + m^2}}. \quad (17)$$

Calculating the new matrix elements arising from the  $q$ -deformation of the NJL Hamiltonian, and adding them up to the non-deformed functional, we obtain the full  $q$ -deformed functional for the total energy

$$\mathcal{W}^q[\theta(p)] = -2N_c N_f \int \frac{d^3 p}{(2\pi)^3} P_q \cos 2\theta(p) - 4G' (N_c N_f)^2 \left[ \int \frac{d^3 p}{(2\pi)^3} \sin 2\theta(p) \right]^2. \quad (18)$$

As in the non-deformed case, the same minimization procedure yields to

$$P_q \tan 2\theta(p) = 4G' N_c N_f \int \frac{d^3 p'}{(4\pi)^3} \sin 2\theta(p'), \quad (19)$$

which becomes

$$M = 4G' N_c N_f \int \frac{d^3 p}{(2\pi)^3} \frac{M}{\sqrt{\mathbf{P}_q^2 + M^2}}. \quad (20)$$

The variational angles have the same old structure but now are  $q$ -dependent

$$\tan 2\theta_q(p) = \frac{M}{P_q}, \quad \sin 2\theta_q(p) = \frac{M}{\sqrt{\mathbf{P}_q^2 + M^2}}, \quad \cos 2\theta_q(p) = \frac{P_q}{\sqrt{\mathbf{P}_q^2 + M^2}}, \quad (21)$$

where the new variables appearing in the deformed equations are defined as

$$P = \left(1 + \frac{Q}{2}\right) p, \quad (22)$$

$$P_0 = \frac{N_c N_f Q}{3\pi^2} G \Lambda^3, \quad (23)$$

$$P_q = P - P_0, \quad (24)$$

$$G' = G \left(1 + \frac{Q}{4}\right). \quad (25)$$

It is easy to see that, when  $q \rightarrow 1$  ( $Q \rightarrow 0$ ), Eqs. (19), (20), and (21) reduce to their non-deformed versions Eqs. (15), (16), and (17), since  $P \rightarrow p$ ,  $P_0 \rightarrow 0$ ,  $P_q \rightarrow p$ , and  $G' \rightarrow G$ .

In analogy with the non-deformed case we can write the gap equation in terms of the quark condensates as

$$M = -2G' \langle \bar{\Psi}\Psi \rangle. \quad (26)$$

Comparing the two forms of the gap equation Eqs. (20) and (26), we find a new deformed condensate given by

$$\langle \bar{\Psi}\Psi \rangle = -\frac{N_c}{\pi^2} \int_0^\Lambda d^3p \frac{M}{\sqrt{\mathbf{P}_q^2 + M^2}} \quad (27)$$

for each quark flavour. This condensate is different from the one obtained in our previous work, where the condensate was explicitly deformed [2]. It also has exactly the same form of the non-deformed one, but is written in terms of the new variables. It is worth to mention that the new condensate is not obtained by calculating the vacuum expectation value of a deformed scalar density, it corresponds to the gap equation which arises from the variational procedure started from the  $q$ -deformed Hamiltonian.

We can also obtain a new pion decay constant in analogy to the non-deformed case <sup>1</sup>

$$F_\pi^2 = N_c M^2 \int_0^\Lambda \frac{d^3p}{(2\pi)^3} \frac{1}{(\mathbf{P}_q^2 + M^2)^{3/2}}. \quad (28)$$

In the non-deformed case, a coupling constant equal to  $G/G_c = 0.75$  leads to a pion decay constant  $F_\pi = 88$  MeV. By setting the deformation parameter to  $q = 1.2$ , the pion decay constant is shifted to  $F_\pi = 92$  MeV which is very close to experimental value  $f_\pi = 93$  MeV. This is in agreement with the results obtained in the context of chiral perturbation theory by Gasser and Leutwyler [9], where the calculated pion decay constant is reduced by 6% becoming  $F_\pi = 88$  MeV when current quark masses  $m_{u,d}$  are set to zero.

The variational angle plays an important role in the chiral symmetry breaking process. When  $2\theta = 0$ , there is no breaking of chiral symmetry and therefore no dynamical mass is generated. The generation of the dynamical mass is associated to a chiral rotation and the presence of a current quark mass can be associated to a non-vanishing angle, namely to a permanent chiral rotation. The study of the behavior of the variational angle provides an interesting way to observe the dynamical chiral symmetry breaking in the Bogoliubov-Valatin approach to the NJL model.

By fixing the dynamical mass we can use Eqs. (21) to obtain the  $q$ -dependence of the variational angle, which is shown in Fig. (1) for different values of the momentum. Then, starting from the fixed value of the dynamical mass, we use the  $q$ -dependence of  $2\theta$  to obtain the mass generated when the fermionic algebra is deformed. In Fig. (2) we show the difference between the fixed value of the dynamical mass and the  $q$ -dependent one, compared to typical values for the current  $u$  and  $d$  masses.

The NJL phase transition can be studied by solving the gap equation Eq. (20) and calculating the quark condensate  $\langle \bar{\Psi}\Psi \rangle$ , which is the chiral order parameter. If we solve this  $q$ -deformed self-consistent equation in terms of the new variables defined above ( $P_q, E_q$ ) the phase transition will look exactly like in the non-deformed case, since Eq. (20) is identical to usual gap equation Eq. (16). However, if we solve it in terms of the old variables ( $p, E$ ) we can see the effect of the  $q$ -deformation in the phase transition. The curves shown in Fig. (3) for different values of  $q$  are compared to the previous approach, where the  $q$ -deformation was performed only in the quark condensates [2]. This feature can be understood as follows. We have two separated scenarios: the non-deformed and the  $q$ -deformed one. In both situations the gap equation has exactly the same form and yields the same results. The effects of the deformation are observed when we express the gap equation of the deformed case in terms of the original physical quantities of the non-deformed one. For  $q > 1$  the value of the quark condensates and the dynamical mass increase with the deformation and are larger than in the case where only the quark condensates were deformed. For  $q \lesssim 1$  we have the opposite effect and the value of the condensate decreases. It is therefore tempting to explore the behavior of the condensate for smaller values of  $q$ , even considering that the truncation at order  $Q$  may not be granted. In this case, we can see that the chiral symmetry is restored in the limit  $q \rightarrow 0$ , since the condensate vanishes. The value of the condensate for  $q < 1$  is shown in Fig. (4), and in Fig. (5) we can see the chiral symmetry restoration at small values of the deformation parameter  $q$  at fixed value of the coupling constant.

The chiral symmetry restoration here is different from the obtained at finite temperature [5,6]. It seems to be important an investigation of the effect of both temperature and  $q$ -deformation in the chiral symmetry restoration process. This study is in progress and will be left for a future publication [10].

<sup>1</sup>In Ref. [2] the factor  $N_c m^2$  is missing in Eq. (24).

So far we have performed the  $q$ -deformation of the NJL Hamiltonian and used with the variational Bogoliubov-Valatin approach to obtain the  $q$ -deformed functional that leads to a new gap equation. As far as effects of the deformation are concerned, our main conclusions can be summarized as follows.

In this approach the variational angles become  $q$ -dependent, meaning that the dynamical mass generation is affected by the  $q$ -deformation. The effect of the deformation is to enhance the condensate and the dynamical mass for  $q > 1$ , and to restore chiral symmetry when  $q \rightarrow 0$ . Here the effect is stronger than in the case where the deformation is performed directly into the gap equation. In terms of the new  $q$ -deformed variables the functional for the total energy, the gap equation, the variational angles, and the quark condensates have the same form of the non-deformed case, which is a consequence of the quantum group invariance of the NJL Lagrangian.

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$\mathcal{H}_{NJL}$	Without	With	New	Order Q
$-i\bar{\psi}\gamma \cdot \nabla\psi$	16	36	20	16
$-G\bar{\psi}\psi\bar{\psi}\psi$	256	1296	1040	520
$\langle NJL   -i\bar{\psi}\gamma \cdot \nabla\psi   NJL \rangle$	256	576	320	256
$\langle NJL   -G\bar{\psi}\psi\bar{\psi}\psi   NJL \rangle$	4096	20736	16640	8320

TABLE I. Number of terms of the Hamiltonian and its matrix elements without deformation, with deformation, the new terms, and the terms of first order in  $Q$ .

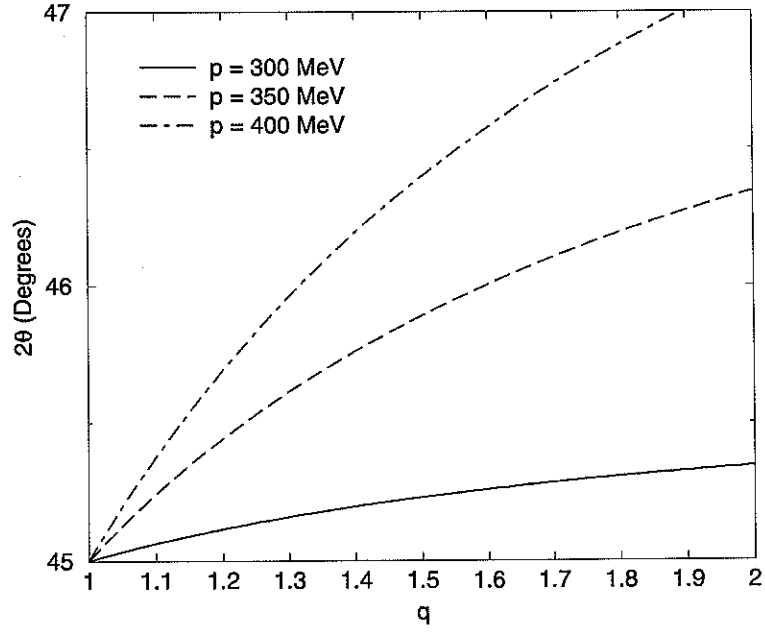


FIG. 1. The variational angle  $2\theta$  as a function of the deformation parameter  $q$  when the dynamical mass is 300 MeV.

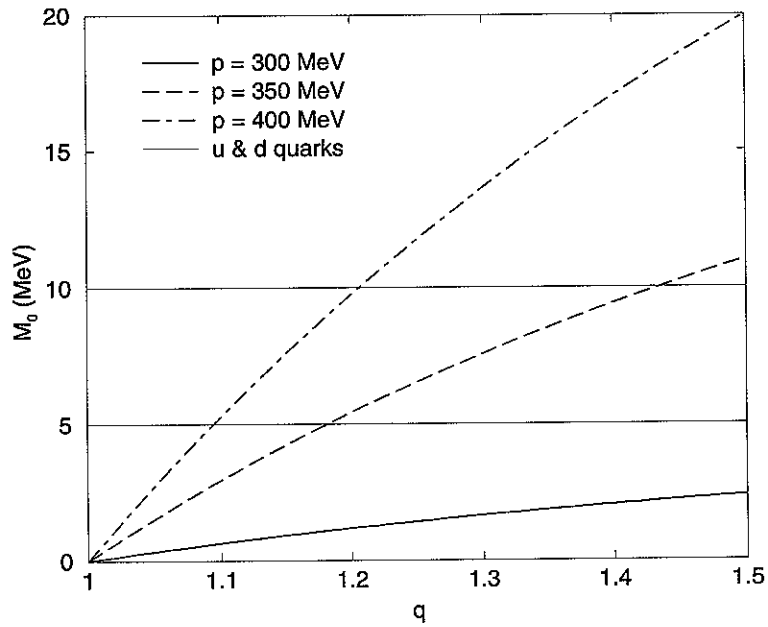


FIG. 2. The corresponding generated mass as a function of the deformation parameter  $q$  when the momentum is fixed. When  $M_0 = 0$  the dynamical mass is 300 MeV.

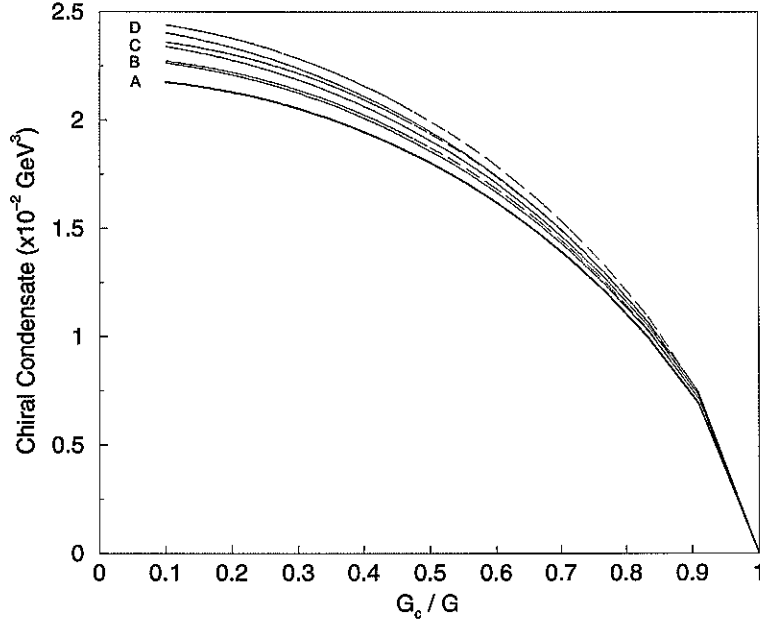


FIG. 3. The NJL phase transition. Four sets of curves are presented. The correspond to: (A) the non-deformed case  $q = 1$ , (B)  $q = 1.1$ , (C)  $q = 1.2$ , and (D)  $q = 1.3$ . In each set (B,C,D) the lower curve (continuous line) is obtained by deforming directly the gap equation, and the upper one (dashed line) corresponds to the results obtained from the deformation of the NJL Hamiltonian.

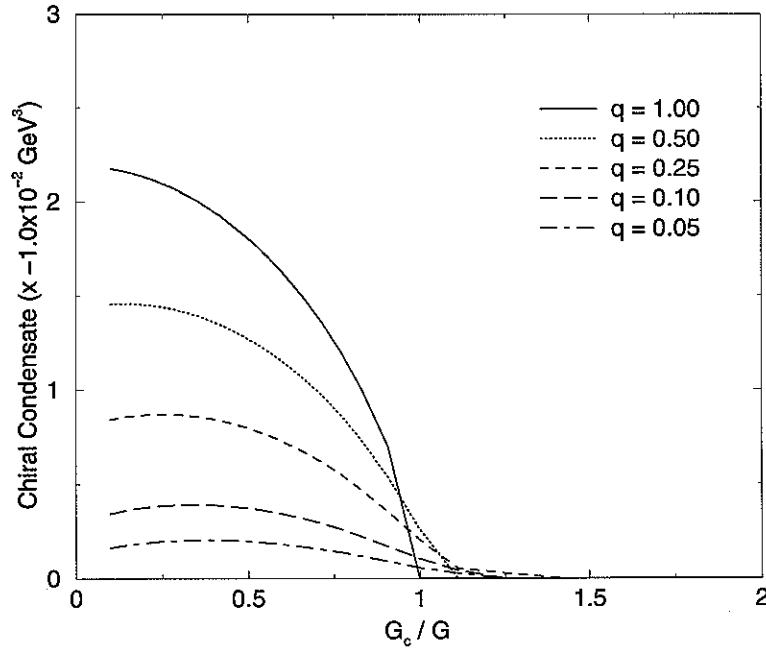


FIG. 4. The condensate for  $0 < q < 1$ .



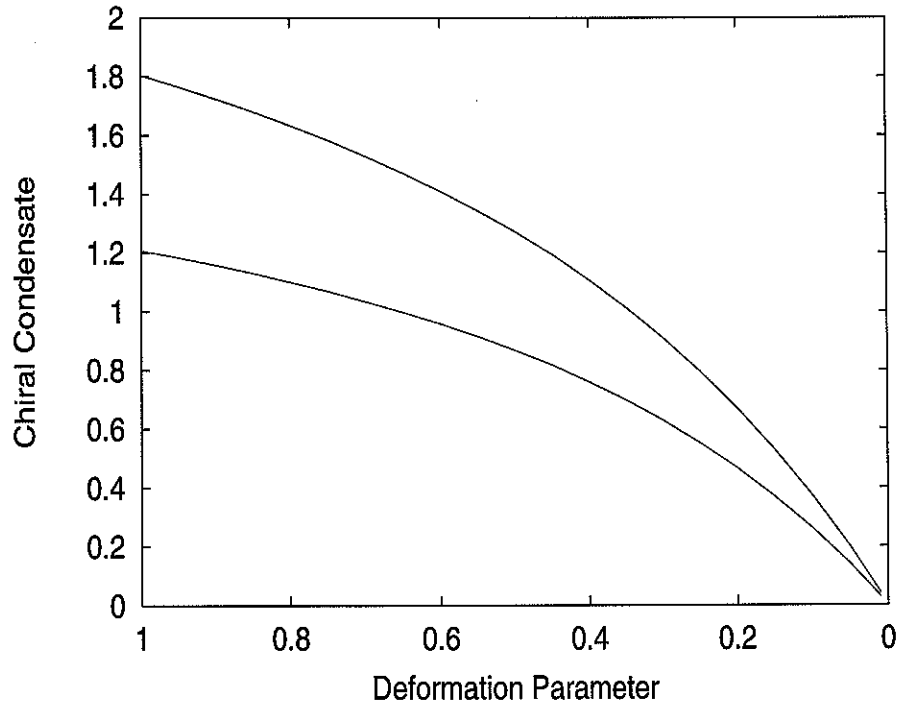


FIG. 5. Chiral symmetry restoration as  $q \rightarrow 0$ . The solid and dashed lines correspond to  $G = 5.94$  and  $G = 9.14$  respectively.