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NEW SOLUTIONS OF RELATIVISTIC WAVE EQUATIONS IN MAGNETIC FIELDS AND LONGITUDINAL FIEDS

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New solutions of relativistic wave equations in magnetic fields and longitudinal fields.

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Abstract

We succeeded to describe explicitly all the arbitrariness in solutions of relativistic wave equations in external electromagnetic fields of special form. On this base we construct new sets of stationary and nonstationary solutions in magnetic field and in some superpositions of electric and magnetic fields.

I. INTRODUCTION

Relativistic wave equations (Dirac and Klein-Gordon) provide a basis for relativistic quantum mechanics and quantum electrodynamics of spinor and scalar particles [1]. In relativistic quantum mechanics, solutions of relativistic wave equations are referred to as one-particle wave functions of fermions and bosons in external electromagnetic fields. In quantum electrodynamics, such solutions allow the development of the perturbation expansion known as the Furry picture, which incorporates the interaction with the external field exactly,

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while treating the interaction with the quantized electromagnetic field perturbatively [2]. The physically most important exact solutions of the Klein-Gordon and the Dirac equations are: an electron in a Coulomb field, a uniform magnetic field, the field of a plane wave, the field of a magnetic monopole, the field of a plane wave combined with a uniform magnetic and electric fields parallel to the direction of wave propagation, crossed fields, and some simple one-dimensional electric fields (for a complete review of solutions of relativistic wave equations see [3]).

Considering, for example, stationary solutions of relativistic wave equations, we can see that in the general case, there exist different sets of stationary solutions for one and the same Hamiltonian. The possibility to get different sets of stationary states reflects the existence of an arbitrariness in the solutions of the eigenvalue problem for a Hamiltonian. Considering nonstationary solutions, we also encounter the possibility of constructing different complete sets of such solutions. There is no regular method of describing such an arbitrariness explicitly. Especially in the presence of an external field the problem appears to be nontrivial.

In the present article we demonstrate how one can describe explicitly the arbitrariness in solutions of the relativistic wave equations for some types of external electromagnetic fields, namely, for uniform magnetic fields and combination of these fields with some electric fields. This arbitrariness is connected to the existence of a transformation, which reduces effectively the number of variables in the initial equations. Then we use the corresponding representations to construct new sets of exact solutions, which may have a physical interest. In Sect.II we consider relativistic wave equations in pure uniform magnetic fields. Here we derive a representation for the exact solutions, in which the above mentioned arbitrariness is described explicitly by an arbitrary function. From a suitable choice of this function, we get both the well-known set of solutions and new ones. This Section contains the most complete (at the present) description of the problem of a uniform magnetic field in relativistic quantum mechanics. Among new sets of solutions there are both stationary, generalized coherent solutions and nonstationary solutions. Then, in Sect.III, we consider more complicated configurations of external electromagnetic fields, namely, longitudinal electromagnetic fields. Here we describe all the arbitrariness in the solutions, and on this base present various sets of new exact solutions. In Sect.IV we interpret the above results from the point of view of the general theory of differential equations.

II. UNIFORM MAGNETIC FIELD

A. Arbitrariness in solutions of relativistic wave equations.

Consider a uniform magnetic field $\mathbf{H} = (0, 0, H)$ directed along the x^3 axis $(H > 0)$. The electromagnetic potentials are chosen in the symmetric gauge

$$A_0 = A_3 = 0, \; A_1 = \frac{1}{2} H x^2, \; A_2 = -\frac{1}{2} H x^1 . \tag{2.1}$$

We write the Klein-Gordon and the Dirac equations in the form

$$\begin{aligned} &\mathcal{K}\Psi = 0, \quad \hbar^2 \mathcal{K} = \mathcal{P}^2 - m_0^2 c^2, \; \mathcal{P}_\mu = i\hbar \partial_\mu - \frac{e}{c} A_\mu , \\ &\mathcal{D}\Psi = 0 , \quad \hbar \mathcal{D} = \gamma^\mu \mathcal{P}_\mu - m_0 c . \end{aligned} \tag{2.2}$$

Here $e = -|e|$ and γ-matrices are chosen in the standard representation [3].

In the field under consideration, the operators $\mathcal{P}_0$ and $\mathcal{P}_3$ are mutually commuting integrals of motion, $[\mathcal{K}, \mathcal{P}_0] = [\mathcal{K}, \mathcal{P}_3] = [\mathcal{D}, \mathcal{P}_0] = [\mathcal{D}, \mathcal{P}_3] = [\mathcal{P}_0, \mathcal{P}_3] = 0$.

In the case of the Klein-Gordon equation, the operator L_z,

$$L_z = i\hbar \left(x^2 \partial_1 - x^1 \partial_2 \right), \; [L_z, \mathcal{P}_0] = [L_z, \mathcal{P}_3] = [\mathcal{K}, L_z] = 0 , \tag{2.3}$$

can be included (together with $\mathcal{P}_0$ and $\mathcal{P}_3$) in the complete set of integrals of motion, whereas for the Dirac equation case, the operator J_z,

$$J_z = L_z + \frac{\hbar}{2} \Sigma_3, \; [J_z, \mathcal{P}_0] = [J_z, \mathcal{P}_3] = [\mathcal{D}, J_z] = 0 , \tag{2.4}$$

can be included (together with $\mathcal{P}_0$ and $\mathcal{P}_3$) in the complete set of integrals of motion. Here $\Sigma_3 = \mathrm{diag}\,(\sigma_3, \sigma_3)$.

We are going to use dimensionless coordinates $-\infty < x < \infty$, $-\infty < y < \infty$ or $0 \leq \rho < \infty$, $0 \leq \varphi < 2\pi$ defined by the relations

$$\sqrt{\frac{\gamma}{2}} x^1 = x = \sqrt{\rho}\cos\varphi, \quad \sqrt{\frac{\gamma}{2}} x^2 = y = \sqrt{\rho}\sin\varphi\,, \quad \gamma = \frac{|e|\,H}{c\hbar} > 0\,,$$
$$dx^1 dx^2 = \frac{2}{\gamma} dx\, dy = \frac{1}{\gamma} d\rho\, d\varphi, \quad x + iy = \sqrt{\rho}\exp i\varphi\,. \tag{2.5}$$

It is useful to introduce the operators a_1, a_2, a_1^+, a_2^+ ,

$$a_2 = \frac{1}{\sqrt{2\gamma\hbar}}\left[\mathcal{P}_2 - i\mathcal{P}_1 + \hbar\gamma\left(x^1 + ix^2\right)\right] = \frac{1}{2}\left(x + iy + \partial_x + i\partial_y\right) = \frac{e^{i\varphi}}{2\sqrt{\rho}}\left(\rho + i\partial_\varphi + 2\rho\partial_\rho\right),$$
$$a_2^+ = \frac{1}{\sqrt{2\gamma\hbar}}\left[\mathcal{P}_2 + i\mathcal{P}_1 + \hbar\gamma\left(x^1 - ix^2\right)\right] = \frac{1}{2}\left(x - iy - \partial_x + i\partial_y\right) = \frac{e^{-i\varphi}}{2\sqrt{\rho}}\left(\rho + i\partial_\varphi - 2\rho\partial_\rho\right).$$
$$a_1 = -\frac{1}{\sqrt{2\gamma\hbar}}\left(i\mathcal{P}_1 + \mathcal{P}_2\right) = \frac{1}{2}\left(x - iy + \partial_x - i\partial_y\right) = \frac{e^{-i\varphi}}{2\sqrt{\rho}}\left(\rho - i\partial_\varphi + 2\rho\partial_\rho\right),$$
$$a_1^+ = \frac{1}{\sqrt{2\gamma\hbar}}\left(i\mathcal{P}_1 - \mathcal{P}_2\right) = \frac{1}{2}\left(x + iy - \partial_x - i\partial_y\right) = \frac{e^{i\varphi}}{2\sqrt{\rho}}\left(\rho - i\partial_\varphi - 2\rho\partial_\rho\right). \tag{2.6}$$

They obey the commutation relations

$$\left[a_k, a_s^+\right] = \delta_{k,s}\,, \quad \left[a_k, a_s\right] = \left[a_k^+, a_s^+\right] = 0\,, \quad k, s = 1, 2\,. \tag{2.7}$$

Thus, we can interpret these operators as creation and annihilation ones. One can also find the following relations

$$\mathcal{P}_1^2 + \mathcal{P}_2^2 = \hbar^2\gamma\left(a_1 a_1^+ + a_1^+ a_1\right) = 2\hbar^2\gamma\mathcal{N} + \hbar^2\gamma\,,$$
$$L_z = \hbar\left(\mathcal{N} - a_2^+ a_2\right)\,, \quad \mathcal{N} = a_1^+ a_1\,. \tag{2.8}$$

Then the Klein-Gordon and the Dirac operators can be written as

$$\mathcal{K} = \hbar^{-2}\left(\mathcal{P}_0^2 - \mathcal{P}_3^2\right) - 2\gamma\mathcal{N} - \gamma - m^2, \quad m = \frac{m_0 c}{\hbar}\,,$$
$$\mathcal{D} = \hbar^{-1}\left(\gamma^0\mathcal{P}_0 + \gamma^3\mathcal{P}_3\right) - \sqrt{\frac{\gamma}{2}}\left[\left(\gamma^2 - i\gamma^1\right)a_1 + \left(\gamma^2 + i\gamma^1\right)a_1^+\right] - m\,. \tag{2.9}$$

The operator $\mathcal{N}$ commutes with $\mathcal{P}_0$, $\mathcal{P}_3$, L_z, plus it is an integral of motion in the case of the Klein-Gordon equation. Its generalization for the Dirac equation has the form $\mathcal{N}_D = \mathcal{N} + \frac{1}{2}\Sigma_3$.

One ought to remark that the operators $\mathcal{K}$ and $\mathcal{D}$ do not contain the operators a_2^+, a_2. Thus, the latter operators are integrals of motion, which commute with $\mathcal{N}$, $\mathcal{N}_D, \mathcal{P}_0$, $\mathcal{P}_3$, but do not commute with L_z and J_z .

The operators of creation and annihilation with different numbers commute. One can find a representation in which these operators are acting on different variables. To this end, we present the wave functions from (2.2) in the following form (we make a Fourier transform in the variable y only, and call such a representation the semi-momentum representation)

$$\Psi(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iky} \tilde{\Psi}(x,k)\, dk\ . \tag{2.10}$$

Of course the functions Ψ and $\tilde{\Psi}$ depend on the variables x^2 and x^3 as well, but we do not indicate this dependence explicitly. In terms of $\tilde{\Psi}$ the multiplication and differentiation have the form $y \to i\partial_k, \quad i\partial_y \to -k$. Then, the expressions for the creation and annihilation operators in the semi-momentum representation take the form

$$\begin{aligned} 2a_1 &= x + k + \partial_x + \partial_k, \quad 2a_1^+ = x + k - \partial_x - \partial_k\ , \\ 2a_2 &= x - k + \partial_x - \partial_k, \quad 2a_2^+ = x - k - \partial_x + \partial_k\ . \end{aligned} \tag{2.11}$$

Now we pass from x, k to new variables ξ, η,

$$\sqrt{2}\xi = x + k, \quad \sqrt{2}\eta = x - k, \quad \sqrt{2}x = \xi + \eta, \quad \sqrt{2}k = \xi - \eta\ . \tag{2.12}$$

Then the creation and annihilation operators can be written as

$$\sqrt{2}a_1 = \xi + \partial_\xi\,, \quad \sqrt{2}a_1^+ = \xi - \partial_\xi\,, \quad \sqrt{2}a_2 = \eta + \partial_\eta\,, \quad \sqrt{2}a_2^+ = \eta - \partial_\eta\ . \tag{2.13}$$

In the new variables,

$$2\mathcal{N} = \xi^2 - \partial_\xi^2 - 1\ , \tag{2.14}$$

and the Klein-Gordon and the Dirac operators read

$$\begin{aligned} \mathcal{K} &= \hbar^{-2}\left(\mathcal{P}_0^2 - \mathcal{P}_3^2\right) + \gamma\left(\partial_\xi^2 - \xi^2\right) - m^2\ , \\ \mathcal{D} &= \hbar^{-1}\left(\gamma^0\mathcal{P}_0 + \gamma^3\mathcal{P}_3\right) - \sqrt{\gamma}\left(\gamma^2\xi - i\gamma^1\partial_\xi\right) - m\ . \end{aligned} \tag{2.15}$$

One can see that the latter operators do not contain the variable η. Notice that both operators L_z and J_z contain variables ξ, η. For example,

$$2L_z = \xi^2 - \partial_\xi^2 - \eta^2 + \partial_\eta^2 \, . \tag{2.16}$$

The integration over k in (2.10) can be replaced by an integration over η,

$$\Psi(x,y) = \frac{e^{ixy}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-i\sqrt{2}y\eta} \tilde{\Psi}(\xi,\eta) \, d\eta, \quad \xi = \sqrt{2}x - \eta \, . \tag{2.17}$$

Besides, one can write

$$(\Psi,\Phi) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, \Psi^*(x,y) \, \Phi(x,y) = \left(\tilde{\Psi},\tilde{\Phi}\right) = \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \, \tilde{\Psi}^*(\xi,\eta) \, \tilde{\Phi}(\xi,\eta) \, . \tag{2.18}$$

The independence of the operators (2.15) on the variable η will allow us to separate explicitly the functional arbitrariness in the solutions (2.17), as will be seen below.

B. Stationary states

Known sets of stationary solutions in a uniform magnetic field (that were found in the first works [4–8]) are eigenfunctions of the operators $\mathcal{P}_0, \mathcal{P}_3, \mathcal{N}$ in the scalar case and of the operators $\mathcal{P}_0$, $\mathcal{P}_3$, $\mathcal{N}_D$ in the spinor case. Thus for scalar wave functions Ψ we have the conditions

$$\mathcal{P}_0\Psi = \hbar k_0 \Psi, \quad \mathcal{P}_3\Psi = \hbar k_3 \Psi \, , \, \mathcal{N}\Psi = n\Psi, \, n = 0,1,2,\ldots, \tag{2.19}$$

and for Dirac wave functions $\boldsymbol{\Psi}$ the conditions

$$\mathcal{P}_0\boldsymbol{\Psi} = \hbar k_0 \boldsymbol{\Psi}, \quad \mathcal{P}_3\boldsymbol{\Psi} = \hbar k_3 \boldsymbol{\Psi}, \quad \mathcal{N}_D\boldsymbol{\Psi} = \left(n - \frac{1}{2}\right)\boldsymbol{\Psi}, \quad n = 0,1,2,\ldots \, . \tag{2.20}$$

Consider first the scalar case. It follows from (2.15) that

$$k_0^2 = m^2 + \gamma + k_3^2 + 2\gamma n = m^{*\,2} + k_3^2 + 2\gamma n, \quad m^{*\,2} = m^2 + \gamma \, , \tag{2.21}$$

and

$$\Psi_{n,k_3}(x^\mu) = N\exp\left(-ik_0x^0 - ik_3x^3\right)\Psi_n(x,y) \ . \tag{2.22}$$

Here N is a normalization factor. In the semi-momentum representation (2.10) the function $\Psi_n(x,y)$ has the following image

$$\widetilde{\Psi}_n(\xi,\eta) = U_n(\xi)\,\Phi(\eta)\,, \quad \xi = \sqrt{2}x - \eta \ . \tag{2.23}$$

Here Eqs. (2.19), (2.14) were used. $U_n(\xi)$ are Hermit functions; they are related to the corresponding polynomials $H_n(\xi)$ as $U_n(x) = (2^n n!\sqrt{\pi})^{-\frac{1}{2}}\exp(-x^2/2)\,H_n(x)$ [14]. The function $\Phi(\eta)$ is arbitrary. The functions $\Psi_n(x,y)$ from (2.22) obey the relations

$$a_1\Psi_n = \sqrt{n}\Psi_{n-1}, \quad a_1^+\Psi_n = \sqrt{n+1}\Psi_{n+1}\,, \quad \Psi_n(x,y) = \frac{\left(a_1^+\right)^n}{\sqrt{\Gamma(n+1)}}\Psi_0(x,y)\ , \tag{2.24}$$

$$\Psi_0(x,y) = \pi^{-\frac{3}{4}}\exp\left(-x^2+ixy\right)\int_{-\infty}^{\infty} d\eta\,\exp\left[-\frac{\eta^2}{2} + \sqrt{2}\eta(x-iy)\right]\Phi(\eta)\ . \tag{2.25}$$

Dirac wave functions are of the form $\boldsymbol{\Psi}_{n,k_3}(x^\mu) = N\exp(-ik_0x^0 - ik_3x^3)\,\boldsymbol{\Psi}_{k_3,n}(x,y)$ with bispinors $\boldsymbol{\Psi}_{k_3,n}(x,y)$ having the structure

$$\boldsymbol{\Psi}^T_{n,k_3}(x,y) = (c_1\Psi_{n-1}(x,y),\ ic_2\Psi_n(x,y),\ c_3\Psi_{n-1}(x,y),\ ic_4\Psi_n(x,y))\ . \tag{2.26}$$

The functions $\Psi_n(x,y)$ are defined by the relations (2.17), (2.23), whereas the constant bispinor C (with the elements c_k) obeys an algebraic system of equations

$$AC = 0, \quad A = \gamma^0k_0 + \gamma^3k_3 - \sqrt{2\gamma n}\gamma^1 - m\ . \tag{2.27}$$

The condition $\det A = (k_0^2 - k_3^2 - 2\gamma n - m^2)^2 = 0$ results in an equation which is an analog of (2.21),

$$k_0^2 = k_3^2 + 2\gamma n + m^2\ . \tag{2.28}$$

Since the rank of the matrix A is equal to 2, a general solution of (2.27) has the form

$$C = \begin{pmatrix} (k_0+m)v \\ (\sqrt{2\gamma n}\sigma_1 - k_3\sigma_3)v \end{pmatrix}, \quad C^+C = 2k_0(k_0+m)v^+v\ , \tag{2.29}$$

where v is an arbitrary constant bispinor and σ are Pauli matrices. We can specify v selecting a spin integral of motion (see [3]). The state $n = 0$ is a special case. Here we must set $c_1 = c_3 = 0$, that corresponds to the choice $v^T = (0,\ c_2)\,, \quad c_2 \neq 0$. The latter means that $\Sigma_3 \Psi_D = -\Psi_D$. Thus, for $n = 0$, the electron spin can only point to the direction opposite to the magnetic field.

Expressions for $\Psi_n(x, y)$ in the semi-momentum representation contain explicitly a functional arbitrariness, which means that every energy level is infinitely degenerated. Let us demand that the scalar and spinor wave functions be eigenvectors of the operators L_z and J_z respectively. According to (2.4) and (2.8) that means that the functions $\Psi_n(x, y)$ have to obey an additional condition

$$
\begin{aligned}
& a_2^+ a_2 \Psi_n(x,y) = s\Psi_n(x,y), \quad s = 0,1,2,\dots\ , \\
& L_z = \hbar\,(n-s) = \hbar l, \quad l = n - s, \quad n \geq l > -\infty, \quad J_z = \hbar\left(l - \frac{1}{2}\right) .
\end{aligned} \tag{2.30}
$$

This condition defines the function $\Phi(\eta)$ according to (2.13), $a_2^+ a_2 \Phi_s(\eta) = s\Phi_s(\eta)$, therefore $\Phi_s(\eta) = U_s(\eta)$. Substituting this result into (2.23) and into (2.17), and doing the integral over η, we find in the coordinate representation,

$$
\Psi_{n,s}(x,y) = \frac{(-1)^n}{\sqrt{2\pi}} e^{il\varphi} I_{s,n}(\rho) = \frac{(-1)^n}{\sqrt{2\pi}} \left(\frac{x+iy}{x-iy}\right)^{\frac{n-s}{2}} I_{s,n}(x^2+y^2)\ . \tag{2.31}
$$

Here $I_{m,n}\,(x)$ are Laguerre functions, which are connected to the corresponding polynomials $L_n^\alpha\,(x)$ by the relations (see [14]) $I_{m,n}\,(x) = \left(\Gamma\,(n+1)\,/\Gamma\,(m+1)\right)^{1/2} e^{-\frac{x}{2}} x^{\frac{\alpha}{2}} L_n^\alpha\,(x)\,, \quad \alpha = m - n$. The states (2.31) were first obtained in the works [4–8]. Besides (2.24) and (2.25), the functions (2.31) obey the following relations as well

$$
\begin{aligned}
& a_2 \Psi_{n,s} = \sqrt{s}\Psi_{n,s-1}, \quad a_2^+ \Psi_{n,s} = \sqrt{s+1}\Psi_{n,s+1}\ , \\
& \Psi_{n,s} = \frac{(a_1^+)^n (a_2^+)^s}{\sqrt{\Gamma(n+1)\Gamma(s+1)}} \Psi_{0,0}\ , \quad \Psi_{0,0}(x,y) = \frac{1}{\sqrt{\pi}} \exp\left[-\frac{1}{2}(x^2+y^2)\right] = \frac{e^{-\frac{\rho}{2}}}{\sqrt{\pi}}\ .
\end{aligned} \tag{2.32}
$$

Below we are going to find new sets of solutions imposing complementary conditions different from (2.30). This results in a different form for the function $\Phi(\eta)$.

Taking into account that the operators a_2^+, a_2 are integrals of motion, we may construct stationary states, which are eigenvectors of a linear combination $A_2^{\alpha,\beta}$ of these operators,

$$A_2^{\alpha,\beta} = \alpha a_2 + \beta a_2^+ \,. \tag{2.33}$$

Here α, β are arbitrary complex numbers. One has to distinguish here three nonequivalent cases:

If $|\alpha|^2 < |\beta|^2$, then do not exist any normalizable eigenvectors of the operator (2.33) . We are not going to consider such case.

If $|\alpha|^2 = |\beta|^2$, then $A_2^{\alpha,\beta}$ is, in fact, reduced to a Hermitian operator

$$A_2^{\mu} = \mu a_2 + \mu^* a_2^+, \quad A_2^{+\mu} = A_2^{\mu}, \quad \mu \neq 0\,, \tag{2.34}$$

where μ is an arbitrary complex number.

If $|\alpha|^2 > |\beta|^2$, then without loss of generality we can assume that operators $A_2^{\alpha,\beta}$ have the form

$$A_2^{\alpha,\beta} = \alpha a_2 + \beta a_2^+, \quad |\alpha|^2 - |\beta|^2 = 1, \quad \left[A_2^{\alpha,\beta},\, A_2^{+\alpha,\beta}\right] = 1. \tag{2.35}$$

Then $A_2^{+\alpha,\beta}$, $A_2^{\alpha,\beta}$ are creation and annihilation operators, which are related to a_2^+, a_2 by a canonical transformation

$$a_2 = \alpha^* A_2^{\alpha,\beta} - \beta A_2^{+\,\alpha,\beta}, \quad a_2^+ = \alpha A_2^{+\,\alpha,\beta} - \beta^* A_2^{\alpha,\beta}\,. \tag{2.36}$$

Consider eigenvectors of the operator (2.34), i.e., $A_2^{\mu}\Psi_{n,z}^{\mu}(x,y) = z\Psi_{n,z}^{\mu}(x,y), \quad z = z^*$. This equation results in the equation $A_2^{\mu}\Phi_z^{\mu}(\eta) = z\Phi_z^{\mu}(\eta)$ for the function $\Phi(\eta)$. Taking into account (2.13), one can find that solutions of the latter equation are

$$\begin{aligned} &\Phi_z^{\mu}(\eta) = \left[\frac{\mu}{\sqrt{2\pi}\,|\mu|\,(\mu - \mu^*)}\right]^{\frac{1}{2}} \exp Q_1, \\ &4(\mu - \mu^*)\,Q_1 = -2(\mu + \mu^*)\,\eta^2 + 4\sqrt{2}z\eta - z^2(\mu + \mu^*)\,|\mu|^{-2}\,. \end{aligned} \tag{2.37}$$

These solutions obey the orthonormality and completeness relations

$$\int_{-\infty}^{\infty} \Phi_{z'}^{*\,\mu}(\eta)\Phi_z^{\mu}(\eta)\,d\eta = \delta(z - z')\,, \ \int_{-\infty}^{\infty} \Phi_z^{*\,\mu}(\eta')\Phi_z^{\mu}(\eta)\,dz = \delta(\eta - \eta')\,. \tag{2.38}$$

Their overlapping has the form

$$R^{\mu\prime,\mu}\left(z',z\right)=\int_{-\infty}^{\infty}\Phi_{z'}^{*\,\mu'}(\eta)\Phi_{z}^{*\,\mu}(\eta)\ d\eta=N_1\exp\left[\frac{Q_2}{4\left(\mu'\mu^*-\mu\mu'^*\right)}\right],$$
$$N_1^2=\frac{\mu'^*\mu}{2\pi^2\left|\mu'\right|\left|\mu\right|\left(\mu\mu'^*-\mu'\mu^*\right)},$$
$$Q_2=\left(z\sqrt{\frac{\mu'}{\mu}}-z'\sqrt{\frac{\mu}{\mu'}}\right)^2+\left(z\sqrt{\frac{\mu'^*}{\mu^*}}-z'\sqrt{\frac{\mu^*}{\mu'^*}}\right)^2 . \tag{2.39}$$

It defines the mutual decomposition

$$\Phi_z^{\mu}\left(\eta\right)=\int_{-\infty}^{\infty}\Phi_{z'}^{\mu'}\left(\eta\right)R^{\mu',\mu}\left(z',z\right)\ dz' . \tag{2.40}$$

The coordinate representation (2.17) for the solutions under consideration has the form

$$\Psi_{n,z}^{\mu}\left(x,y\right)=\left(\sqrt{2}\pi\left|\mu\right|\right)^{-\frac{1}{2}}\left(\frac{\mu^*}{\mu}\right)^{\frac{n}{2}}U_n\left(p_1\right)\exp iQ_3 ,$$
$$4\left|\mu\right|^2Q_3=\left[i\left(\mu^*-\mu\right)x+\left(\mu+\mu^*\right)y\right]\left[\left(\mu+\mu^*\right)x+i\left(\mu-\mu^*\right)y-2z\right],$$
$$\sqrt{2}\left|\mu\right|p_1=\left(\mu+\mu^*\right)x+i\left(\mu-\mu^*\right)y-z . \tag{2.41}$$

Their scalar product (2.18) reads $\left(\Psi_{n',z'}^{\mu},\Psi_{n,z}^{\mu}\right)=\delta_{n,n'}\delta(z-z')$. The relation (2.40) results into the following decomposition in the coordinate representation

$$\Psi_{n,z}^{\mu}\left(x,y\right)=\int_{-\infty}^{\infty}\Psi_{n,z'}^{\mu'}\left(x,y\right)R^{\mu',\mu}\left(z',z\right)\ dz' . \tag{2.42}$$

In particular, in the cases of real or pure imaginary μ, such wave functions were known before [3].

Consider eigenvectors of the operator (2.35), i.e., $A_2^{\alpha,\beta}\Psi_{n,z}^{\alpha,\beta}\left(x,y\right)=z\Psi_{n,z}^{\alpha,\beta}\left(x,y\right)$ where z is a complex number. In fact, we get coherent (squeezed) stationary states. They are labeled by z and by two complex parameters α, β, which are related by the condition (2.35). In the semi-momentum representation the above equation is reduced to the one

$$A_2^{\alpha,\beta}\Phi_z^{\alpha,\beta}\left(\eta\right)=z\Phi_z^{\alpha,\beta}\left(\eta\right) . \tag{2.43}$$

It is well known that such solutions form a complete (overcomplete) set at any fixed α, β. Solutions within each set are not orthogonal. One can use these functions to construct a orthogonal set of solutions.

Since the operators $A_2^{+\,\alpha,\beta}$, $A_2^{\alpha,\beta}$ are integrals of motion (both for the Klein-Gordon equation and for the Dirac equation), they are symmetry operators for the equations. The action of these operators on a solution provides again a solution. For example, applying the operators $(\Gamma(1+s))^{-1/2}\left(A_2^{+\,\alpha,\beta}-z^*\right)^s$, $s=0,1,2,\ldots$ to normalized solutions of the equation (2.43), we get normalized solutions labeled by the index s. These new solutions are orthogonal with respect to s,

$$\Phi_{s,z}^{\alpha,\beta}(\eta)=\frac{\left(A_2^{+\,\alpha,\beta}-z^*\right)^s}{\sqrt{\Gamma(1+s)}}\Phi_z^{\alpha,\beta}(\eta)\,,\quad \Phi_{0,z}^{\alpha,\beta}(\eta)=\Phi_z^{\alpha,\beta}(\eta)\,,$$
$$\int_{-\infty}^{\infty}\Phi_{s',z}^{*\,\alpha,\beta}(\eta)\,\Phi_{s,z}^{\alpha,\beta}(\eta)\,d\eta=\delta_{s,s'}\int_{-\infty}^{\infty}\left|\Phi_z^{\alpha,\beta}(\eta)\right|^2\,d\eta\,. \tag{2.44}$$

We call such states generalized squeezed coherent states. It is possible to get an explicit form for these states,

$$\Phi_{s,z}^{\alpha,\beta}(\eta)=\left[\frac{\alpha}{|\alpha|(\alpha-\beta)}\right]^{\frac{1}{2}}\left(\frac{\alpha^*-\beta^*}{\alpha-\beta}\right)^{\frac{s}{2}}e^{Q_4}U_s(p_2)\,,\;\;4|\alpha-\beta|^2Q_4$$
$$=2(\alpha\beta^*-\alpha^*\beta)\eta^2+2\sqrt{2}\eta\left[z(\alpha^*-\beta^*)-z^*(\alpha-\beta)\right]+z^{*\,2}(\alpha-\beta)^2$$
$$-z^2(\alpha^*-\beta^*)^2\,,\;\;2|\alpha-\beta|p_2=2\eta-\sqrt{2}z(\alpha^*-\beta^*)-\sqrt{2}z^*(\alpha-\beta)\,. \tag{2.45}$$

The functions (2.45) form a complete set for each fixed z ,

$$\sum_{s=0}^{\infty}\Phi_{s,z}^{*\,\alpha,\beta}(\eta')\Phi_{s,z}^{\alpha,\beta}(\eta)=\delta(\eta'-\eta)\,, \tag{2.46}$$

and for each fixed s ,

$$\int\frac{d^2z}{\pi}\Phi_{s,z}^{*\,\alpha,\beta}(\eta')\Phi_{s,z}^{\alpha,\beta}(\eta)=\delta(\eta-\eta'),\quad d^2z=d\,\mathrm{Re}\,z\;d\,\mathrm{Im}\,z\,. \tag{2.47}$$

The overlapping

$$R_{s',s}^{\alpha',\beta';\alpha,\beta}(z',z)=\int_{-\infty}^{\infty}\Phi_{s',z'}^{*\,\alpha',\beta'}(\eta)\Phi_{s,z}^{\alpha,\beta}(\eta)\,d\eta\,, \tag{2.48}$$

allows us to find mutual decompositions

$$\Phi_{s,z}^{\alpha,\beta}(\eta)=\sum_{s'=0}^{\infty}R_{s',s}^{\alpha',\beta';\alpha,\beta}(z',z)\Phi_{s',z'}^{\alpha',\beta'}(\eta),\;\;\Phi_{s,z}^{\alpha,\beta}(\eta)=\int d^2z'\,R_{s',s}^{\alpha',\beta';\alpha,\beta}(z',z)\Phi_{s',z'}^{\alpha',\beta'}(\eta)\,. \tag{2.49}$$

Unfortunately, the overlapping (2.48) has a complicated form via a finite sum of Hermit functions. In some particular cases this sum can be simplified. For example, if $\alpha' = \alpha$, $\beta' = \beta$, then the overlapping does not depend on α, β and has the form

$$R_{s',s}^{\alpha,\beta;\alpha,\beta}(z',z) = R_{s',s}(z',z) = \left(\frac{z-z'}{z^*-z'^*}\right)^{\frac{s'-s}{2}} \exp\left[\frac{1}{2}\left(zz'^* - z^*z'\right)\right] I_{s',s}\left(|z-z'|^2\right). \quad (2.50)$$

For $s = s' = 0$ we get

$$R_{0,0}^{\alpha',\beta';\alpha,\beta}(z',z) = \sqrt{\frac{\alpha\alpha'^*}{|\alpha\alpha'|}}\left(\alpha\alpha'^* - \beta\beta'^*\right)^{-\frac{1}{2}} \exp Q_5\,,$$
$$2Q_5 = \frac{z^2(\alpha'^*\beta^* - \alpha^*\beta'^*) + (z'^*)^2(\alpha\beta' - \alpha'\beta) + 2zz'^*}{\alpha\alpha'^* - \beta\beta'^*} - z'z^* - zz'^* - |z - z'^*|^2\,. \quad (2.51)$$

The wave function $\Psi_{n,s,z}^{\alpha,\beta}(x,y)$ has also a very complicated form in the general case. However, in some particular cases it can be simplified. For example,

$$\Psi_{n,s,z}^{1,0}(x,y) = \frac{(-1)^n}{\sqrt{\pi}}\left(\frac{x+iy-z}{x-iy-z^*}\right)^{\frac{n-s}{2}} e^M I_{s,n}\left(|x+iy-z|^2\right)\,,$$
$$M = z(x-iy) - z^*(x+iy)\,. \quad (2.52)$$

For $z = 0$ we arrive at the set (2.31).

For $s = 0$ we get a compact form for a set of stationary squeezed coherent states

$$\Psi_{n,0,z}^{\alpha,\beta}(x,y) = \Psi_{n,z}^{\alpha,\beta}(x,y) = (-1)^n\frac{\pi^{\frac{1}{4}}}{\sqrt{|\alpha|}}\left(\frac{\beta}{\alpha}\right)^{\frac{n}{2}} U_n\left(\frac{p_3}{\sqrt{2\alpha\beta}}\right)\exp Q_6\,,$$
$$p_3 = z - \alpha(x+iy) - \beta(x-iy)\,, \quad (2.53)$$
$$4\alpha\beta Q_6 = (1+2|\beta|^2)z^2 - 2\alpha\beta|z|^2 + (z+p_3)\left[\beta(x-iy) - \alpha(x+iy)\right]\,.$$

Additional simplifications are available for $\alpha = 1$, $\beta = 0$,

$$\Psi_{n,z}^{1,0}(x,y) = \Psi_{n,z}(x,y) = \varphi_{n,z}(x,y)\exp\left(-\frac{1}{2}|z|^2\right)\,,$$
$$\varphi_{n,z}(x,y) = \frac{(x+iy-z)^n}{\sqrt{\pi\Gamma(n+1)}}\exp\left[z(x-iy) - \frac{1}{2}\left(x^2+y^2\right)\right]\,. \quad (2.54)$$

Namely, these states where found in [9]. However, the meaning of the parameter z was not clarified.

For arbitrary α, β, the functions $\Psi_{n,s,z}^{\alpha,\beta}(x,y)$ obey, besides (2.24), the following relations

$$A_2^{\alpha,\beta}\Psi_{n,s,z}^{\alpha,\beta} = z\Psi_{n,s,z}^{\alpha,\beta} + \sqrt{s}\Psi_{n,s-1,z}^{\alpha,\beta}\ , \quad A_2^{+\ \alpha,\beta}\Psi_{n,s,z}^{\alpha,\beta} = z^*\Psi_{n,s,z}^{\alpha,\beta} + \sqrt{s+1}\Psi_{n,s+1,z}^{\alpha,\beta}\ . \tag{2.55}$$

Taking into account that $a_2^+\varphi_{n,z} = \partial\varphi_{n,z}/\partial z$, we can construct a new set of stationary states by successive differentiations,

$$\begin{aligned}\bar{\Psi}_{n,s,z}(x,y) &= \frac{(-1)^n N}{\sqrt{\pi}}\exp\left[i(n-s)\varphi + \frac{\rho - q}{2}\right]\left(\frac{q}{\rho}\right)^{\frac{n-s}{2}} I_{s,n}(q) \\ &= \frac{(-1)^n N}{\sqrt{\pi}}\exp\left[\frac{z}{2}(x+iy)\right]\left(\frac{x+iy-z}{x-iy}\right)^{\frac{n-s}{2}} I_{s,n}(q)\ , \\ q &= \rho - z\sqrt{\rho}e^{-i\varphi} = (x-iy)(x+iy-z)\ .\end{aligned} \tag{2.56}$$

For $N=1$ the above set obeys (besides (2.24)) the relations

$$a_2\bar{\Psi}_{n,s,z} = z\bar{\Psi}_{n,s,z} + \sqrt{s}\bar{\Psi}_{n,s-1,z}\ , \quad a_2^+\bar{\Psi}_{n,s,z} = \frac{\partial}{\partial z}\bar{\Psi}_{n,s,z} = \sqrt{s+1}\bar{\Psi}_{n,s+1,z}\ . \tag{2.57}$$

The set (2.31) is a particular case of (2.56), it corresponds to $z=0$. The set (2.56) is not orthogonal,

$$\begin{aligned}&\left(\bar{\Psi}_{n',s',z'}, \bar{\Psi}_{n,s,z}\right) = N'^*N\delta_{n,n'}\mathcal{J}_{s,s'}(z,z')\ , \\ &\mathcal{J}_{s,s'}(z,z') = \sqrt{\frac{\Gamma(s+1)}{\Gamma(s'+1)}}z^{s'-s}e^{zz'^*}L_s^{s'-s}(-zz'^*), \quad s \le s'\ , \\ &\mathcal{J}_{s,s'}(z,z') = \sqrt{\frac{\Gamma(s'+1)}{\Gamma(s+1)}}(z'^*)^{s-s'}e^{zz'^*}L_{s'}^{s-s'}(-zz'^*), \quad s' \le s\ .\end{aligned} \tag{2.58}$$

The functions from the set (2.56) are normalized to unity for $N = N_s(z) = \exp\left(-|z|^2/2\right)\left[L_s\left(-|z|^2\right)\right]^{-1/2}$. For $N=1$, the following mutual decompositions take place

$$\begin{aligned}\bar{\Psi}_{n,s+k,z'}(x,y) &= \sqrt{\frac{\Gamma(k+1)}{\Gamma(s+k+1)}}\int\frac{d^2z}{\pi}z^{*s}e^{(z'z^*-|z|^2)}\bar{\Psi}_{n,k,z}(x,y), \\ \bar{\Psi}_{n,s,z'}(x,y) &= \sum_{k=0}^{\infty}\sqrt{\frac{\Gamma(k+s+1)}{\Gamma(s+1)}}\frac{(z'-z)^k}{k!}\bar{\Psi}_{n,s+k,z}(x,y)\ .\end{aligned} \tag{2.59}$$

That means, in particular, that (2.56) is a complete set since the set (2.31) is complete.

Selecting different forms for the function $\Phi(\eta)$, we can get other sets of stationary states for a charge in a uniform magnetic field.

C. Nonstationary states

The most interesting nonstationary solutions of relativistic wave equations for a charge in a uniform magnetic field are coherent states; for the first time such solutions were presented in [10–13], see also [3]. Below we present a new family of nonstationary solutions, which includes the above coherent states as a particular case.

Here we are going to use light-cone variables $u^0 = x^0 - x^3$, $u^3 = x^0 + x^3$, and the corresponding momentum operators

$$\widetilde{\mathcal{P}}_0 = i\hbar\tilde{\partial}_0 = \frac{1}{2}\left(\mathcal{P}_0 - \mathcal{P}_3\right), \quad \widetilde{\mathcal{P}}_3 = i\hbar\tilde{\partial}_3 = \frac{1}{2}\left(\mathcal{P}_0 + \mathcal{P}_3\right) , \tag{2.60}$$

where $\tilde{\partial}_0 = \partial/\partial u^0$, $\tilde{\partial}_3 = \partial/\partial u^3$. Then the Klein-Gordon operator can be presented in the form

$$\mathcal{K} = 4\hbar^{-2}\widetilde{\mathcal{P}}_3\widetilde{\mathcal{P}}_0 - 2\gamma\mathcal{N} - m^{*\,2} , \tag{2.61}$$

whereas the Dirac equation reads ($\mathbf{\Psi}$ is a Dirac bispinor)

$$4\hbar^{-2}\widetilde{\mathcal{P}}_3\widetilde{\mathcal{P}}_0\Psi_{(-)} = \left(2\gamma\mathcal{N}_D + m^{*2}\right)\Psi_{(-)} , \quad 2\widetilde{\mathcal{P}}_3\Psi_{(+)} = \left[(\alpha\mathcal{P}_\perp) + \hbar\rho_3 m\right]\Psi_{(-)} ,$$
$$\mathcal{P}_\perp = -\left(\mathcal{P}_1, \mathcal{P}_2, 0\right) , \quad \mathbf{\Psi} = \Psi_{(+)} + \Psi_{(-)} , \quad \Psi_{(\pm)} = p_\pm\mathbf{\Psi}, \quad 2p_\pm = 1 \pm \alpha_3 . \tag{2.62}$$

Here α and ρ_3 are Dirac matrices [3], and $p_\pm$ projection operators.

In the case of the uniform magnetic field under consideration, the operators $\widetilde{\mathcal{P}}_3, \widetilde{\mathcal{P}}_0$ are integrals of motion. Thus, we will consider solutions that are eigenvectors of $\widetilde{\mathcal{P}}_3$,

$$\widetilde{\mathcal{P}}_3\Psi = \hbar\frac{\lambda}{2}\Psi . \tag{2.63}$$

The scalar wave function obeys (2.63) and can be written as

$$\Psi(x^\mu) = N\exp\left(-i\frac{\lambda}{2}u^3 - i\frac{m^{*\,2}}{2\lambda}u^0\right)\psi(u^0, x, y) . \tag{2.64}$$

It is easy to see that $\psi(u^0, x, y)$ obeys a first order equation, which can be treated as a Schrödinger equation,

$$i\partial_0\psi\left(u^0,x,y\right)=\omega a_1^+a_1\psi\left(u^0,x,y\right), \quad \omega=\frac{\gamma}{\lambda}\,. \tag{2.65}$$

Suppose Eq. (2.63) holds, then $\Psi_{(-)}$ can be presented in the form:

$$\Psi_{(-)}(x^\mu)=N\exp\left(-i\frac{\lambda}{2}u^3-i\frac{m^{*\,2}}{2\lambda}u^0\right)W\left(1-\alpha_3\right)C\psi(u^0,x,y)\,. \tag{2.66}$$

Here C is an arbitrary constant bispinor, and W is a unitary matrix (φ_0 is a constant phase),

$$W=\cos\kappa-i\Sigma_3\sin\kappa, \quad 2\kappa=\omega u^0+\varphi_0, \quad W^+W=I\,, \tag{2.67}$$

and $\psi(u^0,x,y)$ is a scalar function. The latter function obeys the equation (2.65). Then, the $\Psi_{(+)}$ projection can be found from (2.62), $\Psi_{(+)}=(\hbar\lambda)^{-1}\left[(\alpha\mathcal{P}_\perp)+\hbar m\rho_3\right]\Psi_{(-)}$.

Thus, both in the scalar and spinor cases we have to solve the same equation (2.65).

In the semi-momentum representation, the corresponding function $\tilde{\psi}(u^0,\xi,\eta)$ obeys the same equation (2.65), where, however, one has to use the expression (2.14) for the operator $\mathcal{N}=a_1^+a_1$. The relation between the functions $\tilde{\psi}(u^0,\xi,\eta)$ and $\psi(u^0,\xi,\eta)$ still has the form (2.17).

Let us introduce the operators

$$A_1^{f,g}=fa_1+ga_1^+, \quad A_1^{+\,f,g}=f^*a_1^++g^*a_1\,, \tag{2.68}$$

where the complex quantities f and g can depend on u^0. These operators are integrals of motion whenever f, g obey the equations (by dots above are denoted derivatives with respect to u^0)

$$i\dot{f}+\omega f=0\,, \quad i\dot{g}-\omega g=0\,. \tag{2.69}$$

It is easy to find

$$f=f_0\exp\left(i\omega u^0\right)\,, \quad g=g_0\exp\left(-i\omega u^0\right)\,, \tag{2.70}$$

where f_0, g_0 are some complex constants. Bearing in mind considerations related to the operators (2.33), we are going to consider two nonequivalent cases only. The first one corresponds to $|f|^2 = |g|^2$ or equivalently to $|f_0|^2 = |g_0|^2$. In this case we can, in fact, only consider the Hermitian operator

$$A_1^\nu = \nu a_1 + \nu^* a_1^+, \quad \nu = \nu_0 e^{i\omega u^0}, \quad \nu_0 = \text{const}. \tag{2.71}$$

The second case corresponds to $|f|^2 > |g|^2$, and here we can suppose that

$$|f|^2 - |g|^2 = |f_0|^2 - |g_0|^2 = 1 , \tag{2.72}$$

without the loss of generality. In both cases the operators (2.68) are, within constant complex factors, creation and annihilation operators.

Let us include operators (2.71) and (2.34) (they are integrals of motion) into the complete set of operators. Then

$$A_1^\nu \psi_{z_1,z_2}^{\nu,\mu} = z_1 \psi_{z_1,z_2}^{\nu,\mu}, \quad A_2^\mu \psi_{z_1,z_2}^{\nu,\mu} = z_2 \psi_{z_1,z_2}^{\nu,\mu}, \quad z_k^* = z_k, \quad k = 1, 2 . \tag{2.73}$$

In the semi-momentum representation we find

$$\tilde{\psi}\left(u^0, \xi, \eta\right) = \Phi_{z_1}^\nu (\xi) \, \Phi_{z_2}^\mu (\eta) \ , \tag{2.74}$$

where functions $\Phi_{z_1}^\nu$ are defined in (2.37). The corresponding coordinate representation reads

$$\begin{aligned}
&\psi_{z_1,z_2}^{\nu,\mu}(u^0, x, y) = \left[\frac{\mu\nu}{2\pi^2|\mu||\nu|(\mu\nu - \mu^*\nu^*)}\right]^{\frac{1}{2}} \exp\left[\frac{Q_6}{4\left(\mu^*\nu^* - \mu\nu\right)}\right] , \\
&Q_6 = 2(\mu + \mu^*)(\nu + \nu^*)x^2 + 2(\mu - \mu^*)(\nu - \nu^*)y^2 + 4i(\mu\nu^* - \mu^*\nu)xy \\
&\quad -4x\left[z_1(\mu + \mu^*) + z_2(\nu + \nu^*)\right] - 4iy\left[z_1(\mu - \mu^*) - z_2(\nu - \nu^*)\right] \\
&\quad + \left(z_1\sqrt{\frac{\mu^*}{\nu}} + z_2\sqrt{\frac{\nu}{\mu^*}}\right)^2 + \left(z_1\sqrt{\frac{\mu}{\nu^*}} + z_2\sqrt{\frac{\nu^*}{\mu}}\right)^2 .
\end{aligned} \tag{2.75}$$

These solutions are orthogonal at any fixed u^0, $\left(\psi_{z_1',z_2'}^{\nu,\mu}, \psi_{z_1,z_2}^{\nu,\mu}\right) = \delta\left(z_1 - z_1'\right) \delta\left(z_2 - z_2'\right)$, and obey the completeness relation

$$\int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{\infty} dz_2 \, \psi_{z_1,z_2}^{*\,\nu,\mu}(u^0, x', y') \psi_{z_1,z_2}^{\nu,\mu}(u^0, x, y) = \delta\left(x - x'\right) \delta\left(y - y'\right) . \tag{2.76}$$

Consider now generalized squeezed coherent states, which can be constructed by analogy with (2.44) in the semi-momentum representation. We use here the operators (2.68) supposing that the relations (2.69), (2.70), (2.72), and (2.35) hold,

$$\widetilde{\psi}^{f,g;\alpha,\beta}_{n,s;z_1,z_2}(u^0,\xi,\eta) = \Phi^{f,g}_{n,z_1}(\xi)\,\Phi^{\alpha,\beta}_{s,z_2}(\eta)\,. \tag{2.77}$$

The functions $\Phi^{a,b}_{n,z}(x)$ are defined in (2.45). Thus,

$$\psi^{f,g;\alpha,\beta}_{n,s;z_1,z_2}(u^0,x,y) = \frac{\left(A_1^{+\,f,g} - z_1^*\right)^n \left(A_2^{+\,\alpha,\beta} - z_2^*\right)^s}{\sqrt{\Gamma(n+1)\Gamma(s+1)}}\psi^{f,g;\alpha,\beta}_{z_1,z_2}(u^0,x,y),\ \psi^{f,g;\alpha,\beta}_{z_1,z_2} = \psi^{f,g;\alpha,\beta}_{0,0;z_1,z_2}\,. \tag{2.78}$$

The solutions (2.78) obey the relations

$$\begin{aligned}&\left(A_1^{f,g} - z_1\right)\psi^{f,g;\alpha,\beta}_{n,s;z_1,z_2} = \sqrt{n}\psi^{f,g;\alpha,\beta}_{n-1,s;z_1,z_2}\,,\ \left(A_1^{+\,f,g} - z_1^*\right)\psi^{f,g;\alpha,\beta}_{n,s;z_1,z_2} = \sqrt{n+1}\psi^{f,g;\alpha,\beta}_{n+1,s;z_1,z_2}\,,\\ &\left(A_2^{\alpha,\beta} - z_2\right)\psi^{f,g;\alpha,\beta}_{n,s;z_1,z_2} = \sqrt{s}\psi^{f,g;\alpha,\beta}_{n,s-1;z_1,z_2}\,,\ \left(A_2^{+\,\alpha,\beta} - z_2^*\right)\psi^{f,g;\alpha,\beta}_{n,s;z_1,z_2} = \sqrt{s+1}\psi^{f,g;\alpha,\beta}_{n,s+1;z_1,z_2}\,.\end{aligned} \tag{2.79}$$

Eq. (2.78) describes the most general form of relativistic wave equation solutions in a constant uniform magnetic field. All the formerly known solutions can be obtained from this equation by a particular choice of parameters. For instance, by selecting $f_0 = \alpha = 1,\ g = \beta = 0,\ z_1 = 0,\ z_2 = z$ with $z = 0$ we get the states (2.31), on the other hand, if one puts $s = 0,\ z \neq 0$, then one gets the states (2.54). For $n = s = 0,\ f_0 = \alpha = 1,\ g = \beta = 0$, we get coherent states [10–13].

In the general case, an explicit coordinate representation for the solutions (2.78) looks complicated enough. However, some particular cases admit essential simplifications. For example, suppose $f_0 = \alpha = 1,\ g = \beta = 0$, then

$$\begin{aligned}&\Psi^{1,0;1,0}_{n,s;z_1,z_2}(u^0,x,y) = \frac{(-1)^n}{\sqrt{\pi}}\left(\frac{x+iy-\bar{z}_1^*-z_2}{x-iy-\bar{z}_1-z_2^*}\right)^{\frac{n-s}{2}} e^{M_1} I_{s,n}(p_4)\,,\\ &2M_1 = (\bar{z}_1 - z_2^*)(x+iy) - (\bar{z}_1^* - z_2)(x-iy) + \bar{z}_1^* z_2 - \bar{z}_1 z_2^* - 2in\omega u^0\,,\\ &p_4 = |x+iy-\bar{z}_1^*-z_2|^2\,,\quad \bar{z}_1 = z_1\exp(-i\omega u^0)\,,\end{aligned} \tag{2.80}$$

For $n = s = 0$, we get the coordinate representation for the squeezed coherent states in the form

$$\begin{aligned}&\Psi^{f;g;\alpha,\beta}_{z_1,z_2}(u^0,x,y) = \left[\frac{\alpha f}{(\alpha f - \beta g)\pi|\alpha||f|}\right]^{\frac{1}{2}}\exp Q_7\,,\\ &Q_7 = -\frac{1}{2}\left(|z_1|^2 + |z_2|^2\right) + \frac{q}{2(\alpha f - \beta g)}\quad,\end{aligned}$$

$$
\begin{aligned}
q = & -(\alpha+\beta)(f+g)x^2 - (\alpha-\beta)(f-g)y^2 + 2i(\beta f - \alpha g)xy \\
& +2x\left[(\alpha+\beta)z_1 + (f+g)z_2\right] + 2iy\left[(\alpha-\beta)z_1 - (f-g)z_2\right] \\
& +(\alpha g^* - \beta f^*)z_1^2 + (\beta^* f - \alpha^* g)z_2^2 - 2z_1 z_2 \,. \qquad (2.81)
\end{aligned}
$$

Solutions from [10–13] are particular cases of (2.81) for $f_0 = \alpha = 1,\ g = \beta = 0.$

Calculating mean values in the states (2.78), we get[1]

$$
\overline{\mathcal{P}}_1 = i\hbar\sqrt{\frac{\gamma}{2}}\left[(f^*+g^*)\,z_1 - (f+g)\,z_1^*\right], \quad \overline{\mathcal{P}}_2 = -\hbar\sqrt{\frac{\gamma}{2}}\left[(f^*-g^*)\,z_1 + (f-g)\,z_1^*\right]. \quad (2.82)
$$

Here we have taken into account the relations (2.6), (2.36), (2.79), and the orthogonality of the states with respect to the indices n, s. Remember now that in classical theory the corresponding momenta $\mathcal{P}_1^{cl}, \mathcal{P}_2^{cl}$ have the following parametric representation (with u^0 being the evolution parameter, R radius of the classical orbit, and κ is given by (2.67))

$$
\mathcal{P}_1^{cl} = \hbar\gamma R \sin 2\kappa, \quad \mathcal{P}_2^{cl} = -\hbar\gamma R\cos 2\kappa \,. \qquad (2.83)
$$

It is easy to see that (2.82) coincides with (2.83) for $z_1 = (\gamma/2)^{1/2}\, R\,(f_0 e^{-i\varphi_0} + g_0 e^{i\varphi_0})$. Calculating mean values of the coordinates $\overline{x^1}$, $\overline{x^2}$, we find that they evolve as the corresponding classical quantities $x^{1\,cl}$, $x^{2\,cl}$ ($x^1_{(0)}, x^2_{(0)}$ are coordinates of the orbit center)

$$
x^{1\,cl} = R\cos\kappa + x^1_{(0)}, \quad x^{2\,cl} = R\sin\kappa + x^2_{(0)}\,, \qquad (2.84)
$$

for $z_2 = (\gamma/2)^{1/2}\left[(\alpha+\beta)x^1_{(0)} + i(\alpha-\beta)x^2_{(0)}\right]$.

Thus, mean-value trajectories in the plane x^1, x^2 do not depend on quantum numbers n, s. These trajectories have classical forms under a proper choice of z_1, z_2 .

Calculating quadratic fluctuations in the states (2.78), we get

$$
\begin{aligned}
& 2\overline{(\Delta\mathcal{P}_1)^2} = \hbar^2\gamma|f+g|^2(2n+1), \quad 2\overline{(\Delta\mathcal{P}_2)^2} = \hbar^2\gamma\,|f-g|^2\,(2n+1)\,, \\
& 2\gamma\overline{(\Delta x^1)^2} = |f-g|^2(2n+1) + |\alpha-\beta|^2(2s+1)\,, \\
& 2\gamma\overline{(\Delta x^2)^2} = |f+g|^2(2n+1) + |\alpha+\beta|^2(2s+1)\,,
\end{aligned}
$$

[1]One can obtain the same results using spinor wave functions for the calculations.

$$\sigma_1 = -\sigma_2 = i\,(fg^* - gf^*)\,(2n+1)\ ,$$
$$\sigma_k = \overline{(\Delta x^k)\,(\Delta P_k) + (\Delta P_k)\,(\Delta x^k)}, \quad k = 1, 2\ . \tag{2.85}$$

They do not depend on z_1, z_2, but do depend on quantum numbers n, s and on parameters f_0, g_0, α, β. The relations (2.85) imply the generalized Heisenberg inequalities

$$4\mathcal{J}_1 = \hbar^2(2n+1)\left[(2n+1) + (2s+1)\,|(\alpha-\beta)(f+g)|^2\right] \geq \hbar^2\ ,$$
$$4\mathcal{J}_2 = \hbar^2(2n+1)\left[(2n+1) + (2s+1)\,|(\alpha+\beta)(f-g)|^2\right] \geq \hbar^2\ .$$
$$\mathcal{J}_k = \overline{(\Delta x^k)^2}\;\overline{(\Delta \mathcal{P}_k)^2} - \frac{1}{4}\sigma_k^2, \quad k = 1, 2\ . \tag{2.86}$$

One can fix any given $\overline{(\Delta x^k)^2}$ or $\overline{(\Delta \mathcal{P}_k)^2}$ in a given "instant" u^0 by means of a choice of parameters f_0, g_0, α, β . Then they evolve with "time" u^0 according Eqs. (2.85).

Below we present another type of nonstationary states, which are quite different from the above generalized coherent states. Recall that the problem was reduced to solving the equation (2.65) under the condition (2.63). All the integrals of motion for such an equation can be constructed as functional combination of the operators

$$fa_1, \quad ga_1^+, \quad a_2, \quad a_2^+, \tag{2.87}$$

whenever f, g obey the relations (2.69), (2.70). Constructing integrals of motion that are linear combinations of these operators, we get coherent states. Any linear combinations of the operators (2.87) do not commute with the operator L_z (2.8) or J_z (2.4). Thus, coherent states with definite values of these quantities cannot be constructed. The generalized squeezed coherent states (2.44) and (2.77) are eigenvectors of the operators $\mathcal{N}_1, \mathcal{N}_2$ (that follows from (2.79). The latter operators are integrals of motion and are quadratic in creation and annihilation operators,

$$\mathcal{N}_1 = \left(A_1^{+\,f,g} - z_1^*\right)\left(A_1^{f,g} - z_1\right), \quad \mathcal{N}_2 = \left(A_2^{+\,\alpha,\beta} - z_2^*\right)\left(A_2^{\alpha,\beta} - z_2\right). \tag{2.88}$$

The operators $A_1^{f,g}$ are defined in (2.68), and $A_2^{\alpha,\beta}$ are defined in (2.35). The operators (2.88) do not commute with L_z , J_z as well.

One can see that besides the operators $a_1 a_2$, $a_1^+ a_2^+$, the only one quadratic combination that commutes with L_z , J_z is

$$\bar{A} = f a_1 a_2 + g a_1^+ a_2^+ . \tag{2.89}$$

It is known [15] that eigenvectors for such an operator can be normalized only for $|f| > |g|$, or for $|f| = |g|$. In the first case the eigenvectors have a finite norm, and in the second case they can be normalized to a $\delta-$function. Let us consider the case $|f| \geq |g|$ only. Here the operator (2.89) differs from

$$A^p = e^{i\kappa}\left(a_1 a_2 - \bar{p}^2 a_1^+ a_2^+\right), \ \bar{p} = p e^{-i\kappa}, \ -1 \leq p \leq 1, \ \kappa = \omega u^0 + \kappa_0, \ \kappa_0 = \text{const} \tag{2.90}$$

by a complex factor only. Thus, it is enough to consider the latter operator only. Let us demand that functions $\psi\left(u^0, \rho, \varphi\right)$ be solutions of the equation (2.65), and, at the same time, eigenvectors of the operators A^p, L_z,

$$A^p \psi_{q,l}^p = -q \psi_{q,l}^p, \quad L_z \psi_{q,l}^p = \hbar l \psi_{q,l}^p, \quad l = 0, \pm 1, \pm 2, \ldots . \tag{2.91}$$

Such solutions can be constructed in terms of the Laguerre functions $I_{n,m}(x)$ with noninteger indices,

$$\begin{aligned}
&\psi_{q,l}^p\left(u^0, \rho, \varphi\right) = N \exp\left(il\varphi - \Gamma\right)\left(1 + \bar{p}\right)^{-\alpha}\left(1 - \bar{p}\right)^{-\beta} I_{|l|+s,s}(x) \ , \\
&\Gamma = i\frac{l\omega u^0}{2} + \frac{1 + \bar{p}^2}{2\left(1 - \bar{p}^2\right)}\rho, \quad \alpha = \frac{p - q}{2p}, \quad \beta = \frac{p + q}{2p} \ , \\
&s = \frac{q}{2p} - \frac{|l| + 1}{2}, \quad x = \frac{2\bar{p}\rho}{1 - \bar{p}^2} \ , \\
&I_{n,m}(x) = \sqrt{\frac{\Gamma(1+n)}{\Gamma(1+m)\Gamma(1+n-m)}}\frac{\exp\left(-\frac{x}{2}\right)}{}x^{\frac{n-m}{2}}\Phi\left(-m, n - m + 1; x\right) .
\end{aligned} \tag{2.92}$$

Here $\Phi(\alpha, \beta; x)$ is the degenerate hypergeometric function (see [14], 9.210). For $p^2 = 1$, the operator (2.90) is anti-Hermitian and q is imaginary, Re $q = 0$. For $p = 0$, solutions have a very simple form

$$\psi_{q,l}^0\left(u^0, \rho, \varphi\right) = N_0 \exp\left(il\varphi + \bar{q} - \Gamma_0\right) J_{|l|}\left(2\sqrt{\bar{q}\rho}\right) \ , \quad \Gamma_0 = \frac{i}{2} l\omega u^0 + \frac{\rho}{2}, \quad \bar{q} e^{-ik} \ , \tag{2.93}$$

where $J_\nu(x)$ is the Bessel function (see [14], 8.402). The functions (2.93) can be obtained from (2.92) as a limit $p \to 0$, as can be seen with the help of the property

$$\lim_{r\to\infty} I_{r+\alpha,r+\beta}\left(\frac{x^2}{4r}\right) = J_{\alpha-\beta}(x) . \tag{2.94}$$

The functions (2.92) and (2.93) are orthogonal only with respect to quantum numbers l,

$$\begin{aligned}
&\left(\psi^p_{q',l'},\psi^p_{q,l}\right) = \delta_{l,l'} Q F\left(-s,-s'^*;1+|l|;y\right), \quad y = \left(\frac{2p}{1+p^2}\right)^2 , \\
&Q = \left[\frac{\Gamma(1+|l|+s)\,\Gamma\left(1+|l|+s'^*\right)}{p^2\Gamma(1+s)\,\Gamma(1+s'^*)}\right]^{\frac{1}{2}} \frac{\pi N N'^*}{\Gamma(1+|l|)} y^{\frac{1+|l|}{2}} (1-y)^{-\frac{q+q^*}{4p}} , \\
&\left(\psi^0_{q',l'},\psi^0_{q,l}\right) = \delta_{l,l'} 2\pi N_0 N_0'^* I_{|l|}\left(2\sqrt{qq'^*}\right) .
\end{aligned} \tag{2.95}$$

Here $F(\alpha,\beta;\gamma;x)$ is the hypergeometric function (see [14], 9.100), and $I_\alpha(x)$ is the Bessel function of imaginary argument (see [14], 8.404). Calculating (2.95), we have used the integral table (see [14], 6.633.2; 7.622.1).

The states (2.92) are not coherent states, however, they are, in a sense, close to such states. Indeed, let us consider the equations (2.6) on classical trajectories. Then we get a classical relation

$$\rho = \rho\left(u^0\right) = \sqrt{L_z^2\hbar^{-2} + 4\left|a_1 a_2\right|^2} - a_1 a_2 - a_1^+ a_2^+ . \tag{2.96}$$

For $p = 0$, it follows from (2.91) that $a_1 a_2 = -\bar{q}, \quad L_z = \hbar l$. Thus, we can rewrite (2.96) in the form

$$\rho\left(u^0\right) = \rho_0^{cl} + \bar{q} + \bar{q}^*, \quad \rho_0^{cl} = \sqrt{l^2 + 4|q|^2} . \tag{2.97}$$

Calculating the mean value $\bar{\rho}$ by means of the functions (2.93), we find

$$\bar{\rho} = \rho_0 + \bar{q} + \bar{q}^*, \quad \rho_0 = |l| - 1 - 2|q| \frac{I_{|l|-1}(2|q|)}{I_{|l|}(2|q|)} . \tag{2.98}$$

Thus, the time dependence of $\bar{\rho}$ is classical. The only constant which can differ from its classical value is ρ_0 .

III. EXACT SOLUTIONS OF RELATIVISTIC WAVE EQUATIONS IN LONGITUDINAL FIELDS

A. Definition of fields

Consider here longitudinal electromagnetic fields, which have the form

$$\mathbf{E} = \mathbf{n}E, \quad \mathbf{H} = \mathbf{n}H. \tag{3.1}$$

Here $\mathbf{n}$ is a unit vector, $\mathbf{n}^2 = 1$. Suppose $\mathbf{n}$ is directed along x^3 axis. Then, the fields (3.1) obey the free Maxwell equations whenever

$$E = E\left(x^0, x^3\right), \quad H = H\left(x^1, x^2\right),$$

where $E\left(x^0, x^3\right),\ H\left(x^1, x^2\right)$ are arbitrary functions of the indicated arguments. Thus, the fields under consideration can be represented by potentials of the form

$$\begin{aligned} &A_0 = A_0\left(x^0, x^3\right),\ A_1 = A_1\left(x^1, x^2\right),\ A_3 = A_3\left(x^0, x^3\right),\ A_2 = A_2\left(x^1, x^2\right), \\ &E = \partial_0 A_3 - \partial_3 A_0,\ H = \partial_2 A_1 - \partial_1 A_2 . \end{aligned} \tag{3.2}$$

Thus, the operators (2.6) do not depend on the electric field (on $A_0,\ A_3$). Therefore, imposing restrictions only on the magnetic field, we can maintain the relations (2.6-2.11). For a uniform magnetic field (2.1), the commutation relations (2.7) are still valid and we use the semi-momentum representation, where these operators act on different variables (see (2.10-2.12), (2.17)).

Lorentz equations have the following form

$$\begin{aligned} &m\ddot{x}^0 + E\dot{x}^3 = 0, \quad m\ddot{x}^3 + E\dot{x}^0 = 0; \\ &m\ddot{x}^1 + H\dot{x}^2 = 0, \quad m\ddot{x}^2 + H\dot{x}^1 = 0,\ \dot{x}^\mu\dot{x}_\mu = 1 , \end{aligned} \tag{3.3}$$

which implies the following first integrals of motion,

$$m^2\left((\dot{x}_1)^2 + (\dot{x}_2)^2\right) = k_1^2, \quad m^2\left((\dot{x}_0)^2 - (\dot{x}_3)^2\right) = m^2 + (k_1)^2 , \tag{3.4}$$

where k_1 is an integration constant.

B. Klein-Gordon equation

Consider the Klein-Gordon equation in the fields under consideration. Representing the wave function as

$$\Psi = \varphi\left(x^0, x^3\right)\psi\left(x^1, x^2\right), \tag{3.5}$$

we find

$$\left(\mathcal{P}_1^2 + \mathcal{P}_2^2 - k_1^2\right)\psi\left(x^1, x^2\right) = 0,\ \left(\mathcal{P}_0^2 - \mathcal{P}_3^2 - m^2 - k_1^2\right)\varphi\left(x^0, x^3\right) = 0\,. \tag{3.6}$$

Using the variables x, y, η, ξ defined in (2.5) and (2.12), we can rewrite the equation for $\psi\left(x^1, x^2\right)$ in the following form

$$\left(\xi^2 - \partial_\xi^2 - k_1'^2\right)\psi\left(x^1, x^2\right) = 0, \quad k_1'^2 = \frac{k_1^2}{\hbar^2\gamma}\,. \tag{3.7}$$

The operator $\left(\xi^2 - \partial_\xi^2 - k_1'^2\right)$ does not depend on η . Thus, it is convenient to go over to the semi-momentum representation,

$$\psi\left(x, y\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iky}\tilde{\psi}\left(x, k\right)\,dk.$$

Substituting the integration over k by an integration over η, we obtain

$$\psi\left(x, y\right) = \frac{e^{ixy}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-i\sqrt{2}y\eta}\tilde{\psi}\left(\xi, \eta\right)\,d\eta, \quad \xi = \sqrt{2}x - \eta. \tag{3.8}$$

The function ψ is an eigenvector of the operator $\mathcal{N}$ (2.14). The latter operator commutes with the operators from Eqs. (3.6). In the semi-momentum representation we can write

$$\mathcal{N}\tilde{\psi}_n = n\tilde{\psi}_n \Rightarrow \tilde{\psi}_n\left(\xi, \eta\right) = U_n\left(\xi\right)\Phi\left(\eta\right), \tag{3.9}$$

where $\Phi\left(\eta\right)$ is an arbitrary function of η. It follows from (2.8) and the first Eq. (3.6) that

$$\mathcal{P}_1^2 + \mathcal{P}_2^2 = 2\hbar^2\gamma\mathcal{N} + \hbar^2\gamma,\ k_1^2 = 2\hbar^2\gamma n + \hbar^2\gamma\,.$$

Solution of the last equation (3.6) can be found for fields that admit separation of the variables x^0, x^3. For example, let us choose the potentials in the form: $|e|\,A_0 = A\left(x^3\right),\ A_3 = 0,\ |e|\,E = -\partial_3 A$. In this case, stationary solutions of Eq. (3.6) read

$$\varphi\left(x^0,x^3\right)=e^{-ik_0x^0}\chi\left(x^3\right),\quad \chi''+R\chi=0,\quad R\left(x^3\right)=\left(k_0+A\right)^2-m^2-k_1^2. \tag{3.10}$$

Thus, the functions (3.5) being written in the semi-momentum representation take the form

$$\Psi_n=e^{-ik_0x^0}\chi\left(x^3\right)U_n\left(\xi\right)\Phi\left(\eta\right). \tag{3.11}$$

The equation (3.10) for χ can be solved exactly for the following choices of the function $A\left(x^3\right)$:

$$A\left(x^3\right)=\alpha x,\quad A\left(x^3\right)=\alpha\exp\left(\beta x^3\right),\quad A\left(x^3\right)=\frac{\alpha}{x^3},$$
$$A\left(x^3\right)=\alpha\tanh\left(\beta x^3\right),\quad A\left(x^3\right)=\alpha\tan\left(\beta x^3\right),\quad A\left(x^3\right)=\beta\coth\left(\beta x^3\right).$$

The corresponding exact solutions are presented in [3]. Demanding that the Klein-Gordon function be an eigenvector of the operator L_z (2.16), we get an equation for the function $\Phi\left(\eta\right)$ from (3.11),

$$a_2^{+}a_2\Phi\left(\eta\right)=s\Phi\left(\eta\right). \tag{3.12}$$

Thus, $\Phi\left(\eta\right)=U_s\left(\eta\right)$, and $L_Z\Psi_{ns}=\hbar\left(n-s\right)\Psi_{ns}$. Keeping this in mind and doing the integral (3.8), we obtain

$$\Psi_{n,s}=e^{-ik_0x^0}\chi\left(x^3\right)\frac{\left(-1\right)^n}{\sqrt{2\pi}}\left(\frac{x+iy}{x-iy}\right)^{\frac{n-s}{2}}I_{n,s}\left(x^2+y^2\right),$$

where $I_{n,s}$ are the Laguerre functions. Solutions of the equation (3.12) have been analyzed above.

C. Dirac equation

Let us present the Dirac wave function in the form

$$\Psi=\mathbf{M}\begin{pmatrix}\psi_1\\ \psi_{-1}\end{pmatrix}\varphi\left(x^0,x^3\right)v,$$
$$\mathbf{M}=\begin{pmatrix}k_1\left(m+\mathcal{P}_0+\mathcal{P}_3-ik_1\sigma_2\right) & \left(\mathcal{P}_1-i\mathcal{P}_2\right)\left(m+\mathcal{P}_0+\mathcal{P}_3-ik_1\sigma_2\right)\\ \left(\mathcal{P}_1+i\mathcal{P}_2\right)\left[\left(m-\mathcal{P}_0-\mathcal{P}_3\right)\sigma_3-k_1\sigma_1\right] & k_1\left[\left(m-\mathcal{P}_0-\mathcal{P}_3\right)\sigma_3-k_1\sigma_1\right]\end{pmatrix},$$

where ν is an arbitrary spinor, which can be fixed by supplementary conditions. Then the following equations take place

$$\left(\mathcal{P}_0^2 - \mathcal{P}_3^2 - m^2 - k'\right)\varphi\left(x^0, x^3\right) = 0, \quad k' = k_1^2 + ieE, \tag{3.13}$$

$$\left(\mathcal{P}_1 + i\mathcal{P}_2\right)\psi_1\left(x^1, x^2\right) = \hbar k_1 \psi_{-1}\left(x^1, x^2\right), \tag{3.14}$$

$$\left(\mathcal{P}_1 - i\mathcal{P}_2\right)\psi_{-1}\left(x^1, x^2\right) = \hbar k_1 \psi_1\left(x^1, x^2\right). \tag{3.15}$$

As a consequence of Eqs. (3.14) and (3.15), we get

$$a_1\psi_{-1} = -i\sqrt{n}\psi_1\,, \; a_1^+\psi_1 = i\sqrt{n}\psi_{-1}, \quad k_1^2 = 2\gamma n.$$

Thus, we see that $\psi_1 = \psi_{n-1}\,,\ \psi_{-1} = -i\psi_n$, and the problem is reduced to solving the equation (3.13). The latter coincides with the second equation (3.6).

Considering, for example, the constant and uniform magnetic field (2.1) together with the electric field described by potentials $|e|\,A_0 = A\left(x^3\right), \quad A_3 = 0 \Rightarrow |e|\,E = -\partial_3 A$, we get

$$\varphi\left(x^0, x^3\right) = \exp\left(-ik_0x^0\right)\chi\left(x^3\right), \quad \chi'' + \left(i\partial_3 + k_0 + \mathcal{A}\right)\chi = 0\,.$$

All possible solutions of the latter equation for the function χ are presented in [3].

PECULIARITIES OF INTEGRATION OF LINEAR DIFFERENTIAL EQUATIONS WITH NONCOMMUTATIVE SYMMETRIES

Here we are going to return to the above results from a point of view of general theory of differential equation. Recall that we succeeded to find explicitly the transformation (2.10)-(2.13) which has reduced effectively the number of the variables in the initial equations. In fact, that was the main starting point for all further constructions. However, one can see that this "reduction" of variables is a particular example of a general situation, which is described briefly below.

Consider first the case of an integrable classical $2N$-dimensional Hamiltonian system with the Hamiltonian H. Suppose this system has N independent integrals of motion that are in involution. It is well known that in such a case the variables of the type action-angle (J, φ) are available, and the Hamiltonian depends on the action variables only, $H = H(J)$. Let us suppose that for such a system exists one more independent integral of motion Y. Since Y is independent, it cannot commute with the former integrals, and, therefore, Y must depend on the angle variables. One can demonstrate that in such a case the Hamiltonian system is degenerate, $\det ||\partial H(J)/\partial J_i \partial J_j|| = 0$, and, therefore, the Hamiltonian does not depend on some combinations of the action variables. For example, suppose the integral Y does not commute with the integral J_N only. Then the Hamiltonian can depend on the variables $J_1, \ldots, J_{N-1}$ only, otherwise H cannot commute with Y . Thus, we see that the noncommutative algebra of integrals of motion allows one to find canonical variables such that part of the corresponding action variables disappears from the Hamiltonian. This phenomenon is closely related to the topological properties of orbits for the Hamiltonian system. Namely, trajectories of the integrable Hamiltonian system with N-dimensional commutative set of integrals of motion form (in the compact case) a winding of N-torus in $2N$-dimensional phase space. If the set of the integrals is noncommutative then the dimension of the corresponding torus is $r < N$ (see [16]).

The phenomenon of variable "reduction" takes place in the quantum integrable Hamiltonian system as well. As will be demonstrated below, by constructing a special isomorphism of linear functional spaces, we can transform the initial differential operator of an equation into another one with a reduced number of variables. The method which we are going to use for such a demonstration, is, in fact, the harmonic analysis on the noncommutative functional algebras.

Suppose a differential equation

$$H(x, \partial_x)\psi(x) = 0, \quad \psi \in \mathcal{L} \subset C^\infty(R^N) \tag{3.16}$$

with N independent variables $x \in R^N$ admits a noncommutative algebra of functionally independent symmetry operators $\mathcal{F} = \{X_a(x, \partial_x)\}$. The corresponding commutation relations are in the general case nonlinear,

$$\frac{i}{\hbar}[X_a, X_b] = \Omega_{ab}(X), \quad a, b = 1, \ldots, n \equiv \dim \mathcal{F}.. \tag{3.17}$$

Here $\Omega_{ab}(X)$ are symmetric operator functions. The linear case, when $\Omega_{ab}(X) = C^c_{ab}X_c$, corresponds to a Lie algebra, the quadratic symmetric functions $\Omega_{ab}(X)$ correspond to a quadratic algebra, and so on. The algebra $\mathcal{F}$ corresponds to the algebra $\mathcal{F}' = \{Y_\alpha(x, \partial_x)\}$ of the invariant operators on $\mathcal{L}$:

$$[X_a, Y_\alpha] = 0, \quad \frac{i}{\hbar}[Y_\alpha, Y_\beta] = \omega_{\alpha\beta}(Y), \quad \alpha, \beta = 1, \ldots, n' \equiv \dim \mathcal{F}'. \tag{3.18}$$

We denote via $E(\mathcal{F})$ and $E(\mathcal{F}')$ enveloping fields for the algebras $\mathcal{F}$ and $\mathcal{F}'$ respectively. Elements of $E(\mathcal{F})$ and $E(\mathcal{F}')$ are symmetrized operator functions of the generating operators $X_a\, Y_\alpha$. It is clear that the centers of the enveloping fields coincide, i.e. $Z(E(\mathcal{F})) = Z(E(\mathcal{F}'))$. The elements of the center $Z = Z(E(\mathcal{F}))$ are called Casimir operators. The number of the independent Casimir operators, which generate the center Z is called the index of the algebra $\mathcal{F}$: $r \equiv \operatorname{ind}\mathcal{F} = \operatorname{ind}\mathcal{F}'$. If we replace the operators X and Y in the operator functions $\omega_{\alpha\beta}(Y)$ and $\Omega_{ab}(X)$ by arbitrary complex numbers ξ and f, then the index of the algebras $\mathcal{F}$ and $\mathcal{F}'$ can be calculated according to the formula

$$r = \sup_{\xi \in C} \operatorname{corank} \Omega_{ab}(\xi) = \sup_{f \in C} \operatorname{corank} \omega_{\alpha\beta}(f)\,. \tag{3.19}$$

One can show that the following relation takes place

$$n + n' = 2N. \tag{3.20}$$

Let us introduce the notion of the λ-representation of the algebra $\mathcal{F}$ [17]. In fact, the λ-representation is the result of the quantization of the classical Poisson bracket and can be understood as a realization of the algebra $\mathcal{F}$ by an irreducible set of differential operators $\tilde{X} = \tilde{X}(q, \partial_q, j)$, dependent on r parameters $j = (j_1, \ldots, j_r)$, and acting in a space of functions of $[q] = (n - r)/2$ independent variables[2] $q \in Q$,

[2]Via $[q]$ we denote the number of the variables q. Similar notations are used below.

$$\frac{i}{\hbar}[\tilde{X}_a, \tilde{X}_b] = -\Omega_{ab}(\tilde{X}), \quad K_\mu(\tilde{X}(q, \partial_q, j)) = \kappa_\mu(j), \quad \det\left\|\frac{\partial\kappa_\mu(j)}{\partial j_\nu}\right\| \neq 0\,. \tag{3.21}$$

Here K_μ are all the independent Casimir operators of $\mathcal{F}$. In a similar manner, we construct the λ-representation $\{\tilde{Y}\}$ of $\mathcal{F}'$ in a space of functions of $[q'] = (n' - r)/2$ independent variables $q' \in Q'$,

$$\frac{i}{\hbar}[\tilde{Y}_\alpha, \tilde{Y}_\beta] = \omega_{\alpha\beta}(\tilde{Y}), \quad K'_\mu(\tilde{Y}(q', \partial_{q'}, j)) = K_\mu(\tilde{X}(q, \partial_q, j)) = \kappa_\mu(j)\,. \tag{3.22}$$

Suppose that in the spaces of functions of x, of q, and of q' are defined scalar products

$$(\varphi, \psi) = \int\limits_{R^N} \overline{\varphi(x)}\psi(x)d\mu(x),\ (\tilde{\varphi}, \tilde{\psi}) = \int\limits_{Q} \overline{\tilde{\varphi}(q)}\tilde{\psi}(q)d\mu(q),\ (\tilde{\varphi}, \tilde{\psi})' = \int\limits_{Q'} \overline{\tilde{\varphi}(q')}\tilde{\psi}(q')d\mu(q'), \tag{3.23}$$

where $d\mu(x)$, $d\mu(q)$, and $d\mu(q')$ are some measures on R^N, Q, and Q' respectively. And suppose that the operators $X_a(x, \partial_x)$, $Y_\alpha(x, \partial_x)$ and the operators $\tilde{X}_a(q, \partial_q, j), \tilde{Y}(q', \partial_{q'}, j)$ are self-conjugate with respect to the corresponding scalar products[3]. Now we define the set of distributions $D^j_{qq'}(x)$ as a solution of the overdetermined system of the equations:

$$\left[X_a(x, \partial_x) - \tilde{X}_a(q, \partial_q, j)\right] D^j_{qq'}(x) = 0; \ \left[Y_\alpha(x, \partial_x) - \tilde{Y}_a(q', \partial_{q'}, j)\right] D^j_{qq'}(x) = 0\,. \tag{3.24}$$

The distributions $D^j_{qq'}(x)$ obey the completeness and orthogonality relations:

$$\int \overline{D^j_{qq'}(x)} D^{\tilde{j}}_{\tilde{q}\tilde{q}'}(x)\, d\mu(x) = \delta(j, \tilde{j})\delta(q, \tilde{q})\delta(q', \tilde{q}'); \tag{3.25}$$

$$\int \overline{D^j_{qq'}(x)} D^j_{qq'}(\tilde{x})\, d\mu(j) d\mu(q) d\mu(q') = \delta(x, \tilde{x}). \tag{3.26}$$

Here $d\mu(j)$ is the spectral measure of the Casimir operators $K(X)$ $(= K'(Y))$. Due to Eqs. (3.21), (3.22) (3.24) the distributions $D^j_{qq'}(x)$ are eigenfunctions of all the Casimir operators,

$$K_\mu(X(x, \partial_x)) D^j_{qq'}(x) = \kappa_\mu(j) D^j_{qq'}(x), \quad \mu = 1, \ldots, r\,. \tag{3.27}$$

Usually one can find $D^j_{qq'}(x)$ by an integration, at least in the case when $\mathcal{F}$ is a Lie algebra. As a consequence of (3.26) and (3.25) we can define the direct and the inverse Fourier transforms:

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[3]This supposition is not necessary and is introduced to simplify the consideration.

$$\tilde{\psi}(q,q',j) = \int D^j_{qq'}(x)\overline{\psi(x)}\,d\mu(x)\,, \tag{3.28}$$

$$\psi(x) = \int D^j_{qq'}(x)\overline{\tilde{\psi}(q,q',j)}d\mu(j)d\mu(q)d\mu(q')\,. \tag{3.29}$$

The Eqs. (3.28) and (3.29) establish an isomorphism of the spaces $\mathcal{L}$ and $\tilde{\mathcal{L}} = \{\tilde{\psi}\}$. It is important to stress that under such an isomorphism the operators X, Y are transformed into the differential operators $\tilde{X}, \tilde{Y}$ that act in the spaces of functions, which depend on a fewer number of variables,

$$X(x,\partial_x)\psi(x) \leftrightarrow \tilde{X}(q,\partial_q,j)\tilde{\psi}(q,q',j), \quad Y(x,\partial_x)\psi(x) \leftrightarrow \tilde{Y}(q',\partial_{q'},j)\tilde{\psi}(q,q',j). \tag{3.30}$$

Let us return to the equation (3.16). Here we can conclude that $H \in E(\mathcal{F}')$ since the operator H commutes with all the operators of the algebra $\mathcal{F} = \{X_a\}$. In turn, that means that there exists an operator function $H(Y)$ such that $H(x,\partial_x) = H(Y(x,\partial_x))$. Let us look for solutions of the equation (3.16) in the form (3.29). Using the isomorphism (3.30), we get:

$$H(\tilde{Y}(q',\partial_{q'},j))\tilde{\psi}(q,q',j) = 0. \tag{3.31}$$

Thus, departing from the equation (3.16) without loss of any information, we have arrived to a differential equation with $\tilde{N} = [q'] = (n'-r)/2$ independent variables. Taking into account (3.20), one can obtain

$$\tilde{N} = N - \frac{1}{2}(\dim\mathcal{F} + \operatorname{ind}\mathcal{F}). \tag{3.32}$$

Thus, the existence of a noncommutative symmetry algebra results into the phenomenon of variable reduction. Indeed, we started with $N = [q'] + [q] + r$ variables. $[q]$ variables have disappeared completely, and $r = [j]$ variables remain in the equations as some parameters. Respectively, the solution of the equation (3.31) contains as a factor an arbitrary function of the variables q, j. The number of variables q is equal to $[q] = (n-r)/2 = (\dim\mathcal{F} - \operatorname{ind}\mathcal{F})/2$. In the commutative case: $\operatorname{ind}\mathcal{F} = \dim\mathcal{F}$, and $[q] = 0$. In the noncommutative case: $\dim\mathcal{F} > \operatorname{ind}\mathcal{F}$, and $[q] > 0$. Thus, the reduction of variables always takes place when there exists a noncommutative algebra of the integrals of motion.

Let us apply the above consideration to the Klein-Gordon equation (2.2) in a uniform magnetic field. In this case we have four ($N = 4$) variables and five ($n = 5$) independent symmetry operators: $\mathcal{F} = \{P_0, P_3, a_2, a_2^+, L_z = \hbar L\}$,

$$P_0 = i\hbar\partial_0,\; P_3 = i\hbar\partial_3,\; L = u\partial_u - \bar{u}\partial_{\bar{u}}\,,\; a_2 = \partial_{\bar{u}} + u/2,\; a_2^+ = -\partial_u + \bar{u}/2\,,\; u \equiv x + iy\,.$$

The non-zero commutation relations are $[a_2, a_2^+] = 1, \quad [L, a_2] = a_2, \quad [L, a_2^+] = -a_2^+$. It follows from (3.19) that $r = \operatorname{ind}\mathcal{F} = 3$. $\tilde{N} = [q'] = 0\ \mathcal{K} \in Z$ according to (3.32). Thus, the equations (3.31) present algebraic relations on the parameters j (and on the parameters of the equation itself). Besides, $\dim\mathcal{F}' = 3 = r$, due to (3.20). Thus, the algebra of the invariant operators is placed completely in the center. Or more simply, there are no operators $Y, \tilde{Y}$ and variables q' in the case under consideration. The center Z is generated by three Casimir operators, those are $K_1 = P_0,\; K_2 = P_3,\; K_3 = \mathcal{N} = L + a_2^+ a_2 = L + \frac{1}{2}(a_2^+ a_2 + a_2 a_2^+ - 1)$.

Let us construct the λ-representation of the algebra $\mathcal{F}$:

$$\tilde{P}_0 = j_1 = \hbar k_0\,,\; \tilde{P}_3 = j_2 = \hbar k_3\,,\; \tilde{a}_2 = q\,,\; \tilde{a}_2^+ = \partial_q\,,\; \tilde{L} = -q\partial_q + n\,,\; n = j_3 = 0, 1, \ldots\,.$$

The operators $\tilde{a}_2$ and $\tilde{a}_2^+$ are mutually conjugate with respect to the scalar product (3.23) with the measure $d\mu(q) = \exp(-q\bar{q})d^2q/\pi, \quad (d^2q \equiv dq_1 dq_2,\; q = q_1 + iq_2)$. The operator $\tilde{L}$ is self-conjugate, and the space $\tilde{\mathcal{L}}$ is built up from analytic functions dependent on the variable q and on the parameters $j = (k_0, k_3, n)$. Here the Casimir operator $\tilde{\mathcal{N}}$ has the following form $\tilde{\mathcal{N}} = \tilde{L} + \frac{1}{2}(\tilde{a}_2^+\tilde{a}_2 + \tilde{a}_2\tilde{a}_2^+ - 1) = n$.

From the equations (3.24) we can find the set $D_q^j(x)$ that obeys the completeness and orthogonality relations. Such a set has the form

$$D_q^j(x) = e^{i(k_0 x^0 + k_3 x^3)} e^{q\bar{u} - u\bar{u}/2}(u - q)^n/(2\pi\sqrt{\pi n!})\,, \tag{3.33}$$

$$\int \overline{D_q^j(x)} D_{\tilde{q}}^{\tilde{j}}(x)\, dx = \delta(k_0 - \tilde{k}_0)\delta(k_3 - \tilde{k}_3)\delta(\tilde{q}, q)\delta_{n\tilde{n}}\,, \tag{3.34}$$

$$\sum_{n=0}^{\infty} \int \overline{D_q^j(x)} D_q^j(\tilde{x})\, dk_0 dk_3 d\mu(q) = \delta(x^0 - \tilde{x}^0)\delta(x^3 - \tilde{x}^3)\delta(x - \tilde{x})\delta(y - \tilde{y}). \tag{3.35}$$

In Eq.(3.34) $\delta(\tilde{q}, q) = \exp(\bar{q}\tilde{q})$ is a δ-function with respect to the measure $d\mu(q)$. To justify the validity of (3.34) and (3.35) one can use the following relations [18]

$$\int \overline{v_n(q)} v_m(q)\, d\mu(q) = \delta_{nm},\ \sum_{n=0}^{\infty} \overline{v_n(q)} v_n(\tilde{q}) = \delta(\tilde{q}, q) = \exp(\bar{q}\tilde{q})\,,\ v_n(q) \equiv q^n/\sqrt{n!}\,.$$

As was already mentioned above, the Klein-Gordon operator $\mathcal{K}$ belongs to the center Z , and , therefore, can be presented in a polynomial form of the Casimir operators $P_0, P_3, \mathcal{N}$, which generate this center. Such a representation is given by Eq. (2.9). The Klein-Gordon equation in the space $\tilde{\mathcal{L}}$, i.e. the equation (3.31) is, in fact, the relation (2.21) for the parameters $j = (k_0, k_3, n)$. Thus, the functions (3.33) form a basis of the Klein-Gordon equation (2.2) (with allowance made for (2.21).

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