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Radiative noise in circuits with inductance: Poynting vector, radiation emitted by the solenoid and stability of the spectrum

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Abstract

We clarify the relationship between the microscopic current fluctuations and the macroscopic concept of radiation resistance R_{rad}. The fluctuations are generated by absorptions and emissions of radiation within the solenoid coils, which exchange energy with the surrounding vacuum. This fact is explained by a detailed calculation of the Poynting vector generated by the solenoid of a simple RLC circuit without batteries. Our study also includes the Nyquist current fluctuations, associated with the ohmic resistance R_{ohmic} of the circuit. We show that the average value of the Poynting vector is zero, in any direction and at any frequency of the electromagnetic spectrum, provided that total resistance R of the circuit is $R_{rad} + R_{ohmic}$. Consequently, as is expected physically, the vacuum radiation pattern is stable and no radiation energy can be detected above the zero-point and thermal background.

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1 Introduction

In a recent paper Blanco, França, Santos and Sponchiado [1] have studied the magnetic noise generated by the vacuum in the coils of the solenoid of a simple RLC circuit without batteries. It is well known from the work of Nyquist [2], Johnson [3] and others (see C. W. Gardiner [4]) that the spectrum of the voltage fluctuations (Nyquist noise) is given by

$$S_N(\omega, T) = \frac{R_N}{\pi}\frac{\hbar\omega}{2} coth\left(\frac{\hbar\omega}{2KT}\right). \tag{1}$$

Here $R_N = R_{ohmic}(\omega, T)$ is simply the ohmic resistance of the circuit, which is a function of frequency ω and temperature T. Blanco et al. [1] have shown that this expression has to be modified if the circuit has a large enough solenoid. The resistance in (1) has to be replaced by

$$R = R_N + \frac{2\pi^2 N^2}{3c}\left(\frac{a\omega}{c}\right)^4, \tag{2}$$

where a is the radius of the solenoid and N is the number of coils. The last term in (2) was called radiation resistance R_{rad} in complete analogy with the resistance which appears in circuits with an external source of power.

The dynamical equation of the circuit can be written as

$$L\dot{I}(t) + RI(t) + \frac{1}{C}\int I(t')dt' = \varepsilon(t), \tag{3}$$

where

$$\varepsilon(t) = \varepsilon_N(t) + \varepsilon_B(t). \tag{4}$$

Here $\varepsilon_N(t)$ is the familiar Nyquist e.m.f. and $\varepsilon_B(t)$ is the new prediction of reference [1]

$$\varepsilon_B(t) = -\frac{\pi a^2 N}{c}\frac{\partial B_z(t)}{\partial t}, \tag{5}$$

where $B_z(t)$ is the component of vacuum magnectic field $\mathbf{B}_{VF}(t)$ in the direction of the axis of the solenoid. The long wavelength approximation was considered by Blanco et al. [1].

The immediate consequence of (5) is that the spectral distribution of the noise will be such that

$$S_\varepsilon(\omega, T) = \left[R_N + \frac{2\pi^2 N^2}{3c}\left(\frac{a\omega}{c}\right)^4\right]\frac{\hbar\omega}{2} coth\left(\frac{\hbar\omega}{2KT}\right), \tag{6}$$

and

$$\langle\varepsilon(t)\varepsilon(0)\rangle = \int_0^\infty d\omega\cos(\omega t) S_\varepsilon(\omega, T) \tag{7}$$

is the correlation function of the random e.m.f..

An important point of the above approach, which we shall address here, is the detailed justification of the real existence of the radiation resistance R_{rad} in the case of microscopic fluctuations of the voltage. In other words, we want to justify why the solenoid presents a radiation resistance despite the fact that we do not observe any radiation coming from the inductance. Notice that the circuit has no batteries, only noise.

We shall show that the associated Poynting vector is zero on average. However, it will be clear from our analysis, that the solenoid of the circuit is continuously absorbing and emitting radiation from the surrounding vacuum, which acts as an energy reservoir. This fact keeps the circuit in dynamical equilibrium with its environment. Moreover, an striking consequence of this processes is that the spectrum of the thermal and zero-point vacuum electromagnetic fields is stable, that is, the electromagnetic vacuum remains isotropic and homogeneous with the same pattern in despite of the presence of the RLC circuit. A similar conclusion was achieved by T. H. Boyer [5] by considering the charged harmonic oscillator in dynamical equilibrium with the vacuum fields.

Our paper is organized as follows. We shall present within section 2 the calculation of the electric and magnectic fields generated by solenoid in the radiation zone. The calculation of the average value of the Poynting vector will be given within section 3. The discussion of our conclusions will be presented in the last part of our paper. As in references [1], [5], [6] and [7] our approach will be based on Stochastic Electrodynamics.

2 Calculations of electric and magnetic fields generated by the solenoid

The magnetic dipole $\boldsymbol{\mu}$ of the solenoid (with radius a and N coils) is given by

$$\boldsymbol{\mu} = \frac{1}{2c}\int \mathbf{r} \times \mathbf{J} d^3r \tag{8}$$

where $\mathbf{J}$ is the current density in the coils. From (8) we get

$$\boldsymbol{\mu} = \frac{\pi N a^2}{c}[I_B(t) + I_N(t)]\hat{\mu}, \tag{9}$$

where $\hat{\mu}$ is the unit vector in the direction of the solenoid axis (z direction).

The random currents $I_N(t)$ and $I_B(t)$ are given by

$$\begin{aligned} \tilde{I}_N(\omega)Z(\omega) &= \tilde{\varepsilon}_N(\omega) \\ \tilde{I}_B(\omega)Z(\omega) &= \tilde{\varepsilon}_B(\omega), \end{aligned} \tag{10}$$

where $\tilde{\varepsilon}_N(\omega)$ is the Fourier transform of the Nyquist e.m.f.

$$\varepsilon_N(t) = \int_{-\infty}^{\infty} d\omega \tilde{\varepsilon}_N(\omega) e^{-i\omega t}. \tag{11}$$

A similar notation is valid for the magnetic e.m.f. $\tilde{\varepsilon}_B(\omega)$, $\tilde{I}_N(\omega)$ and $\tilde{I}_B(\omega)$. The impedance of the circuit is

$$Z(\omega) = R - i\left(L\omega - \frac{1}{C\omega}\right). \tag{12}$$

Notice that the currents $I_N(t)$ and $I_B(t)$ are assumed to be statistically independent, that is

$$\begin{aligned} \langle \tilde{I}_N(\omega)\tilde{I}_B(\omega')\rangle &= 0 \\ \langle \tilde{\varepsilon}_N(\omega)\tilde{\varepsilon}_B(\omega')\rangle &= 0. \end{aligned} \tag{13}$$

The electric field generated by $I_N(t)$ is given by (see Boyer [5])

$$\mathbf{E}_N(\mathbf{r},t) = -2\int_0^\infty d\omega \frac{\pi N a^2}{c}\tilde{I}_N(\omega)e^{-i\omega t}\mathbf{H}, \tag{14}$$

where

$$\mathbf{H} \equiv k^3 e^{ikr}\hat{n}\times\hat{\mu}\left[\frac{1}{kr}+\frac{i}{(kr)^2}\right]. \tag{15}$$

Here $k = \frac{\omega}{c}$ and $\hat{n} = \frac{\mathbf{r}}{r}$ is the unit vector of the direction of observation.

The corresponding magnetic field is given by

$$\mathbf{B}_N(\mathbf{r},t) = 2\int_0^\infty d\omega \frac{\pi N a^2}{c}\tilde{I}_N(\omega)e^{-i\omega t}\mathbf{G}, \tag{16}$$

where

$$\mathbf{G} \equiv k^3 e^{ikr}\left\{(\hat{n}\times\hat{\mu})\times\hat{n}\left(\frac{1}{kr}\right) + [3\hat{n}(\hat{n}\cdot\hat{\mu})-\hat{\mu}]\left[\frac{1}{(kr)^3}-\frac{i}{(kr)^2}\right]\right\}. \tag{17}$$

Notice that $\tilde{\varepsilon}_N(\omega)$ is random so that $\mathbf{E}_N(\mathbf{r},t)$ and $\mathbf{B}_N(\mathbf{r},t)$ are also random fields.

The electromagnetic fields generated by $\tilde{\varepsilon}_B(\omega)$ (see [5]) will be given by

$$\mathbf{E}_B(\mathbf{r},t) = -\Re e \sum_{\alpha=1}^{2} \int d^3k \; i\omega \left(\frac{\pi N a^2}{c}\right)^2 \frac{\mathfrak{h}(\mathbf{k},T)}{Z(\omega)} e^{-i\omega t + i\theta(\mathbf{k},\alpha)} (\hat{\mu}\cdot\epsilon(\mathbf{k},\alpha))\mathbf{H}, \tag{18}$$

where $\epsilon(\mathbf{k},\alpha)$ are the polarization vectors, $\mathbf{H}$ is the same function defined in (14) and $\theta(\mathbf{k},\alpha)$ are the random phases characteristics of the vacuum fields (see Boyer [5]). The function $\mathfrak{h}(\mathbf{k},T)$ is such that

$$\pi^2\mathfrak{h}^2(\mathbf{k},T) = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\frac{\hbar\omega}{KT}} - 1} = \frac{\hbar\omega}{2} coth\left(\frac{\hbar\omega}{2KT}\right). \tag{19}$$

The corresponding magnetic field is

$$\mathbf{B}_B(\mathbf{r},t) = \Re e \sum_{\alpha=1}^{2} \int d^3 \; i\omega \left(\frac{\pi N a^2}{c}\right)^2 \frac{\mathfrak{h}(\mathbf{k},T)}{Z(\omega)} e^{-i\omega t + i\theta(\mathbf{k},\alpha)} (\hat{\mu}\cdot\epsilon(\mathbf{k},\alpha))\mathbf{G}. \tag{20}$$

These fluctuating electromagnetic fields are statistically independent of the fields $\mathbf{E}_N(\mathbf{r},t)$ and $\mathbf{B}_N(\mathbf{r},t)$. However, $\mathbf{E}_B(\mathbf{r},t)$ and $\mathbf{B}_B(\mathbf{r},t)$ are not statistically independent of the vacuum fields, namely $\mathbf{E}_{VF}(\mathbf{r},t)$ and $\mathbf{B}_{VF}(\mathbf{r},t)$. The expressions for these fields are [5]

$$\mathbf{E}_{VF}(\mathbf{r},t) = -\Re e \sum_{\alpha=1}^{2} \int d^3k\mathfrak{h}(\mathbf{k},T) e^{-i\omega t + i\mathbf{k}\cdot\mathbf{r} + i\theta(\mathbf{k},\alpha)} \frac{\mathbf{k}\times\epsilon(\mathbf{k},\alpha)}{k} \tag{21}$$

and

$$\mathbf{B}_{VF}(\mathbf{r},t) = \Re e \sum_{\alpha=1}^{2} \int d^3k\mathfrak{h}(\mathbf{k},T) e^{-i\omega t + i\mathbf{k}\cdot\mathbf{r} + i\theta(\mathbf{k},\alpha)} \epsilon(\mathbf{k},\alpha). \tag{22}$$

Therefore the total fields at a distance r from the center of the solenoid are such

$$\begin{aligned} \mathbf{E}(\mathbf{r},t) &= \mathbf{E}_{VF}(\mathbf{r},t) + \mathbf{E}_{N}(\mathbf{r},t) + \mathbf{E}_{B}(\mathbf{r},t) \\ \mathbf{B}(\mathbf{r},t) &= \mathbf{B}_{VF}(\mathbf{r},t) + \mathbf{B}_{N}(\mathbf{r},t) + \mathbf{B}_{B}(\mathbf{r},t) \end{aligned} \tag{23}$$

The Poynting vector is given by

$$\mathbf{S} = \frac{c}{4\pi}\left(\mathbf{E}(\mathbf{r},t) \times \mathbf{B}(\mathbf{r},t)\right). \tag{24}$$

This Poynting vector is fluctuating for obvious reasons. The average over the fluctuating variables will explained in what follows.

3 Summary of the calculation of the average value of the Poynting vector

The vector $\mathbf{S}$ has an average denoted by symbol $\langle \mathbf{S} \rangle$ and we shall take

$$\langle \mathbf{S} \rangle = \frac{c}{8\pi}\mathfrak{Re}\langle \mathbf{E}^* \times \mathbf{B} \rangle, \tag{25}$$

where the time average is also being considered [5].

Since $\mathbf{E}$ and $\mathbf{B}$ are given by (23) the above expression has several terms. Some of the results are trivial as, for instance,

$$\begin{aligned} \langle \mathbf{E}^*_{VF} \times \mathbf{B}_N \rangle &= 0 \qquad & \langle \mathbf{E}^*_N \times \mathbf{B}_{VF} \rangle &= 0 \\ \langle \mathbf{E}^*_B \times \mathbf{B}_N \rangle &= 0 \qquad & \langle \mathbf{E}^*_N \times \mathbf{B}_B \rangle &= 0. \end{aligned} \tag{26}$$

This happens because the fluctuations associated with the vacuum fields are statistically independent from the Nyquist fluctuations. We also have

$$\langle \mathbf{E}^*_{VF} \times \mathbf{B}_{VF} \rangle = 0. \tag{27}$$

There are, however, several terms present in (25) which are nonzero. We shall give their expressions bellow, the details of the calculation are explained in reference [5].

One can show that the following results are obtained

$$\langle \mathbf{E}^*_B \times \mathbf{B}_B \rangle = -\int dk k^2 \frac{8\pi\omega^2}{3} \left(\frac{\pi N a^2}{c}\right)^4 \frac{\mathfrak{h}^2}{|Z(\omega)|^2} \mathbf{H}^* \times \mathbf{G}, \tag{28}$$

$$\langle \mathbf{E}^*_{VF} \times \mathbf{B}_B \rangle = -\int dk k^2 i\omega \left(\frac{\pi N a^2}{c}\right)^2 \frac{\mathfrak{h}^2}{|Z(\omega)|^2} (-Z^*(\omega)\mathbf{G}) \times (\frac{4\pi i}{k^3} \Re e\mathbf{H}), \tag{29}$$

$$\langle \mathbf{E}^*_B \times \mathbf{B}_{VF} \rangle = -\int dk k^2 i\omega \left(\frac{\pi N a^2}{c}\right)^2 \frac{\mathfrak{h}^2}{|Z(\omega)|^2} (Z(\omega)\mathbf{H}^*) \times (\frac{4\pi}{k^3} \Im m\mathbf{G}) \tag{30}$$

and

$$\langle \mathbf{E}^*_N \times \mathbf{B}_N \rangle = -\int_0^\infty d\omega \left(\frac{\pi N a^2}{c}\right)^2 \frac{4\pi R_N \mathfrak{h}^2}{|Z(\omega)|^2} \mathbf{H}^* \times \mathbf{G}. \tag{31}$$

Collecting these expressions we get

$$\langle \mathbf{S} \rangle = -\frac{c}{2}\left(\frac{\pi N a^2}{c}\right)^2 \int_0^\infty d\omega \frac{\mathfrak{h}^2}{|Z(\omega)|^2}\left[\frac{2\pi^2 N^2}{3c}\left(\frac{a\omega}{c}\right)^4 + R_N - \Re e Z(\omega)\right] \cdot$$
$$\cdot(\Re e\mathbf{H} \times \Re e\mathbf{G} + \Im m\mathbf{H} \times \Im m\mathbf{G}). \quad (32)$$

One can see that the first term inside the square brackets in (32) is related to the contribution of (28), the second term is the contribution of (31) and the last term is the contribution of (29) and (30).

The result $\langle \mathbf{S} \rangle = 0$ follows immediately, because

$$\Re e Z(\omega) = \frac{2\pi^2 N^2}{3c}\left(\frac{a\omega}{c}\right)^4 + R_N = R_{rad} + R_N. \quad (33)$$

4 Discussion

The magnetic e.m.f. acting within the solenoid coils, namely

$$\begin{aligned}\varepsilon_B(t) &= -\frac{d}{cdt}\int_S \mathbf{B}_{VF} \cdot \hat{\mu}\, ds \\ &= \frac{\pi a^2 N}{c}\Re e \sum_{\alpha=1}^{2} \int d^3k\; i\omega \mathfrak{h}(\mathbf{k},T) e^{-i\omega t + i\theta(\mathbf{k},\alpha)} \hat{\mu} \cdot \epsilon(\mathbf{k},\alpha)\end{aligned} \quad (34)$$

have definite statistical properties and may be detected experimentally. Inductances with toroidal shape give $\varepsilon_B(t) = 0$, so that the effect discovered by Blanco et al. [1] can be easely tested experimentally.

It was shown in reference [1] that the spectral distribution of the total voltage fluctuation, namely

$$S_\varepsilon(\omega, T) = \left[\frac{2\pi^2 N^2}{3c}\left(\frac{a\omega}{c}\right)^4 + R_N\right]\left(\frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\frac{\hbar\omega}{KT}} - 1}\right), \quad (35)$$

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