Instituto de Física
Universidade de São Paulo

Quantization of Spinning Particle in 2+1 Dimensions

Fresneda R.,[a]; Gavrilov S., Gitman D.,[a]; Moshin P.[a,b]

[a] Instituto de Física, Universidade de São Paulo, São Paulo, Brasil
[b] Tomsk State Pedagogical University, 634041 Tomsk, Russia

Publicação IF - 1563/2003

Abstract

We consider the quantization problem for a pseudoclassical model of the (2+1) spinning relativistic particle in arbitrary electromagnetic backgrounds respecting vacuum stability. The presented quantization is a combination of the canonical and Dirac schemes, since the model contains a first-class constraint which does not admit gauge-fixing. The presence of this constraint at the quantum level leads to an extension of the number of state-vector components in comparison with the (3+1) case. The constructed quantum mechanics describes both particles and antiparticles with positive energy levels, in accordance with quantum field theory.

1 Introduction

Several years ago, there appeared a number of papers [1, 2] with various models of relativistic particles. Quantization schemes for these models were also proposed [1, 2, 3]; however, a consistent quantization, in our opinion, was presented only in [3], using the example of a $(3+1)$ spinning particle. The main point of [3] is that in the course of quantization one reproduces both particle and antiparticle states. One may think that the application of this scheme to other dimensions is trivial; nevertheless, it is not the case with the $(2+1)$ spinning particle. In arbitrary electromagnetic backgrounds, the above problem is not a simple task due to complications related to the ordering problem. Indeed, as shown in this paper, using the most consistent, in our opinion, model of a $(2+1)$ spinning particle [2], the quantization in $(2+1)$ is more complicated than in $(3+1)$. A priori, one would expect that the Hilbert space could be realized by state vectors whose number of components is smaller than 8, being the corresponding number in the $(3+1)$ case. However, a successful realization in $(2+1)$ turns out to be a 16-component one. Technically, this is connected with the fact that the model under consideration contains a first-class constraint related to an additional gauge symmetry absent in $(3+1)$. As observed in [2], this constraint cannot be gauge-fixed. Therefore, one needs to use a combination of the canonical and Dirac quantization schemes. The physical sector is selected at both classical and quantum levels, and the appearance of the additional first-class constraint at the quantum level leads to the extension of the number of state-vector components. The problem of ordering turns out to be so involved that the construction of a consistent realization satisfying all the requirements (including quasiclassics) appears to be very instructive and may prove useful in solving other problems. After solving all technical difficulties, we arrive at a clear and physically grounded result: the one-particle physical sector contains both particles and antiparticles with positive energy levels. Note that the realization proves successful in arbitrary electromagnetic backgrounds, which, however, do not create particle-antiparticle pairs from vacuum and thus do not violate vacuum stability. The performed quantization confirms once again that first quantization respecting vacuum stability exactly reproduces the corresponding one-particle sector of the quantum field theory – in the case under consideration, the quantum theory of the spinor field in $(2+1)$ dimensions.

The paper is organized as follows. In Section 2, we study the classical properties of the given pseudoclassical model and obtain its hamiltonian formulation in a reduced phase space, restricted by constraints and gauges. In Section 3, we apply a modification of canonical quantization and construct a relativistic quantum mechanics leading to the $(2+1)$ Dirac equation for the wave functions; we analyze the one-particle sector of the quantum theory and prove the consistency of the presented operator realization. In Section 4, we summarize the results of the paper. In Appendix A, we consider the (2+1) gamma-matrices. In Appendix B, we analyze the one-particle sector of the quantum theory of the $(2+1)$ spinor field.

2 Classical Description

2.1 *Lagrangian and Hamiltonian Formulations*

The pseudoclassical action [2] reads

$$S = \int_0^1 \left(-\frac{z^2}{2e} - e\frac{m^2}{2} - q\dot{x}^\mu A_\mu + ieqF_{\mu\nu}\xi^\mu\xi^\nu - im\xi^3\chi - \frac{1}{2}sm\kappa - i\xi_n\dot{\xi}^n \right) d\tau$$
$$\equiv \int_0^1 L\, d\tau\,,\ z^\mu = \dot{x}^\mu - i\xi^\mu\chi + i\varepsilon^{\mu\nu\lambda}\xi_\nu\xi_\lambda\kappa\,, \tag{1}$$

where summation over repeated indices $\mu = \overline{0,2}$, $n = (\mu, 3)$ is assumed, with $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1)$, $\eta_{mn} = \mathrm{diag}(1,-1,-1,-1)$; x^μ, e, κ are even variables, while ξ^n, χ are odd variables; $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

is the strength tensor of the external electromagnetic field; $\varepsilon^{\lambda\mu\nu}$ is the completely antisymmetric Levi–Civita tensor in $(2+1)$ dimensions. We assume that x^μ and ξ^μ are $(2+1)$ Lorentz vectors, e, κ, χ are Lorentz scalars, and s is an even constant subject to the condition $s^2=1$.

The above action is reparameterization-invariant [2]

$$\delta x^\mu = \dot{x}^\mu \xi\,, \quad \delta e = \frac{d}{d\tau}(e\xi)\,, \quad \delta\psi^a = \dot{\psi}^a\xi\,, \quad \delta\chi = \frac{d}{d\tau}(\chi\xi)\,, \quad \delta\kappa = \frac{d}{d\tau}(\kappa\xi)\,,$$

with an even gauge parameter $\xi(\tau)$, and thus we expect the Hamiltonian to vanish on the equations of motion.

The action is also invariant under two types of gauge supertransformations [2], namely,

$$\delta x^\mu = i\psi^\mu\varepsilon\,, \quad \delta e = i\chi\varepsilon\,, \quad \delta\psi^\mu = \frac{z^\mu}{2e}\varepsilon\,, \quad \delta\psi^3 = \frac{m}{2}\varepsilon\,, \quad \delta\chi = \dot{\varepsilon}\,, \quad \delta\kappa = 0,$$
$$\delta x^\mu = -i\varepsilon^{\mu\nu\lambda}\psi_\nu\psi_\lambda\theta, \quad \delta\psi^\mu = \frac{1}{e}\varepsilon^{\mu\nu\lambda}z_\nu\psi_\lambda\theta\,, \quad \delta\kappa = \dot{\theta}\,, \quad \delta e = \delta\psi^3 = \delta\chi = 0,$$

where $\varepsilon(\tau)$ is an odd parameter, and $\theta(\tau)$ is an even parameter.

Notice that the variables e, χ, κ do not carry time derivatives in the action. Such variables are called degenerate [4]. To analyze theories with degenerate coordinates, it is convenient to apply the procedure of generalized hamiltonization [4], in which momenta conjugate to degenerate coordinates are not introduced. To perform the hamiltonization [4], we first introduce the velocities $v^\mu = \dot{x}^\mu$, $\vartheta^{\,\mu} = \dot{\xi}^\mu$ and then write the action (1) in the first-order formalism as

$$S^v = \int_0^1 \left[L^v + p_\mu(\dot{x}^\mu - v^\mu) + \pi_n\left(\dot{\xi}^n - \vartheta^n\right)\right]d\tau = \int_0^1\left(p_\mu\dot{x}^\mu + \pi_n\dot{\xi}^n - H^v\right)d\tau\,,$$

where

$$L^v = -\frac{\bar{z}^2}{2e} - e\frac{m^2}{2} - qv^\mu A_\mu + iqeF_{\mu\nu}\xi^\mu\xi^\nu - im\psi^3\xi - \frac{1}{2}sm\kappa - i\xi_n\vartheta^n,$$
$$\bar{z}^\mu = v^\mu - i\xi^\mu\chi + i\varepsilon^{\mu\nu\lambda}\xi_\nu\xi_\lambda\kappa\,, \quad H^v = p_\mu v^\mu + \pi_n\vartheta^n - L^v.$$

The ordering of variables in the first-order Hamiltonian complies with the usual convention for the choice of derivatives with respect to coordinates as right-hand ones and those with respect to momenta as left-hand ones.

The variational equations with respect to the velocities and degenerate coordinates read as follows:

$$\frac{\delta S^v}{\delta v^\mu} = -p_\mu - \frac{1}{e}\bar{z}_\mu - qA_\mu = 0, \quad \frac{\delta S^v}{\delta e} = \frac{\bar{z}^2}{2e^2} - \frac{m^2}{2} + iqF_{\mu\nu}\xi^\mu\xi^\nu = 0,$$
$$\frac{\delta S^v}{\delta\kappa} = -\frac{i}{e}\bar{z}^\mu\varepsilon_{\mu\nu\lambda}\xi^\nu\xi^\lambda - \frac{1}{2}sm = 0, \quad \frac{\delta S^v}{\delta\vartheta^a} = \pi_n + i\xi_n = 0,$$
$$\frac{\delta S^v}{\delta\chi} = \frac{1}{e}\left(v^\mu\xi_\mu + i\varepsilon^{\mu\nu\lambda}\xi_\mu\xi_\nu\xi_\lambda\kappa\right) - im\xi^3 = 0.$$

The equations $\delta S^v/\delta\vartheta^n = 0$ lead to the constraints

$$\varphi_n = \pi_n + i\xi_n\,, \tag{2}$$

and the equations $\delta S^v/\delta v^\mu = 0$ can be used to express the velocities v^μ, namely,

$$v^\mu = -e\left(p^\mu + qA^\mu\right) + i\xi^\mu\chi - i\varepsilon^{\mu\nu\lambda}\xi_\nu\xi_\lambda\,\kappa\,. \tag{3}$$

Substituting this relation into the other equations, we obtain new constraints:

$$\phi_1 = P^\mu\xi_\mu + m\xi^3, \quad \phi_2 = P^2 - m^2 + 2iqF_{\mu\nu}\xi^\mu\xi^\nu, \quad \phi_3 = \varepsilon_{\mu\nu\lambda}P^\mu\xi^\nu\xi^\lambda + \frac{i}{2}sm\,,$$

where $P_\mu = p_\mu + qA_\mu$.

Upon substituting (3) into the Hamiltonian, we get

$$H^{(1)} = -\frac{e}{2}\left(P^2 - m^2 + 2iqF_{\mu\nu}\xi^\mu\xi^\nu\right) - i\kappa\left(\varepsilon^{\mu\nu\lambda}P_\mu\xi_\nu\xi_\lambda + \frac{i}{2}sm\right)$$
$$-i\chi\left(P_\mu\xi^\mu + m\xi^3\right) - \vartheta^n\left(\pi_n + i\xi_n\right) = \Lambda_1\phi_1 + \Lambda_2\phi_2 + \Lambda_3\phi_3 + \lambda^n\varphi_n\,, \tag{4}$$

where

$$\Lambda_1 = -i\chi\,, \;\; \Lambda_2 = -\frac{e}{2},\;\; \Lambda_3 = -i\kappa\,, \;\; \lambda^n = -\vartheta^n .$$

As expected, the Hamiltonian is proportional to the constraints. The action becomes

$$S^{(1)} = \int_0^1 \left(p_\mu \dot{x}^\mu + \pi_n \dot{\xi}^n - H^{(1)} \right) d\tau \,.$$

In the following, we shall make use of the Dirac bracket, which is constructed from the complete set of second-class constraints. As is well-known, the Dirac bracket of two dynamical (parity-definite) variables is defined as

$$\{A, B\}_{D(\varphi)} = \{A, B\} - \{A, \varphi_a\}\, C^{ab}\, \{\varphi_b, B\}\,,$$

where C^{ab} is the inverse matrix of the Poisson matrix between the second-class constraints φ: $C^{ab}\,\{\varphi_b, \varphi_c\} = \delta^a_c$. The Poisson bracket of two phase-space functions F and G of definite parity is given by

$$\{F, G\} = \frac{\partial F}{\partial x^\mu}\frac{\partial G}{\partial p_\mu} + \frac{\partial F}{\partial \xi^n}\frac{\partial G}{\partial \pi_n} - (-1)^{\varepsilon(F)\varepsilon(G)}\,(F \leftrightarrow G), \tag{5}$$

where $\varepsilon(F)$ is the parity of F.

2.2 *Constraint Reorganization and Gauge Fixing*

Following the approach [2, 5], we shall fix as many first-class constraints as possible, using the choice of gauge conditions. To this end, we construct a set of first-class constraints $\tilde{\phi}$, obtained from the original constraints ϕ by means of the shift $\xi^n \to \xi^n + \frac{i}{2}\varphi^n$, namely,

$$\begin{aligned}
\tilde{\phi}_1 &= P_\mu \left(\pi^\mu - i\xi^\mu\right) + m\left(\pi^3 - i\xi^3\right),\\
\tilde{\phi}_2 &= P^2 - m^2 - 2qF_{\mu\nu}\xi^\mu\pi^\nu,\\
\tilde{\phi}_3 &= \varepsilon_{\mu\nu\lambda}P^\mu\xi^\nu\pi^\lambda + \frac{1}{2}sm\,,
\end{aligned} \tag{6}$$

where we have omitted trivial terms, quadratic in the second-class constraints φ^n.

We further rewrite the set (6) in the form, analogous to [3],

$$\begin{aligned}
\tilde{T}_1 &= P_\mu \left(\pi^\mu - i\xi^\mu\right) + m\left(\pi^3 - i\xi^3\right),\\
\tilde{T}_2 &= P_0 + \zeta r\,,\\
\tilde{T}_3 &= \varepsilon_{\mu\nu\lambda}P^\mu\xi^\nu\pi^\lambda + \frac{1}{2}sm\,,
\end{aligned} \tag{7}$$

where $\zeta = -\mathrm{sign}(P_0)$ and $r = \sqrt{m^2 - P_kP^k + 2qF_{\mu\nu}\xi^\mu\pi^\nu}$. The constraints $\tilde{T}_1$ and $\tilde{T}_3$ are identical to $\tilde{\phi}_1$, $\tilde{\phi}_3$, whereas the constraints $\tilde{T}_2$ and $\tilde{\phi}_2$ are related by $\tilde{\phi}_2 = -2\zeta r\tilde{T}_2 + \left(\tilde{T}_2\right)^2$. By construction, the $\tilde{T}$-constraints are first-class ones:

$$\left\{\tilde{T}, \tilde{T}\right\}\Big|_{\tilde{T},\varphi=0} = \left\{\tilde{T}, \varphi\right\}\Big|_{\tilde{T},\varphi=0} = 0\,,$$

which follows from the coincidence of $\tilde{T}_1$, $\tilde{T}_3$ with the old constraints $\tilde{\phi}_1$, $\tilde{\phi}_3$, and from the fact that

$$\{\tilde{T}_2, F\} = -\frac{1}{2\zeta r}\{\tilde{\phi}_2, F\} + \{\tilde{T}_2\}\,,$$

where $\{\tilde{T}_2\}$ denotes a term proportional to the constraint $\tilde{T}_2$.

In terms of the $\tilde{T}$-constraints, the Hamiltonian (4) becomes

$$H_{\tilde{T}} = \Lambda_1\tilde{T}_1 + \Lambda_2\tilde{T}_2 + \Lambda_3\tilde{T}_3\,, \tag{8}$$

with redefined Lagrange multipliers.

Let us consider the gauge-fixing of the first-class constraints $\tilde{T} = (\tilde{T}_1, \tilde{T}_2, \tilde{T}_3)$, given by (7), using the corresponding gauges $\phi^G = (\phi^G_1, \phi^G_2, \phi^G_3)$.

In [2], it was observed that one cannot choose a gauge condition ϕ_3^G to fix the constraint $\tilde{T}_3$, which thus remained a first-class one within canonical quantization. The difference between the paper [2] and the present approach is that in [2] one did not introduce the reparameterization of the constraint ϕ_2 in terms of $\tilde{T}_2$. However, we can prove that $\tilde{T}_3$ cannot be gauge-fixed in the present constraint parameterization, as well. To demonstrate this fact, we will proceed by reduction to absurdity. To this end, consider the matrix M of Poisson brackets between all second-class constraints $\Psi = (\tilde{T}, \phi^G, \varphi)$, including the gauge ϕ_3^G, corresponding to $\tilde{T}_3$,

$$M = \|\{\Psi, \Psi\}\|_{\Psi=0} = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}, \tag{9}$$

were the matrix M is arranged so that the block M_1 consists of brackets between the even constraints and gauges, namely, it has rows and columns labelled by $\tilde{T}_2$, $\tilde{T}_3$, ϕ_2^G, ϕ_3^G; whereas the block M_4 consists of brackets between the odd constraints and gauges, namely, it has rows and columns labelled by $\tilde{T}_1$, ϕ_1^G, φ.

Due to the quadratic structure of $\tilde{T}_3$ in Grassmann variables (7), its Poisson bracket with any quantity,

$$\{\tilde{T}_3, F\} = \varepsilon_{\mu\nu\lambda}\xi^\nu\pi^\lambda\{P^\mu, F\} + \varepsilon_{\mu\nu\lambda}P^\mu\left(\xi^\nu\{\pi^\lambda, F\} - \pi^\lambda\{\xi^\nu, F\}\right),$$

is zero on the surface of vanishing Grassmann variables $(\xi, \pi \to 0)$. This means that on the surface of vanishing Grassmann variables M_1 is degenerate $(\det M_1|_{\xi,\pi\to 0} = 0)$, because the block $M_1|_{\xi,\pi\to 0}$ contains a vanishing row (column). However, if we require that the entire matrix M be non-degenerate, it is necessary and sufficient that the even blocks M_1 and M_4 be non-degenerate on the surface of vanishing Grassmann variables (see, e.g., [6]). Thus, due to the particular structure of $\tilde{T}_3$, independently of the choice of ϕ^G, one cannot fix the gauge freedom related to $\tilde{T}_3$.

Following the approach [2], we impose gauge conditions on all first-class constraints, except $\tilde{T}_3$, so that the resulting set of constraints and gauges will consist of second-class constraints and a first-class constraint. This is a modification of the canonical quantization procedure in which we retain one of the first-class constraints and realize it as a condition on state vectors in Hilbert space, by analogy with the Dirac quantization method. Namely, we impose the following gauge-fixing conditions:

$$\phi_1^G = \pi^0 - i\xi^0 + \zeta\left(\pi^3 - i\xi^3\right), \tag{10}$$

$$\phi_2^G = x_0 - \zeta\tau. \tag{11}$$

The gauge (10), chosen to fix the gauge freedom related to $\tilde{T}_1$, is analogous to the gauge [3] used in the case of the spinning particle in $(3+1)$ dimensions. This gauge leads to simplifications, in particular, reducing the set of independent spin variables. The gauge (11), chosen to fix the gauge freedom related to $\tilde{T}_2$, is the chronological gauge [3, 6], which relates the physical time x_0 to τ or $-\tau$, according to the classical interpretation of particles with charge q or $-q$, respectively [3, 6].

Taking into account the explicit time-dependence of the gauge (10), it is convenient, following the approach of [6], to extend the phase space by introducing the momentum ϵ conjugate to the time parameter τ, which allows one to retain the general Poisson bracket structure of the formalism. Therefore, henceforth Poisson brackets $\{\ ,\ \}$ are calculated in the extended phase space including the canonical pair (τ, ϵ).

By imposing the gauge conditions (10) and (11), we arrive at a non-degenerate matrix of Poisson brackets calculated on the constraint surface between the constraints $\Psi = (\tilde{T}_2, \phi_2^G, \tilde{T}_1, \phi_1^G, \varphi^n)$, with the rows labelled by $\tilde{T}_2$, ϕ_2^G, $\tilde{T}_1$, ϕ_1^G, φ^n, respectively:

$$\|\{\Psi, \Psi\}\|_{\Psi=0} = \begin{pmatrix} M_1 & M_2 & 0 \\ M_3 & M_4 & 0 \\ 0 & 0 & 2i\eta \end{pmatrix}, \tag{12}$$

where the diagonal blocks read

$$M_1 = \left\|\left\{\tilde{T}_2, \phi_2^G\right\}\right\|_{\Psi=0} = -i\sigma_2\,, \quad M_4 = \left\|\left\{\tilde{T}_1, \phi_1^G\right\}\right\|_{\Psi=0} = 2i\zeta\,\mathrm{antidiag}\,(r+m)\,,$$

and $2i\eta$ corresponds to the matrix $\|2i\eta^{nm}\|$, whose rows are labelled by φ^n. The non-diagonal terms are

$$M_2 = \begin{pmatrix} 0 & \zeta q r^{-1} F_{k0}\left(\pi^k - i\xi^k\right) \\ \pi^0 - i\xi^0 & 0 \end{pmatrix}, \quad M_3 = -M_2^T\,.$$

Furthermore, using the constraints $\tilde{T}_1$, $\tilde{T}_2$ and ϕ_1^G, it is possible to write $\pi^0 - i\xi^0$ as a function of π_k and ξ^k:

$$\pi^0 - i\xi^0 = \frac{\zeta}{\tilde{\omega}_0 + m} P_k \left(\pi^k - i\xi^k\right), \quad \tilde{\omega}_0 = \sqrt{m^2 - P_k P^k + 2qF_{ik}\xi^i\pi^k}\,.$$

The non-degeneracy of the matrix $\|\{\Psi, \Psi\}\|_{\Psi=0}$ follows from the non-degeneracy of its even blocks, which implies that the constraints Ψ are of second class. The consistency conditions $\dot{\phi}_1^G = 0$ and $\dot{\phi}_2^G = 0$ read

$$\dot{\phi}_1^G = -\Lambda_1 \left\{\phi_1^G, \tilde{T}_1\right\} + \Lambda_2 \left\{\phi_1^G, \tilde{T}_2\right\} + \Lambda_3 \left\{\phi_1^G, \tilde{T}_3\right\} = 0,$$
$$\dot{\phi}_2^G = -\zeta + \Lambda_1 \left\{\phi_2^G, \tilde{T}_1\right\} + \Lambda_2 \left\{\phi_2^G, \tilde{T}_2\right\} + \Lambda_3 \left\{\phi_2^G, \tilde{T}_3\right\} = 0,$$

which we write in a matrix form:

$$M \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix} = \begin{pmatrix} \Lambda_3 f_1 \\ \zeta + \Lambda_3 f_2 \end{pmatrix}, \quad f_1 \equiv \left\{\phi_1^G, \tilde{T}_3\right\}, \quad f_2 \equiv -\left\{\phi_2^G, \tilde{T}_3\right\},$$

where the matrix M, given by

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = \begin{pmatrix} \left\{\phi_1^G, \tilde{T}_1\right\} & -\left\{\phi_1^G, \tilde{T}_2\right\} \\ -\left\{\phi_2^G, \tilde{T}_1\right\} & \left\{\phi_2^G, \tilde{T}_2\right\} \end{pmatrix},$$

is invertible due to the non-degeneracy of its even blocks M_1, M_4. As a consequence, there exists a matrix N such that

$$N = \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix}, \quad NM = I,$$

with $\varepsilon(N_1) = \varepsilon(N_4) = 0$ and $\varepsilon(N_2) = \varepsilon(N_3) = 1$, namely,

$$\begin{aligned} &M_1 N_1 + M_2 N_3 = 1, \quad M_1 N_2 + M_2 N_4 = 0, \\ &M_3 N_1 + M_4 N_3 = 0, \quad M_3 N_2 + M_4 N_4 = 1. \end{aligned} \tag{13}$$

Using the matrix N, one can resolve Λ_1 and Λ_2 in terms of Λ_3:

$$\begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix} = N \begin{pmatrix} \Lambda_3 f_1 \\ \zeta + \Lambda_3 f_2 \end{pmatrix} = \begin{pmatrix} \zeta N_2 + \Lambda_3 (N_1 f_1 + N_2 f_2) \\ \zeta N_4 + \Lambda_3 (N_3 f_1 + N_4 f_2) \end{pmatrix}.$$

Then, inserting the expressions for the determined Lagrange multipliers (Λ_1, Λ_2) into the Hamiltonian (8), we get

$$\bar{H}_T = \zeta N_2 \tilde{T}_1 + \zeta N_4 \tilde{T}_2 + \Lambda_3 \bar{T}_3\,,$$

where $\bar{T}_3$ is a first-class constraint, being a linear combination of the old constraints:

$$\bar{T}_3 = \bar{\lambda}_1 \tilde{T}_1 + \bar{\lambda}_2 \tilde{T}_2 + \tilde{T}_3\,, \quad \bar{\lambda}_1 = N_1 f_1 + N_2 f_2\,, \quad \bar{\lambda}_2 = N_3 f_1 + N_4 f_2\,.$$

The fact that the above constraint is a first-class one can be demonstrated, on the one hand, by noticing that it is a linear combination of the T-constraints, which commute with themselves and with φ^n on the constraints surface. On the other hand, the explicit form of the constraint $\bar{T}_3$, with allowance for (13), ensures that its Poisson brackets with the gauge conditions vanish identically on the constraints surface:

$$\{\phi_1^G, \bar{T}_3\} = -\bar{\lambda}_1 M_1 - \bar{\lambda}_2 M_2 + f_1 = -(M_1 N_1 + M_2 N_3) f_1 - (M_1 N_2 + M_2 N_4) f_2 + f_1 \equiv 0\,,$$
$$\{\phi_2^G, \bar{T}_3\} = -\bar{\lambda}_1 M_3 + \bar{\lambda}_2 M_4 - f_2 = (M_3 N_1 + M_4 N_3) f_1 + (M_3 N_2 + M_4 N_4) f_2 - f_2 \equiv 0\,.$$

Thus, the total set of constraints is divided into the second-class ones $\Psi = \left(\tilde{T}_2, \phi_2^G, \tilde{T}_1, \phi_1^G, \varphi^n\right)$ and the first-class one $\bar{T}_3$. It is easily seen that the new set of constraints $\Theta' = (\Psi, \bar{T}_3)$ is equivalent to the old one $\Theta = \left(\Psi, \tilde{T}_3\right)$, since the transformation matrix M, $\Theta \to \Theta' = M\Theta$, has non-degenerate even blocks M_1 and M_4, in terms of the notation (9).

Following the approach [3], we replace the constraint $\tilde{T}_2$ by

$$\Phi_1 = P_0 + \zeta\tilde{\omega}, \quad \tilde{\omega} = \sqrt{\tilde{\omega}_0^2 + \frac{2\zeta q F_{k0} P_l}{\tilde{\omega}_0 + m}\left(\xi^k \pi^l + \pi^k \xi^l\right)}\,.$$

Indeed, the sets of constraints Ψ and $\Psi' = \left(\Phi_1, \phi_2^G, \tilde{T}_1, \phi_1^G, \varphi^n\right)$ are equivalent, i.e. there exists a matrix $M = M(\eta)$, satisfying $\det M|_{\Psi=0} \neq 0$, such that $\Psi = M\Psi'$. The equivalence follows from the fact that the constraint surface specified by $\Psi(\eta) = 0$ is a first-order zero of the system of independent equations $\Psi'(\eta) = 0$ (see, e.g., [6]), which is implied by the observation that the equations $\Psi'(\eta) = 0$ are identically satisfied on the constraint surface $\Psi(\eta) = 0$, and besides

$$\operatorname{rank} \left.\frac{\partial \Psi'}{\partial \eta}\right|_{\Psi=0} = (n_1, n_2)\,, \quad [\Psi] = [\Psi'] = n_1 + n_2\,,$$

where $n_1 = 2$ and $n_2 = 6$, i.e. the rank is maximal on the surface $\Psi(\eta) = 0$. The latter property is a consequence of the fact that one can use the conditions $\Psi'(\eta) = 0$ to resolve $n_1 + n_2$ variables in terms of the remaining ones, namely, the dependent variables $\underline{\eta} = \left(x^0, p_0; \xi^0, \pi_0; \xi^3, \pi_3; \pi_k\right)$ in terms of the independent ones $\bar{\eta} = \left(x^k, p_k; \xi^k\right)$.

Let us now introduce a new set of second-class constraints $\Phi = (U, V)$, obtained by linear combinations of the set of second-class constraints Ψ', such that the Dirac bracket constructed from the new set has a simple form. The subset U reads

$$\Phi_1 = P_0 + \zeta\tilde{\omega}, \quad \Phi_2 = \phi_2^G\,, \quad \Phi_3 = \varphi_1\,, \quad \Phi_4 = \varphi_2\,,$$

and the subset V is given by

$$\Phi_5 = -\frac{i}{2}\tilde{T}_1 + b\tilde{T}_2 + c\phi_2^G\,, \quad \Phi_6 = \phi_1^G\,, \quad \Phi_7 = \varphi_0, \quad \Phi_8 = \varphi_3\,,$$
$$b = \frac{i}{2}\frac{\left\{\phi_2^G, \tilde{T}_1\right\}}{\left\{\phi_2^G, \tilde{T}_2\right\}} = \xi^0, \quad c = -\frac{\left\{-\frac{i}{2}\tilde{T}_1 + b\phi_2^G, \Phi_1\right\}}{\left\{\phi_2^G, \Phi_1\right\}},$$

where the brackets are calculated on the constraint surface. It is easily seen that the sets Φ and Ψ' are equivalent, which also implies the equivalence of Φ and Ψ.

The constraint matrix $\|\{\Phi_a, \Phi_b\}\|_{\Phi=0}$ is semi-diagonal, with the non-zero entries

$$\{\Phi_2, \Phi_1\} = -\{\Phi_1, \Phi_2\} = 1, \;\; \{\Phi_3, \Phi_3\} = \{\Phi_4, \Phi_4\} = -2i\,,$$
$$\{\Phi_5, \Phi_6\} = \{\Phi_6, \Phi_5\} = \zeta\left(\tilde{\omega}_0 + m\right), \;\; \{\Phi_7, \Phi_7\} = -\{\Phi_8, \Phi_8\} = 2i\,.$$

The above constraint algebra implies that the subsets U and V are orthogonal in the sense of Poisson brackets, calculated on the constraint surface:

$$\{U, V\} = 0, \quad \Phi = 0. \tag{14}$$

2.3 *Equations of Motion in Reduced Phase Space*

In what follows, we shall consider the set of variables

$$\eta = \left(x^k, p_k; \xi^k, \pi_k; \zeta\right), \quad k = 1, 2, \tag{15}$$

which consists of the independent canonical set $\left(x^k, p_k; \xi^k, \pi_k\right)$ and the discrete parameter ζ.

The time-evolution of the η-variables is given by

$$\dot{\eta} = \left\{\eta, \bar{H}_T + \epsilon\right\}_{D(\Phi)} = \left\{\eta, \Lambda_3\bar{T}_3 + \epsilon\right\}_{D(\Phi)}\,, \quad \Phi = 0, \;\; \bar{T}_3 = 0, \tag{16}$$

where the Dirac bracket $\{\;,\;\}_{D(\Phi)}$ is constructed with respect to the second-class constraints Φ. In the above equations of motion, we have used the well-known fact that any terms proportional to second-class constraints can be eliminated prior to calculating the Dirac brackets. In our case, from the previous consideration of the equivalence of the constraints Ψ and Φ, we deduce that $\tilde{T}_1$ and $\tilde{T}_2$ in $\bar{H}_T$ can be expressed in terms of the constraints Φ. Moreover, from the stated equivalence of Ψ and Φ it follows that $\bar{T}_3$ is a first-class constraint with respect to the set Φ.

Due to the fact that the constraint sets U and V are mutually orthogonal in the sense of Poisson brackets, resulting in a block-diagonal matrix $\|\{\Phi,\Phi\}\|$, it is possible to apply the rule of successive calculation of Dirac brackets (see, e.g., [6]). Namely, we can calculate the Dirac bracket of two (parity-definite) dynamical variables A, B with respect to a subset V in the total set of second-class constraints $\Phi=(U,V)$, the subset V being of second-class:

$$\{A,B\}_{D(\Phi)} = \{A,B\}_{D(U)} - \{A,V_a\}_{D(U)} C^{ab} \{V_b,B\}_{D(U)}\,, \quad C^{ac}\{V_c,V_b\}_{D(U)} = \delta^a_b\,. \tag{17}$$

Let us obtain the evolution equation (16) for the variables η (15), namely,

$$\dot{\eta} = \{\eta, \Lambda_3\bar{T}_3 + \epsilon\}_{D(\Phi)} = \{\eta, \Lambda_3\bar{T}_3\}_{D(\Phi)} + \{\eta,\epsilon\}_{D(\Phi)}\,, \tag{18}$$

The second term in the r.h.s of this equation can be explicitly calculated by the rule (17),

$$\{\eta,\epsilon\}_{D(\Phi)} = \{\eta,\epsilon\}_{D(U)} - \{\eta,V_a\}_{D(U)} C^{ab}\{V_b,\epsilon\}_{D(U)}\,.$$

The bracket $\{V_b,\epsilon\}_{D(U)}$ on the r.h.s. of the above expression is easily seen to vanish on the constraint surface due to the orthogonality relation (14). Therefore, the bracket $\{\eta,\epsilon\}_{D(\Phi)}$ reduces to $\{\eta,\epsilon\}_{D(U)}$. Let us divide the subset U into two sets $u=(\Phi_3,\Phi_4)$ and $v=(\Phi_1,\Phi_2)$. By applying the rule (17) to the second-class sets u and v, we obtain

$$\{\eta,\epsilon\}_{D(U)} = \{\eta,\epsilon\}_{D(u)} - \{\eta,v_a\}_{D(u)} c^{ab}\{v_b,\epsilon\}_{D(u)}\,, \quad c^{ab}\{v_b,v_c\}_{D(u)} = \delta^a_c\,.$$

A new simplification takes place due to the vanishing of $\{\eta,\epsilon\}_{D(u)}$, and thus we are finally left with

$$\{\eta,\epsilon\}_{D(\Phi)} = -\{\eta,v_a\}_{D(u)} c^{ab}\{v_b,\epsilon\}_{D(u)} = \zeta\{\eta,\Phi_1\}_{D(u)}\,,$$

where we have used the fact that $c^{ab}=\{v_b,v_a\}=i\sigma_2$. Because neither the independent variables nor the constraints u involve the coordinate x^0, we can eliminate the momentum p_0 from Φ_1 and write the bracket $\{\eta,\epsilon\}_{D(\Phi)}$ as follows:

$$\{\eta,\epsilon\}_{D(\Phi)} = \{\eta,\zeta qA_0 + \tilde{\omega}\}_{D(u)}\,.$$

Furthermore, due to the absence of the momentum p_0 from the above bracket, and due to the particular form of the constraint $\Phi_2 = x^0 - \zeta\tau$, we can substitute $x^0=\zeta\tau$ into the same bracket, obtaining

$$\{\eta,\epsilon\}_{D(\Phi)} = \left\{\eta, [\zeta qA_0 + \tilde{\omega}]_{x^0=\zeta\tau}\right\}_{D(u)}\,. \tag{19}$$

The first term in the r.h.s of (18) can be presented in a similar form, first, by noting that

$$\{\eta,\Lambda_3\bar{T}_3\}_{D(\Phi)} = \{\eta,\Lambda_3 T_3\}_{D(\Phi)}\,,$$

where

$$T_3 = \bar{T}_3\big|_{\Phi=0} = -m\zeta\left(\xi^1\pi^2 - \xi^2\pi^1\right) + \frac{1}{2}sm\,, \tag{20}$$

that is, the first-class constraint $\bar{T}_3$ differs from T_3 by a linear combination of constraints, $\bar{T}_3 = T_3 + \{\Phi\}$. Second, taking into account the fact that T_3 has non-vanishing Poisson brackets only with Φ_5, we write

$$\begin{aligned}\{\eta,\Lambda_3 T_3\}_{D(\Phi)} &= \{\eta,\Lambda_3 T_3\}_{D(U)} - \{\eta,V_a\}_{D(U)} C^{ab}\{V_b,\Lambda_3 T_3\} \\ &= \{\eta,\Lambda_3 T_3\}_{D(U)} - \{\eta,\Phi_6\}\{\Phi_5,\Phi_6\}^{-1}\{\Phi_5,\Lambda_3 T_3\} = \{\eta,\Lambda_3 T_3\}_{D(U)}\,,\end{aligned}$$

where we have used the fact that $\{\eta,\Phi_6\}=0$. Owing to the orthogonality relations $\{v,T_3\}=0$, one can write

$$\{\eta,\Lambda_3 T_3\}_{D(\Phi)} = \{\eta,\Lambda_3 T_3\}_{D(u)}\,. \tag{21}$$

Equations (19) and (21) yield

$$\dot{\eta} = \left\{\eta, [\zeta qA_0 + \tilde{\omega}]_{x^0=\zeta\tau} + \Lambda_3 T_3\right\}_{D(u)}\,.$$

Finally, we express π_k in terms of ξ^k in $\tilde{\omega}$, using the constraints u, so that the equations of motion become

$$\dot{\eta} = \{\eta, \mathcal{H}_{\text{eff}} + \Lambda_3 T_3\}_{D(u)}\,, \quad u_k = \pi_k + i\xi_k = 0, \quad T_3 = 0, \tag{22}$$

where the effective Hamiltonian $\mathcal{H}_{\text{eff}}$ and the effective first-class constraint T_3 are given by

$$\mathcal{H}_{\text{eff}} = [\zeta q A_0 + \omega]_{x^0=\zeta\tau}\,, \quad \omega = \tilde{\omega}|_{\pi_k=-i\xi^k} = \sqrt{\omega_0^2 + \rho}\,,$$
$$\omega_0 = \sqrt{m^2 + (p_k + qA_k)^2 - 2iqF_{kl}\xi^k\xi^l}\,, \quad \rho = \frac{-4i\zeta q F_{k0}}{\omega_0 + m}(p_l + qA_l)\,\xi^k\xi^l\,,$$
$$T_3 = -m\zeta\left(\xi^1\pi^2 - \xi^2\pi^1\right) + \frac{1}{2}sm\,. \tag{23}$$

Remarks are in order regarding the evolution equation (22). This equation takes place on the surface of all second-class constraints Φ, except u, i.e. the ones we have previously denoted by (v, V). The constraints (v, V) are identically satisfied once we have expressed the dependent variables via η. The fact that the constraint T_3 is a first-class one with respect to the remaining constraint set (u, T_3) can be observed from

$$\{T_3, \varphi^k\} = -m\zeta\left(\eta^{k2}\varphi^1 - \eta^{k1}\varphi^2\right).$$

Since T_3 is a first-class constraint in the reduced set (u, T_3), the corresponding constraint equation $T_3 = 0$ is understood as being applied after the Dirac bracket in (22) is calculated.

2.4 *Algebra of Variables in Reduced Phase Space*

Let us calculate the Dirac brackets of the variables η. To this end, we shall apply the rule (17) to calculate $\{x^k, p_r\}_{D(\Phi)}$, namely,

$$\{x^k, p_r\}_{D(\Phi)} = \{x^k, p_r\}_{D(U)} - \{x^k, V_a\}_{D(U)}\, C^{ab}\, \{V_b, p^k\}_{D(U)}\,.$$

Due to the orthogonality relation (14), the term $\{x^k, V_a\}_{D(U)}\, C^{ab}\, \{V_b, p^k\}_{D(U)}$ reduces to

$$\{x^k, V_a\}_{D(U)}\, C^{ab}\, \{V_b, p^k\}_{D(U)} = \{x^k, V_a\}\, C^{ab}\, \{V_b, p^k\} = \{x^k, V_1\}\, C^{11}\, \{V_1, p^k\} = 0,$$

since $C^{11} = 0$. Thus, we are left with

$$\{x^k, p_r\}_{D(U)} = \{x^k, p_r\}_{D(u)} - \{x^k, v_a\}_{D(u)}\, c^{ab}\, \{v_b, p_r\}_{D(u)}\,, \quad c^{ac}\{v_c, v_b\}_{D(u)} = \delta^a_b\,.$$

The second term on the r.h.s. of the above expression for $\{x^k, p_r\}_{D(U)}$ vanishes, since only Φ_1 has non-vanishing brackets with the coordinates x^k and momenta p_r. Thus, we finally obtain

$$\{x^k, p_r\}_{D(\Phi)} = \{x^k, p_r\}_{D(u)} = \{x^k, p_r\} - \{x^k, u_a\}\, d^{ab}\, \{u_b, p_r\} = \delta^k_r, \quad d^{ab}\{u_b, u_c\} = \delta^a_c\,,$$

Summarizing, we observe that the brackets between the variables η are given by

$$\{x^k, p_r\}_{D(u)} = \delta^k_r\,, \quad \{\xi_k, \xi_r\}_{D(u)} = -\{\pi_k, \pi_r\}_{D(u)} = i\{\xi_k, \pi_r\}_{D(u)} = \frac{i}{2}\eta_{kr}\,,$$
$$\{\zeta, x^k\}_{D(u)} = \{\zeta, p_k\}_{D(u)} = \{\zeta, \xi^k\}_{D(u)} = 0. \tag{24}$$

3 Quantization

3.1 *Quantization of Classical Variables*

The equal-time (anti)commutation relations for the operators $\hat{X}^k$, $\hat{P}_k$, $\hat{\Xi}^k$, $\hat{\zeta}$, corresponding to the variables x^k, p_k, ξ^k, ζ, are defined according to their Dirac brackets (24), using the correspondence $[\hat{A}, \hat{B}]_\pm = i\hbar\{A, B\}_{D(u)}$ and the requirement $\hat{\zeta}^2 = \mathbf{I}$, related to $\zeta^2 = 1$. Additionally, we assume $\hat{\zeta}$ to have eigenvalues $\zeta = \pm 1$ in correspondence with the pseudoclassical theory. The non-vanishing quantum (anti)commutation relations are

$$\left[\hat{X}^k, \hat{P}_r\right] = i\hbar\delta^k_r\,, \quad [\hat{\Xi}^k, \hat{\Xi}^l]_+ = -\frac{\hbar}{2}\eta^{kl}\,, \quad \hat{\zeta}^2 = \mathbf{I}\,. \tag{25}$$

We now introduce a Hilbert space $\mathcal{R}$ of x-dependent 16-component columns $\Psi(\mathbf{x})$:

$$\Psi(\mathbf{x}) = \begin{pmatrix} \Psi_{+1}(\mathbf{x}) \\ \Psi_{-1}(\mathbf{x}) \end{pmatrix}, \tag{26}$$

where $\Psi_\zeta(\mathbf{x})$, $\zeta = \pm 1$, are 8-component columns.

The inner product in $\mathcal{R}$ is defined as follows:

$$(\Psi, \Psi') = (\Psi_{+1}, \Psi'_{+1})_D + (\Psi'_{-1}, \Psi_{-1})_D, \quad (\Psi, \Psi')_D = \int \Psi^\dagger(\mathbf{x})\Psi'(\mathbf{x})d\mathbf{x}. \tag{27}$$

This inner product is Lorentz-invariant.

We realize all operators in a block-diagonal form:

$$\begin{gathered} \hat{X}^k = x^k \mathbf{I}, \quad \hat{P}_k = \hat{p}_k \mathbf{I}, \quad \hat{p}_k = -i\hbar\partial_k, \\ \hat{\Xi}^k = \mathrm{bdiag}\left(\hat{\xi}^k, \hat{\xi}^k\right), \quad \hat{\zeta} = \mathrm{bdiag}(I, -I), \quad \hat{S} = \mathrm{bdiag}(\hat{s}, \hat{s}), \end{gathered} \tag{28}$$

where $\hat{S}$ is an operator corresponding to the classical quantity s; $\mathbf{I}$ and I are 16×16 and 8×8 unit matrices, respectively, whereas $\hat{\xi}^k$ are 8×8 matrices which obey the equal time commutation relations $\left[\hat{\xi}^k, \hat{\xi}^l\right]_+ = -\frac{\hbar}{2}\eta^{kl}$.

3.2 *Hamiltonian, First-class Constraint and Spin Variables*

Let us realize the quantum Hamiltonian $\hat{H}_\tau + \hat{\Lambda}_3\hat{T}_3$ (here $\hat{\Lambda}_3$ is an arbitrary operator corresponding to the undetermined Lagrange multiplier Λ_3) in the following way:

$$\hat{H}_\tau = \hat{\zeta} q \hat{A}_0 + \hat{\Omega} = \mathrm{bdiag}\left(\hat{H}_{+1}, \hat{H}_{-1}\right), \tag{29}$$

where $\hat{A}_0 = \mathrm{bdiag}\left(A_0|_{x^0=\tau} I, A_0|_{x^0=-\tau} I\right)$ and $\hat{\Omega} = \mathrm{bdiag}\left(\hat{\Omega}_{+1}, \hat{\Omega}_{-1}\right)$, so that

$$\hat{H}_\zeta = \zeta q \; A_0|_{x^0=\zeta\tau} I + \hat{\Omega}_\zeta,$$

where

$$\hat{\Omega}_\zeta = \hat{\omega}_0|_{x^0=\zeta\tau}.$$

Then the 8×8 blocks $\hat{H}_\zeta$ become

$$\hat{H}_\zeta = (\zeta q A_0 I + \hat{\omega}_0)|_{x^0=\zeta\tau}. \tag{30}$$

As already mentioned concerning the modified scheme of canonical quantization, we realize the first-class constraint (20) as a restriction on the state vectors $\Psi \in \mathcal{R}$, $\hat{T}_3\Psi = 0$. The operator corresponding to the classical function $T_3(q,p)$ is realized by the general prescription $\hat{T}_3 = T_3|_{q=\hat{q},p=\hat{p}}$ (q and p symbolize the relevant coordinates and momenta) and maintains a block-diagonal structure:

$$\hat{T}_3 = m\left(2i\hat{\zeta}\hat{\Xi}^1\hat{\Xi}^2 + \frac{1}{2}\hat{S}\right) = \mathrm{bdiag}\left(\hat{t}_1, \hat{t}_{-1}\right), \quad \hat{t}_\zeta = m\left(2i\zeta\hat{\xi}^1\hat{\xi}^2 + \frac{1}{2}\hat{s}\right). \tag{31}$$

Therefore, the condition $\hat{T}_3\Psi = 0$ implies

$$\hat{t}_\zeta\Psi_\zeta = 0. \tag{32}$$

The conservation of this condition in time is guaranteed by $\left[\hat{T}_3, \hat{H}_\tau\right] = 0$, which follows from the corresponding relation in the pseudoclassical theory $\{T_3, \mathcal{H}_{\mathrm{eff}}\}_{D(u)} = 0$. Thus, the quantum condition of conservation of the constraint $\hat{T}_3$ reads

$$\left[\hat{t}_\zeta, \hat{\Omega}_\zeta\right] = 0. \tag{33}$$

Consequently, the specific form of $\hat{\Omega}_\zeta$ and $\hat{t}_\zeta$ is restricted by the above condition.

In the following, we shall need the operator $\hat{\Omega}^2$. For the choice

$$\hat{\omega}_0 = \begin{pmatrix} 0 & m - \gamma^k(\hat{p}_k + qA_k) \\ m + \gamma^k(\hat{p}_k + qA_k) & 0 \end{pmatrix}, \quad [\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}, \quad \mu = 0, 1, 2, \tag{34}$$

where γ^μ are four-dimensional γ-matrices, we have

$$\hat{\omega}_0^2 = \begin{pmatrix} m^2 + (\hat{p}_k + qA_k)^2 + \frac{i\hbar}{2} qF_{kl}\gamma^k\gamma^l & 0 \\ 0 & m^2 + (\hat{p}_k + qA_k)^2 + \frac{i\hbar}{2} qF_{kl}\gamma^k\gamma^l \end{pmatrix}. \tag{35}$$

It remains to realize the spin operators $\hat{\xi}$ and the operator $\hat{s}$, corresponding to the quantity s. This can be accomplished by regarding the relation (33) for $\hat{t}_\zeta$, which, in turn, determines the conditions on $\hat{\xi}^1\hat{\xi}^2$ and $\hat{s}$. These operators will be chosen to commute with themselves and with $\hat{\omega}_0$. Besides, we shall impose additional restrictions on the realization of $\hat{s}$, namely, that it commute with all dynamical variables (including $\hat{\xi}^k$), and that its eigenvalues be $\pm\hbar$, which is dictated by dimensionality reasons explained below.

Let us consider a general nontrivial 8×8 matrix M. The condition that this matrix commute with $\hat{\omega}_0$ implies the following restriction on its form:

$$M = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad [a,\gamma^k]_- = [b,\gamma^k]_+ = 0, \tag{36}$$

where a and b are 4×4 matrices.

Consider the following basis in the space of four-dimensional matrices:

$$\begin{gathered} \Gamma_1 = \mathbf{1},\ \Gamma_2 = i\gamma^1,\ \Gamma_3 = i\gamma^2,\ \Gamma_4 = i\gamma^3,\ \Gamma_5 = \gamma^0,\ \Gamma_6 = i\gamma^2\gamma^3,\ \Gamma_7 = i\gamma^3\gamma^1, \\ \Gamma_8 = i\gamma^1\gamma^2,\ \Gamma_9 = \gamma^0\gamma^1,\ \Gamma_{10} = \gamma^0\gamma^2,\ \Gamma_{11} = \gamma^0\gamma^3,\ \Gamma_{12} = i\gamma^0\gamma^2\gamma^3, \\ \Gamma_{13} = i\gamma^0\gamma^1\gamma^3,\ \Gamma_{14} = i\gamma^0\gamma^1\gamma^2,\ \Gamma_{15} = \gamma^1\gamma^2\gamma^3,\ \Gamma_{16} = i\gamma^0\gamma^1\gamma^2\gamma^3. \end{gathered} \tag{37}$$

The matrices Γ_i, $i=\overline{1,16}$, form an algebra under matrix multiplication. For each $i\neq 1$, we have $\operatorname{tr}\Gamma_i = 0$, and there exists a number j such that $\Gamma_j\Gamma_i\Gamma_j = -\Gamma_i$. We note that among these matrices the ones which commute with γ^1, γ^2 are Γ_i, $i\in\{1,2,3,11,14,15\}$, while those which anticommute with γ^1, γ^2 are Γ_i, $i\in\{4,5,8,16\}$.

Let us expand the matrices a and b from (36) in the basis (37), $a = a^i\Gamma_i$, $b = b^i\Gamma_i$. Then non-trivial solutions of (36) can be constructed as follows:

$$a = a^i\Gamma_i\,,\quad i\in\{1,2,3,11,14,15\},\quad b = b^j\Gamma_j\,,\quad j\in\{4,5,8,16\}, \tag{38}$$

with arbitrary a^i and b^i.

We divide the matrix M into linearly independent parts:

$$M_1 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}, \tag{39}$$

where the expressions for a and b are given by (38). We will realize the product $\hat{\xi}^1\hat{\xi}^2$ using the matrices M_1 and M_2 defined by (39). Further restrictions on the realization based on M_1 and M_2 (39) will be considered below.

For our purposes, it is convenient to choose the realization of the γ-matrices [7]

$$\gamma^0 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & i\sigma^1 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{40}$$

where σ^k are the Pauli matrices.

We realize the operators $\hat{\xi}^k$ as block-diagonal (or antidiagonal) 8×8 matrices whose blocks are chosen as γ-matrices (40). Then the product $\hat{\xi}^1\hat{\xi}^2$ can only consist of blocks which contain the product of two γ-matrices. As a consequence, the expansions of M_1 and M_2 must contain only the terms corresponding to a^{11} and b^8, respectively. In the following, it is sufficient to make the choice $a^{11}=0$, $b^8\neq 0$. Then we have

$$\hat{\xi}^1\hat{\xi}^2 = -\frac{\hbar}{4}\begin{pmatrix} 0 & \gamma^1\gamma^2 \\ \gamma^1\gamma^2 & 0 \end{pmatrix} = \frac{i\hbar}{4}\begin{pmatrix} 0 & \Sigma^3 \\ \Sigma^3 & 0 \end{pmatrix}, \quad \Sigma^3 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix},$$

where the spin matrices $\hat{\xi}^k$ are chosen as

$$\hat{\xi}^1 = \frac{i}{2}\hbar^{1/2}\begin{pmatrix} 0 & \gamma^1 \\ \gamma^1 & 0 \end{pmatrix}, \quad \hat{\xi}^2 = \frac{i}{2}\hbar^{1/2}\begin{pmatrix} \gamma^2 & 0 \\ 0 & \gamma^2 \end{pmatrix}, \quad [\hat{\xi}^k,\hat{\xi}^l]_+ = -\frac{\hbar}{2}\eta^{kl}. \tag{41}$$

Taking into account the above operator realization, the conditions previously imposed on $\hat{s}$ can be satisfied by its choice as the following block-diagonal matrix:

$$\hat{s} = \hbar \begin{pmatrix} \gamma^0 \Sigma^3 & 0 \\ 0 & \gamma^0 \Sigma^3 \end{pmatrix}, \tag{42}$$

where the factor $\hbar$ is included to assign a definite dimensionality to $\hat{T}_3$, according to (31).

Now that we have realized the operator $\hat{t}_\zeta$ in a suitable way, we can determine the structure of physical states Ψ_ζ, namely, those which satisfy the condition (32):

$$\Psi_\zeta(\tau, \mathbf{x}) = \begin{pmatrix} \psi_\zeta(\tau, \mathbf{x}) \\ \zeta \gamma^0 \psi_\zeta(\tau, \mathbf{x}) \end{pmatrix}. \tag{43}$$

3.3 *Dirac Equation in* $2+1$ *Dimensions*

The Schrödinger equation

$$i\hbar \partial_\tau \Psi = \left(\hat{H}_\tau + \hat{\Lambda}_3 \hat{T}_3\right) \Psi, \tag{44}$$

with $\hat{H}_\tau$ given by (29), for vectors Ψ subject to $\hat{T}_3 \Psi = 0$, has the form

$$i\hbar \partial_\tau \Psi = \hat{H}_\tau \Psi. \tag{45}$$

Solutions of the above equation can be chosen as eigenstates of the operator $\hat{S}$ and of the operator $\hat{\Xi}^1 \hat{\Xi}^2$, since $[\hat{s}, \hat{\omega}_0] = 0$, $\left[\hat{\xi}^1 \hat{\xi}^2, \hat{\omega}_0\right] = 0$, and $\left[\hat{\xi}^1 \hat{\xi}^2, \hat{s}\right] = 0$, by construction. Let us denote eigenstates of the operator $\hat{S}$ as Ψ_s, which are subject to $\hat{S} \Psi_s = s \Psi_s$. The latter implies that these solutions have the particular structure

$$\Psi_{\zeta,s}(\tau, \mathbf{x}) = \begin{pmatrix} \psi_{\zeta,s}(\tau, \mathbf{x}) \\ \zeta \gamma^0 \psi_{\zeta,s}(\tau, \mathbf{x}) \end{pmatrix}, \quad \psi_{\zeta,1}(\tau, \mathbf{x}) = \begin{pmatrix} \psi_\zeta^{(u)}(\tau, \mathbf{x}) \\ 0 \end{pmatrix}, \quad \psi_{\zeta,-1}(\tau, \mathbf{x}) = \begin{pmatrix} 0 \\ \sigma^1 \psi_\zeta^{(d)}(\tau, \mathbf{x}) \end{pmatrix}. \tag{46}$$

One can check that these states satisfy the eigenvalue equation $\hat{\xi}^1 \hat{\xi}^2 \Psi_{\zeta,s} = \frac{i\hbar}{4} s \zeta \Psi_{\zeta,s}$.

As we shall see below, the Schrödinger equation (45) leads to the Dirac equation. Therefore, in view of the original Schrödinger equation (44), the resulting Dirac equation leads to $\hat{T}_3 \Psi = 0$, since $\hat{\Lambda}_3$ is an arbitrary operator. Consequently, the Dirac equation is equivalent to the condition $\hat{T}_3 \Psi = 0$ in the space of states $\Psi = \Psi(\tau, \mathbf{x})$ subject to the Schrödinger equation (44).

In components, the Schrödinger equation (45) implies

$$i\hbar \partial_\tau \Psi_{\zeta,s}(\tau) = \left(\zeta q A_0 I + \hat{\omega}_0\right)\big|_{x^0 = \zeta\tau} \Psi_{\zeta,s}(\tau),$$

and, in terms of the physical time x^0, it becomes

$$i\hbar \zeta \partial_0 \Psi_{\zeta,s}\left(\zeta x^0\right) = \left(\zeta q A_0 I + \hat{\omega}_0\right) \Psi_{\zeta,s}\left(\zeta x^0\right).$$

In view of the explicit form of $\hat{\omega}_0$ (34) and $\Psi_{\zeta,s}$ (46), the last equation is rewritten as

$$\left[\gamma^\mu \left(i\hbar \partial_\mu - q A_\mu\right) - m\right] \psi_{\zeta,s}\left(\zeta x^0, \mathbf{x}\right) = 0. \tag{47}$$

Consider states Ψ_ζ with definite charge ζq. They satisfy the eigenvalue equation $\hat{\zeta} \Psi_\zeta = \zeta \Psi_\zeta$. Therefore, states with charge $+q$ satisfy $\Psi_{-1} = 0$, and states with charge $-q$ satisfy $\Psi_{+1} = 0$ in (26). Note that classical trajectories are parameterized by the physical time $x^0 = \tau$. The wave function $\Psi_+\left(x^0\right)$ is also parameterized by x^0. The wave functions $\Psi_{\pm 1}$ are parameterized by the physical time $\tau = \pm x^0$.

It is clear that (47) for $\zeta = +1$ is the $(3+1)$ Dirac equation (without the third spatial coordinate) for a particle with charge $+q$. The solutions of (47) with $\zeta = -1$ can be brought into correspondence with the $(3+1)$ Dirac equation for a particle $\psi^c\left(x^0\right)$ with charge $-q$, by the rule $\psi^c\left(x^0\right) = \gamma^2 \psi_-^*\left(-x^0\right)$.

In order to arrive at the $(2+1)$ Dirac equations for particles with charge $\pm q$, we use the decomposition (46) of the four-component column $\psi_{\zeta,s}$ into two-component columns $\psi_\zeta^{(u/d)}$. For $\zeta = +1$, equation (47) decomposes into

$$\left[\Gamma_{u/d}^\mu \left(i\hbar \partial_\mu - q A_\mu\right) - m\right] \psi^{(u/d)}(x) = 0, \quad \psi^{(u/d)}(x) \equiv \psi_{+1}^{(u/d)}\left(x^0, \mathbf{x}\right), \tag{48}$$

where $\Gamma^{\mu}_{u/d}$ are two inequivalent sets of gamma-matrices in $(2+1)$ dimensions, $\Gamma^0_u = \Gamma^0_d = \sigma^3$, $\Gamma^1_u = \Gamma^1_d = i\sigma^2$, $\Gamma^2_u = -\Gamma^2_d = -i\sigma^1$. The analogous equation for $\zeta = -1$ has the form

$$\left[\Gamma^{\mu}_{u/d}\left(i\hbar\partial_{\mu} + qA_{\mu}\right) - m\right]\psi^{(u/d)c}(x) = 0, \quad \psi^{(u/d)c}(x) \equiv \Gamma^2_{u/d}\psi^{(u/d)*}_{-1}\left(-x^0, \mathbf{x}\right), \tag{49}$$

which is the $(2+1)$ Dirac equation for a particle with charge $-q$.

Thus, the physical states are

$$\begin{aligned}
\boldsymbol{\Psi}_s(x) &= \begin{pmatrix} \Psi(x) \\ \Psi^c(x) \end{pmatrix}, \quad \Psi_s(x) = \begin{pmatrix} \psi_s(x) \\ \gamma^0(x) \end{pmatrix}, \quad \Psi^c_s(x) = \begin{pmatrix} \psi^c_s(x) \\ \gamma^0(x) \end{pmatrix}, \\
\psi_1(x) &= \begin{pmatrix} \psi^{(u)}(x) \\ 0 \end{pmatrix}, \quad \psi_{-1}(x) = \begin{pmatrix} 0 \\ \sigma^1\psi^{(d)}(x) \end{pmatrix}, \\
\psi^c_1(x) &= \begin{pmatrix} \psi^{(u)c}(x) \\ 0 \end{pmatrix}, \quad \psi^c_{-1}(x) = \begin{pmatrix} 0 \\ \sigma^1\psi^{(d)c}(x) \end{pmatrix}.
\end{aligned} \tag{50}$$

The operation of charge-conjugation (49) for the components $\Psi^c_s(x)$ is

$$\Psi^c_s(x) = \begin{pmatrix} 0 & \gamma^0\gamma^2 \\ \gamma^0\gamma^2 & 0 \end{pmatrix}\Psi^*_{\zeta=-1,s}\left(-x^0, \mathbf{x}\right).$$

The states $\boldsymbol{\Psi}_s(x)$ satisfy the evolution equation

$$i\hbar\partial_0\boldsymbol{\Psi}_s\left(x^0, \mathbf{x}\right) = \hat{H}_{x^0}\boldsymbol{\Psi}_s\left(x^0, \mathbf{x}\right), \quad \hat{H}_{x^0} = \mathrm{bdiag}\left(\hat{H}\left(x^0\right), \hat{H}^c\left(x^0\right)\right), \tag{51}$$

where

$$\begin{aligned}
\hat{H}\left(x^0\right) &= qA_0 I + \hat{\omega}_0, \quad \hat{H}^c\left(x^0\right) = \hat{H}\left(x^0\right)\Big|_{q\to -q} = -qA_0 I + \hat{\omega}^c_0, \\
\hat{\omega}^c_0 &= \hat{\omega}^c_0\big|_{q\to -q} = \begin{pmatrix} 0 & \gamma^0\gamma^2 \\ \gamma^0\gamma^2 & 0 \end{pmatrix}\hat{\omega}^*_0\begin{pmatrix} 0 & \gamma^0\gamma^2 \\ \gamma^0\gamma^2 & 0 \end{pmatrix}.
\end{aligned} \tag{52}$$

3.4 *One-particle Sector*

In the one-particle sector, it is sufficient to restrict the consideration to the case of a time-independent electromagnetic field, which makes the Hamiltonian also time-independent.

As indicated in Appendix B (Section 5), devoted to the quantum theory of the $(2+1)$ spinor field, physical states are those which are eigenvectors of the charge operator. Thus, let us introduce the physical subspace $\mathcal{R}_{\mathrm{ph}}$ of $\mathcal{R}$ which is spanned by eigenvectors of the charge operator $q\hat{\zeta}$, namely,

$$\boldsymbol{\Psi}_s = \begin{pmatrix} \Psi_s \\ 0 \end{pmatrix}, \quad \boldsymbol{\Psi}^c_s = \begin{pmatrix} 0 \\ \Psi^c_s \end{pmatrix}, \quad q\hat{\zeta}\boldsymbol{\Psi}_s = q\boldsymbol{\Psi}_s, \quad q\hat{\zeta}\boldsymbol{\Psi}^c_s = -q\boldsymbol{\Psi}^c_s, \tag{53}$$

where the components Ψ_s and Ψ^c_s are given by (50). States in this representation satisfy the Schrödinger equation with the Hamiltonian (51).

The scalar product (27) between equally charged states is

$$\left(\boldsymbol{\Psi}_s, \boldsymbol{\Psi}'_s\right) = \left(\Psi_s, \Psi'_s\right)_D, \quad \left(\boldsymbol{\Psi}^c_s, \boldsymbol{\Psi}^{c\prime}_s\right) = \left(\Psi^c_s, \Psi^{c\prime}_s\right)_D,$$

and for differently charged states it is vanishing.

Following the decomposition performed in the previous section of the 16-component columns $\boldsymbol{\Psi}_s$ and $\boldsymbol{\Psi}^c_s$ in terms of two-component spinors $\psi^{(u/d)}$ and $\psi^{(u/d)c}$, we define the one-particle sector of the quantum mechanics (QM) constructed as the space of states

$$\boldsymbol{\psi}^{(u/d)} = \begin{pmatrix} \psi^{(u/d)} \\ 0 \end{pmatrix}, \quad \boldsymbol{\psi}^{(u/d)c} = \begin{pmatrix} 0 \\ \psi^{(u/d)c} \end{pmatrix}. \tag{54}$$

The dynamics in this representation is governed by the Hamiltonian

$$\hat{H}_{u/d} = \mathrm{bdiag}\left(\hat{h}_{u/d}, \hat{h}^c_{u/d}\right), \quad \hat{h}_{u/d} = qA_0 + \Gamma^0_{u/d}\left[m + \Gamma^k_{u/d}\left(\hat{p}_k + qA_k\right)\right], \quad \hat{h}^c_{u/d} = \hat{h}_{u/d}\Big|_{q\to -q}. \tag{55}$$

By comparison with Appendix B, we conclude that for a fixed representation (u or d), the above Hamiltonian is precisely the Hamiltonian (B.19) in the coordinate representation of the one-particle sector of the $(2+1)$ spinor field for the same representation of the gamma-matrices. Moreover, the physical vector space (54) coincides with that of the quantum field theory (QFT) in the one-particle sector (B.20). In the time-independent backgrounds under consideration, we can see that under certain restrictions the constructed QM coincides exactly with QFT in the one-particle sector. The mentioned restrictions are related only to the appropriate definition of the Hilbert space of QM.

Let us establish the positivity of the operator the $\hat{\Omega}$ in the one-particle sector (54). Consider the expectation value of $\hat{\Omega}$ on the definite-charge states (53)

$$\left(\boldsymbol{\Psi}_s, \hat{\Omega}\boldsymbol{\Psi}_s\right) = (\Psi_s, \hat{\omega}\Psi_s)_D\,,\quad \left(\boldsymbol{\Psi}_s^c, \hat{\Omega}\boldsymbol{\Psi}_s^c\right) = (\Psi_s^c, \hat{\omega}^c\Psi_s^c)_D\,.$$

With allowance made for the form of Ψ_s, Ψ_s^c, $\hat{\omega}$, $\hat{\omega}^c$, given by (50), (34), (52), we have

$$\begin{aligned}(\Psi_s, \hat{\omega}_0\Psi_s)_D &= \int d^2x \begin{pmatrix} \psi_s \\ \gamma^0\psi_s \end{pmatrix}^\dagger \begin{pmatrix} \left[m - \gamma^k\left(\hat{p}_k + qA_k\right)\right]\gamma^0\psi_s \\ \left[m + \gamma^k\left(\hat{p}_k + qA_k\right)\right]\psi_s \end{pmatrix} \\ &= 2\int d^2x \psi_s^\dagger\gamma^0\left[m + \gamma^k\left(\hat{p}_k + qA_k\right)\right]\psi_s\,,\end{aligned}$$

and

$$\begin{aligned}(\Psi_s^c, \hat{\omega}_0^c\Psi_s^c)_D &= \int d^2x \begin{pmatrix} \psi_s^c \\ \gamma^0\psi_s^c \end{pmatrix}^\dagger \begin{pmatrix} \left[m - \gamma^k\left(\hat{p}_k - qA_k\right)\right]\gamma^0\psi_s^c \\ \left[m + \gamma^k\left(\hat{p}_k - qA_k\right)\right]\psi_s^c \end{pmatrix} \\ &= 2\int d^2x \psi_s^{c\dagger}\gamma^0\left[m + \gamma^k\left(\hat{p}_k - qA_k\right)\right]\psi_s^c\,,\end{aligned}$$

For $s = +1$, we have

$$\begin{aligned}(\Psi_{s=+1}, \hat{\omega}\Psi_{s=+1})_D &= 2\int d^2x \psi^{(u)\dagger}\Gamma_u^0\left[m + \Gamma_u^k\left(\hat{p}_k + qA_k\right)\right]\psi^{(u)}, \\ \left(\Psi_{s=+1}^c, \hat{\omega}_0^c\Psi_{s=+1}^c\right)_D &= 2\int d^2x \psi^{(u)c\dagger}\Gamma_u^0\left[m + \Gamma_u^k\left(\hat{p}_k - qA_k\right)\right]\psi^{(u)c},\end{aligned}$$

and, similarly, for $s = -1$,

$$\begin{aligned}(\Psi_{s=-1}, \hat{\omega}\Psi_{s=-1})_D &= 2\int d^2x \psi^{(d)\dagger}\Gamma_d^0\left[m + \Gamma_d^k\left(\hat{p}_k + qA_k\right)\right]\psi^{(d)}, \\ \left(\Psi_{s=-1}^c, \hat{\omega}_0^c\Psi_{s=-1}^c\right)_D &= 2\int d^2x \psi^{(d)c\dagger}\Gamma_d^0\left[m + \Gamma_d^k\left(\hat{p}_k - qA_k\right)\right]\psi^{(d)c}.\end{aligned}$$

Using the $(2+1)$ Dirac equation for particles with charge q (48) and $-q$ (49), we write

$$\begin{aligned}(\Psi_s, \hat{\omega}_0\Psi_s)_D &= 2\int d^2x \psi^{(u/d)\dagger}(x)\left(i\hbar\partial_0 - qA_0\right)\psi^{(u/d)}(x)\,, \\ (\Psi_s^c, \hat{\omega}_0^c\Psi_s^c)_D &= 2\int d^2x \psi^{(u/d)c\dagger}(x)\left(i\hbar\partial_0 + qA_0\right)\psi^{(u/d)c}(x)\,.\end{aligned}$$

Recall that in the one-particle sector it is sufficient to restrict the consideration to the case of a time-independent Hamiltonian. In this case, for stationary solutions $\psi^{(u/d)}$ and $\psi^{(u/d)c}$ there hold the relations

$$i\hbar\partial_0\psi^{(u/d)} = \varepsilon_+^{(u/d)}\psi^{(u/d)},\quad i\hbar\partial_0\psi^{(u/d)c} = \varepsilon_+^{(u/d)c}\psi^{(u/d)c}$$

with $\varepsilon_+^{(u/d)} > 0$ and $\varepsilon_+^{(u/d)c} > 0$. Thus,

$$\begin{aligned}(\Psi_s, \hat{\omega}_0\Psi_s)_D &= 2\int d^2x \psi^{(u/d)\dagger}(x)\left(\varepsilon_+^{(u/d)} - qA_0\right)\psi^{(u/d)}(x)\,, \\ (\Psi_s^c, \hat{\omega}_0^c\Psi_s^c)_D &= 2\int d^2x \psi^{(u/d)c\dagger}(x)\left(\varepsilon_+^{(u/d)c} + qA_0\right)\psi^{(u/d)c}(x)\,.\end{aligned}$$

The coincidence of our QM with QFT in the one-particle sector implies that we can use the results of Section 4.4. Taking into account the form of ε in the general case (B.11), we have $\varepsilon_{+}^{(u/d)} - qA_0 > 0$, and hence

$$\left(\Psi_{\zeta=+1,s}, \hat{\Omega}\Psi_{\zeta=+1,s}\right) > 0.$$

Similarly, due to (B.12) and $\varepsilon_{+}^{(u/d)c} = -\varepsilon_{-}^{(u/d)}$, we have $\varepsilon_{+}^{(u/d)c} + qA_0 > 0$, and therefore

$$\left(\Psi_{s}^{c}, \hat{\Omega}\Psi_{s}^{c}\right) > 0.$$

3.5 *Quasiclassical Limit*

In this section, we shall prove that the quantum Hamiltonian $\hat{H}_{\tau}$ (29) is a consistent realization of its classical analogue $\mathcal{H}_{\text{eff}}$. The analysis will be restricted to the case of time-independent electromagnetic field, sufficient for the consideration of the one-particle sector. For simplicity, we also assume that only magnetic field is present. The combination of magnetic and electric fields can be analyzed by direct analogy with the $(3+1)$ case studied in [3].

To prove that $\hat{H}_{\tau}$ is a consistent quantum realization of $\mathcal{H}_{\text{eff}}$, it is sufficient to show that the operator $\hat{\Omega}$ (35) has the correct quasiclassical limit. The correspondence with quasiclassics requires, in the first place, that the operator $\hat{\Omega}$ be positive in the one-particle sector, since the classical root ω_0 is positive. We note that the required positivity of $\hat{\Omega}$ has been established in the previous section, using exact one-particle states. In view of this, to prove that $\hat{\Omega}$ is a consistent quantum realization of ω_0, it suffices to show that the squared operator $\hat{\Omega}^2$ (35) has the correct quasiclassical limit, i.e., it has the same expectation value on quasiclassical states Ψ_{s}^{cl} as the operator $\tilde{\Omega}^2$, obtained by direct quantization of ω_0^2,

$$\tilde{\Omega}^2 = \omega_0^2\big|_{\eta\to\hat{\eta}}\,, \quad \omega_0 = \sqrt{m^2 + \left(p_k + qA_k\right)^2 - 2iqF_{kl}\xi^k\xi^l}\,.$$

Therefore,

$$\tilde{\Omega}^2 = \text{bdiag}\left(\hat{\omega}^2, \hat{\omega}^2\right), \quad \hat{\omega}^2 = m^2 + \left(\hat{p}_k + qA_k\left(x\right)\right)^2 - 2iqF_{kl}\hat{\xi}^k\hat{\xi}^l.$$

Thus, we need to show that

$$\left(\Psi_{s}^{cl}, \hat{\Omega}^2\Psi_{s}^{cl}\right) = \left(\Psi_{s}^{cl}, \tilde{\Omega}^2\Psi_{s}^{cl}\right), \tag{56}$$

where the inner product is defined by (27).

The component structure of quasiclassical states Ψ_{s}^{cl} coincides with that of the states Ψ_s from the one-particle physical subspace (53),

$$\Psi_{s}^{cl} = \begin{pmatrix} \Psi_{s}^{cl} \\ 0 \end{pmatrix}, \quad \Psi_{s}^{cl} = \begin{pmatrix} \psi_{s}^{cl} \\ \gamma^0\psi_{s}^{cl} \end{pmatrix},$$

$$\psi_{+1}^{cl} = \begin{pmatrix} \psi^{(u/d)cl} \\ 0 \end{pmatrix}, \quad \psi_{-1}^{cl} = \begin{pmatrix} 0 \\ \sigma^1\psi^{(u/d)cl} \end{pmatrix}.$$

In terms of the two-component spinors $\psi^{(u/d)cl}$, the condition (56) becomes

$$\left(\psi^{(u/d)cl}, \hat{h}_{u/d}^2\psi^{(u/d)cl}\right) = \left(\psi^{(u/d)cl}, \hat{\epsilon}_{+,s}^2\psi^{(u/d)cl}\right), \tag{57}$$

where

$$\hat{\epsilon}_{+,s}^2 = m^2 + \left(\hat{p}_k + qA_k\right)^2 + s\hbar qF_{12}\,, \tag{58}$$

and $\hat{h}_{u/d}$ is given by (55).

By analogy with solutions of the Dirac equation for a time-independent magnetic field, let us choose the quasiclassical wave packet (QWP) for $\zeta = +1$ in the form

$$\psi_{+}^{(u/d)cl} = \left[\Gamma_{u/d}^0\hat{\epsilon}_{+,s} - \Gamma_{u/d}^k\left(\hat{p}_k + qA_k\right) + m\right]\varphi_{+}^{(u/d)cl}, \tag{59}$$

where $\hat{\epsilon}_{+,s}$ is the square root of the operator (58). The action of the Hamiltonian operator for the $(2+1)$ Dirac equation on the states $\psi_{+}^{cl(s)}$ is given by

$$\hat{h}_{u/d}\psi_{+}^{(u/d)cl} \equiv \Gamma_{u/d}^0\left[\Gamma_{u/d}^k\left(\hat{p}_k + qA_k\right) + m\right]\psi_{+}^{(u/d)cl} = \left[\Gamma_{u/d}^0\hat{h}_{u/d}^2 - \Gamma_{u/d}^k\left(\hat{p}_k + qA_k\right)\hat{\epsilon}_{+,s} + m\hat{\epsilon}_{+,s}\right]\varphi_{+}^{(u/d)cl}.$$

With allowance made for the identity

$$-\Gamma^k_{u/d}\left(\hat{p}_k+qA_k\right)\hat{\epsilon}_{+,s}=\left[\hat{\epsilon}_{+,s},\Gamma^k_{u/d}\left(\hat{p}_k+qA_k\right)\right]-\hat{\epsilon}_{+,s}\Gamma^k_{u/d}\left(\hat{p}_k+qA_k\right),$$

we have

$$\hat{h}_{u/d}\psi^{cl(s)}_{+}=\left[\Gamma^0_{u/d}\hat{h}^2_{u/d}-\hat{\epsilon}_{+,s}\Gamma^k_{u/d}\left(\hat{p}_k+qA_k\right)+m\hat{\epsilon}_{+,s}\right]\varphi^{(u/d)cl}_{+}+\left[\hat{\epsilon}_{+,s},\Gamma^k_{u/d}\left(\hat{p}_k+qA_k\right)\right]\varphi^{(u/d)cl}_{+}. \quad (60)$$

The function $\varphi^{cl(s)}_{+}$ is chosen to be an eigenvector of σ^3 with eigenvalue $+1$. This can be seen as follows. The QWP (59) can be expanded in terms of eigenvectors of the Hamiltonian operator $\hat{h}_s$,

$$\psi^{(u/d)cl}_{+}=\sum_n c^{(u/d)}_{+,n}\psi^{(u/d)}_{+,n},\quad \hat{h}_s\psi^{(u/d)}_{+,n}=\varepsilon^{(u/d)}_{+,n}\psi^{(u/d)}_{+,n}.$$

Let us represent $\varphi^{(u/d)cl}_{+}$ in the form

$$\psi^{(u/d)}_{+,n}=\left[\Gamma^0_{u/d}\varepsilon^{(u/d)}_{+,n}-\Gamma^k_{u/d}\left(\hat{p}_k+qA_k\right)+m\right]\varphi^{(u/d)}_{+,n}$$

Therefore, on the one hand,

$$\psi^{(u/d)cl}_{+}=\left[m-\Gamma^k_{u/d}\left(\hat{p}_k+qA_k\right)\right]\sum_n c^{(u/d)}_{+,n}\varphi^{(u/d)}_{+,n}+\Gamma^0_{u/d}\sum_n\varepsilon^{(u/d)}_{+,n}c_{+,n}\varphi^{(u/d)}_{+,n}$$

and, on the other hand, $\psi^{(u/d)cl}_{+}$ is given by (59). If

$$\hat{\epsilon}_{+,s}\sum_n c^{(u/d)}_{+,n}\varphi^{(u/d)}_{+}=\sum_n\varepsilon^{(u/d)}_{+,n}c^{(u/d)}_{+,n}\varphi^{(u/d)}_{+,n}, \quad (61)$$

then

$$\varphi^{(u/d)cl}_{+}=\sum_n c^{(u/d)}_{+,n}\varphi^{(u/d)}_{+,n},$$

so that

$$\sigma^3\varphi^{cl(u/d)}_{+}=\varphi^{cl(u/d)}_{+},$$

since $\varphi^{(u/d)}_{+,n}$ can always be chosen as an eigenvector of σ^3.

From (60) it follows that

$$\hat{h}_{u/d}\psi^{(u/d)cl}_{+}=\hat{\epsilon}_{+,s}\psi^{(u/d)cl}_{+}+\left[\hat{\epsilon}_{+,s},\Gamma^k_{u/d}\left(\hat{p}_k+qA_k\right)\right]\varphi^{(u/d)cl}_{+},$$

since $\hat{h}^2_{u/d}\varphi^{(u/d)cl}_{+}=\hat{\epsilon}^2_{+,s}\varphi^{(u/d)cl}_{+}$.

As we shall see, the second term on the r.h.s of the above expression can be neglected with quasiclassical accuracy. Therefore, we identify the Hamiltonian operator $\hat{h}_{u/d}$ with $\hat{\epsilon}_{+,s}$ in the space of quasiclassical states $\psi^{(u/d)cl}_{+}$. Since $\hat{h}_{u/d}\psi^{(u/d)cl}=\hat{\epsilon}_{+,s}\psi^{(u/d)cl}$, we arrive at the desired result (57).

Let us analyze the condition which allows one to neglect the term

$$\left[\hat{\epsilon}_{+,s},\Gamma^k_{u/d}\left(\hat{p}_k+qA_k\right)\right]\varphi^{(u/d)cl}_{+} \quad (62)$$

in (60). Note that

$$\left[\hat{\epsilon}^2_{+,s},\Gamma^k_{u/d}\left(\hat{p}_k+qA_k\right)\right]_{\hbar=0}=0,$$

then

$$\hat{\epsilon}_{+,s}\left[\hat{\epsilon}_{+,s},\Gamma^k_{u/d}\left(\hat{p}_k+qA_k\right)\right]+\left[\hat{\epsilon}_{+,s},\Gamma^k_{u/d}\left(\hat{p}_k+qA_k\right)\right]\hat{\epsilon}_{+,s}\Big|_{\hbar=0}=0.$$

Since $\hat{\epsilon}_{+,s}|_{\hbar=0}$ is a c-number, we have

$$\left[\hat{\epsilon}_{+,s},\Gamma^k_{u/d}\left(\hat{p}_k+qA_k\right)\right]_{\hbar=0}\hat{\epsilon}_{+,s}|_{\hbar=0}=0,$$

which implies

$$\left[\hat{\epsilon}_{+,s},\Gamma^k\left(\hat{p}_k+qA_k\right)\right]_{\hbar=0}=0,$$

since $\hat{\epsilon}_{+,s}|_{\hbar=0}\neq 0$. Therefore, we can write with quasiclassical accuracy,

$$\hat{h}_{u/d}\psi^{(u/d)cl}=\hat{\epsilon}_{+,s}\psi^{(u/d)cl}+O\left(\hbar\right).$$

This completes the proof of the fact $\hat{\Omega}$ has the correct quasiclassical limit, which establishes the consistency of the constructed QM.

4 Conclusion

In this paper, we have presented a consistent quantization of the $(2+1)$ spinning relativistic particle in arbitrary electromagnetic backgrounds. At the classical level, the theory contains a first-class constraint which does not admit gauge-fixing. At the quantum level, the presence of this constraint leads to an extension of the Hilbert space to 16-component state-vectors. The consistency of the found operator realization is established in the framework of quasiclassical analysis. The constructed relativistic QM leads to the $(2+1)$ Dirac equation for the wave functions in electromagnetic backgrounds. In arbitrary electromagnetic backgrounds respecting vacuum stability, the one-particle sector of the quantum theory contains both particles and antiparticles with positive energy levels, and exactly reproduces the one-particle sector of the quantum theory of the $(2+1)$ spinor field.

Acknowledgement R.F. is grateful to FAPESP, grant 00/14534-7. D.G. acknowledges the support of FAPESP, CNPq and DAAD. The work of P.M. was supported by FAPESP, grant 02/00423-4.

A Dirac Matrices in $2+1$ Dimensions

In $(2+1)$ dimensions there are two inequivalent representations for the Dirac matrices:

$$\gamma_s^0 = \sigma^3, \ \gamma_s^1 = i\sigma^2, \ \gamma_s^2 = -is\sigma^1, \ s = \pm 1,$$

where $\boldsymbol{\sigma} = (\sigma^i)$ are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The γ_s-matrices satisfy the defining commutation relations

$$[\gamma_s^\mu, \gamma_s^\nu]_+ = 2\eta^{\mu\nu}$$

and the properties

$$\gamma_s^{0\dagger} = \gamma_s^0 , \ \gamma_s^{i\dagger} = \gamma_s^0 \gamma_s^i \gamma_s^0 = -\gamma_s^i , \ i = 1, 2.$$

Then the equivalence of the two representations means, in particular, that there exists a matrix M such that

$$\gamma_{-1}^\mu M = M \gamma_{+1}^\mu$$

for all $\mu = 0, 1, 2$. Thus, M must satisfy

$$[M, \sigma^1]_+ = [M, \sigma^2] = [M, \sigma^3] = 0.$$

It is clear there is no non-trivial matrix M satisfying the above requirements.

B $(2+1)$ Quantum Spinor Field in Electromagnetic Background, One-particle Sector

In this section, we set $\hbar = 1$ and denote the rest mass by M. Unless otherwise specified, we shall always make use of the $s = +1$ representation for γ-matrices (see Appendix A), since the consideration of the case $s = -1$ is analogous.

B.1 *Lagrangian Formulation*

The classical $(2+1)$ Dirac field is described by fermionic fields $\Psi_\alpha(x)$, $\alpha = 1, 2$, which are understood as generating elements of an infinite-dimensional Grassmann algebra:

$$\Psi_\alpha(x)\, \Psi_\beta(y) + \Psi_\beta(y)\, \Psi_\alpha(x) = 0.$$

The action functional of the spinor field in a general electromagnetic background is given by

$$S_{\mathrm{FT}}\left(\Psi,\bar{\Psi}\right)=\int d^3x\mathcal{L},\;\; \mathcal{L}=\bar{\Psi}\left(\gamma^{\mu}P_{\mu}-M\right)\Psi,\;\; P_{\mu}=\left(i\partial_{\mu}-qA_{\mu}\right), \tag{B.1}$$

where summation over repeated indices $\mu=0,1,2$ is assumed; q is the charge (for electrons, $q=-e<0$); γ^{μ} are the γ-matrices in $(2+1)$ dimensions, and $\bar{\Psi}=\Psi^{+}\gamma^{0}$.

The Euler–Lagrange equation following from (B.1) are

$$\begin{aligned}\frac{\delta S_{\mathrm{FT}}}{\delta\Psi}&=\bar{\Psi}\left[\left(i\overleftarrow{\partial}_{\mu}+qA_{\mu}\right)\gamma^{\mu}+M\right]=0,\\ \frac{\delta S_{\mathrm{FT}}}{\delta\bar{\Psi}}&=\left[\left(i\partial_{\mu}-qA_{\mu}\right)\gamma^{\mu}-M\right]\Psi=0.\end{aligned} \tag{B.2}$$

The invariance of the action (B.1) under the global $U(1)$ transformations

$$\Psi\left(x\right)\rightarrow\Psi'\left(x\right)=e^{-iq\alpha}\Psi\left(x\right),\;\; \bar{\Psi}\left(x\right)\rightarrow\bar{\Psi}'\left(x\right)=e^{iq\alpha}\bar{\Psi}\left(x\right), \tag{B.3}$$

leads to the Noether current $j^{\mu}=q\bar{\Psi}\gamma^{\mu}\Psi$ and the corresponding conserved charge

$$Q=\int d^2xj^0=q\int d^2x\Psi^{+}\Psi. \tag{B.4}$$

B.2 *Charge Conjugation*

Let us consider the charge conjugation of solutions of the Dirac equation in a general electromagnetic field. The $(2+1)$ Dirac equation for a particle with charge q in a general electromagnetic field (B.2) is

$$\left[\left(i\partial_{\mu}-qA_{\mu}\right)\gamma^{\mu}-M\right]\Psi_{\varkappa,n}\left(x\right)=0, \tag{B.5}$$

whereas the Dirac equation for a particle with charge $-q$ is

$$\left[\left(i\partial_{\mu}+qA_{\mu}\right)\gamma^{\mu}-M\right]\Phi_{\varkappa,\alpha}\left(x\right)=0. \tag{B.6}$$

Note that we consider positive ($\varkappa=+1$) and negative ($\varkappa=-1$) energy solutions for both cases. Without loss of generality, let us consider the stationary case, for which the (now stationary) solutions $\Psi_{\varkappa}$ and $\Phi_{\varkappa}$ satisfy

$$\hat{h}\Psi_{\varkappa}=\varepsilon_{\varkappa}\Psi_{\varkappa}\,,\;\; \hat{h}_{\Phi}\Phi_{\varkappa}=\varepsilon_{\varkappa}^{\Phi}\Phi_{\varkappa}\,,\;\; \hat{h}_{\Phi}=\gamma^2\hat{h}^{*}\gamma^2.$$

Thus, one can easily see that $\gamma^2\Psi_{\varkappa}^{*}$ is an eigenvector of $\hat{h}_{\Phi}$ with eigenvalue $-\varepsilon_{\varkappa}$. Since $-\varepsilon_{+}<0$, the eigenvector $\gamma^2\Psi_{+}^{*}$ corresponds to the negative-energy and positively charged solution Φ_{-},

$$\Phi_{-}=\gamma^2\Psi_{+}^{*}\,.$$

Similarly, the positive-energy eigenvector $\gamma^2\Psi_{-}^{*}$ of $\hat{h}_{\Phi}$ (with eigenvalue $-\varepsilon_{-}>0$), corresponds to

$$\Phi_{+}=\gamma^2\Psi_{-}^{*}\,. \tag{B.7}$$

Therefore, one can use the negative-energy solutions of equation (B.5) to construct positive-energy solution of equation (B.6).

B.3 *Hamiltonization*

To quantize the theory (B.1) It is convenient to apply the generalized Hamiltonization procedure for degenerate theories [4]. Starting from the Lagrange density (B.1), let us perform the procedure [4] with the choice $N_{\Psi}=1$ and $N_{\bar{\Psi}}=0$. The generalized Hessian matrix is given by

$$\begin{aligned}\tilde{M}_{\Psi\bar{\Psi}}\left(x,y\right)&=\left\|\begin{array}{cc}\frac{\delta^2L}{\delta\bar{\Psi}(x)\delta\bar{\Psi}(y)} & \frac{\delta^2L}{\delta\bar{\Psi}(x)\delta\Psi(y)}\\ \frac{\delta^2L}{\delta\Psi(x)\delta\bar{\Psi}(y)} & \frac{\delta^2L}{\delta\Psi(x)\delta\Psi(y)}\end{array}\right\|=\left\|\begin{array}{cc}\frac{\partial^2\mathcal{L}}{\partial\dot{\bar{\Psi}}\partial\dot{\bar{\Psi}}} & \frac{\partial^2\mathcal{L}}{\partial\dot{\bar{\Psi}}\partial\dot{\Psi}}\\ \frac{\partial^2\mathcal{L}}{\partial\dot{\Psi}\partial\dot{\bar{\Psi}}} & \frac{\partial^2\mathcal{L}}{\partial\dot{\Psi}\partial\dot{\Psi}}\end{array}\right\|\delta\left(\mathbf{x}-\mathbf{y}\right)\\ &=\left\|\begin{array}{cc}0 & -i\gamma^0\\ i\gamma^0 & 0\end{array}\right\|\delta\left(\mathbf{x}-\mathbf{y}\right)\equiv\tilde{M}_{\Psi\Psi^{+}}\delta\left(\mathbf{x}-\mathbf{y}\right).\end{aligned}$$

Since the matrix $\bar{M}_{\Psi\Psi^+}$ is non-degenerate, the theory is nonsingular in the sense of [4].

Hamiltonization is straightforward. We introduce the velocity $v = \dot{\Psi}$, and also treat the degenerate coordinate $\bar{\Psi}$ as an undetermined velocity. The first-order action reads

$$S^v = \int d^4x \left[\mathcal{L}^v + \pi\left(\dot{\Psi} - v\right)\right] = \int d^4x \left(\pi\dot{\Psi} - \mathcal{H}^v\right),$$
$$\mathcal{L}^v = \mathcal{L}|_{\dot{\Psi}=v}\,, \quad \mathcal{H}^v = \pi v - \mathcal{L}^v.$$

Since the theory is nonsingular, all velocities are auxiliary variables, whose expressions are obtained from the following variation principle:

$$\frac{\delta S^v}{\delta v} = \frac{\partial \mathcal{L}^v}{\partial v} - \pi = i\bar{\Psi}\gamma^0 - \pi = 0 \Rightarrow \bar{\Psi} = -i\pi\gamma^0,$$
$$\frac{\delta S^v}{\delta \bar{\Psi}} = \frac{\partial \mathcal{L}^v}{\partial \bar{\Psi}} = -i\gamma^0 v + \left(q\gamma^0 A_0 - \gamma^k P_k + M\right)\Psi = 0 \Rightarrow v = -i\gamma^0\left(q\gamma^0 A_0 - \gamma^k P_k + M\right)\Psi.$$

Thus, we have expressed the velocities in terms of the coordinates and momenta, $\bar{\Psi} = \bar{\Psi}(\pi)$, $v = v(\Psi)$. Substituting these functions into $\mathcal{H}^v$, we get

$$\mathcal{H}_{\text{ph}} = -i\pi\gamma^0\left(q\gamma^0 A_0 - \gamma^k P_k + M\right)\Psi = -i\pi\hat{h}\Psi,$$

where

$$\hat{h} = qA_0 + \gamma^0\left(M - \gamma^k P_k\right). \tag{B.8}$$

The Hamiltonian action of the first-order formalism becomes

$$S^H = \int d^4x \left(\pi\dot{\Psi} - \mathcal{H}_{\text{ph}}\right),$$

with

$$\frac{\delta S^H}{\delta \pi} = 0 \Rightarrow \dot{\Psi} = \{\Psi, H_{\text{ph}}\}, \quad \frac{\delta S^H}{\delta \Psi} = 0 \Rightarrow \dot{\pi} = \{\pi, H_{\text{ph}}\},$$
$$H_{\text{ph}} = \int d^3x \mathcal{H}_{\text{ph}},$$

where the Poisson brackets between two definite-parity functionals $F(\Psi,\pi)$, $G(\Psi,\pi)$ are given by

$$\{F, G\} = \int d\mathbf{x} \left(\frac{\delta F}{\delta \Psi^\alpha}\frac{\delta G}{\delta \pi_\alpha} - (-1)^{\varepsilon(F)\varepsilon(G)}\frac{\delta G}{\delta \Psi^\alpha}\frac{\delta F}{\delta \pi_\alpha}\right).$$

In the course of quantization of the model (B.1), the fields Ψ and π become Heisenberg operators $\hat{\Psi}$, $\hat{\pi}$ with equal-time anticommutation relations given by the general rule $\left[\hat{F}, \hat{G}\right]_+ = i\{F, G\}$ for Grassmann-odd F, G,

$$\left[\hat{\Psi}_\alpha(x), \hat{\pi}_\beta(y)\right]_+\Big|_{x_0=y_0} = i\left[\hat{\Psi}_\alpha(x), \hat{\Psi}^+_\beta(y)\right]_+\Big|_{x_0=y_0}$$
$$= i\{\Psi_\alpha(x), \pi_\beta(y)\} = i\delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{y}). \tag{B.9}$$

B.4 *Spectrum of Dirac Hamiltonian in Stationary Case*

Let us consider the eigenvalue problem for the Hamiltonian (B.8) in a time-independent background, for which the Hamiltonian does not depend on x^0,

$$\hat{h}\Psi(\mathbf{x}) = \varepsilon\Psi(\mathbf{x}), \tag{B.10}$$

where $\Psi(\mathbf{x})$ is a solution of the stationary Schrödinger equation (B.10). Let us write

$$\Psi(\mathbf{x}) = \left[\gamma^0(\varepsilon - qA_0) + \gamma^k(i\partial_k - qA_k) + m\right]\psi(\mathbf{x}).$$

Then $\psi(\mathbf{x})$ satisfies the equation

$$\left[(\varepsilon - qA_0)^2 - D\right]\psi(\mathbf{x}) = 0,$$

where $D = m^2 - P_k P^k + \frac{i}{4} q F_{\mu\nu} [\gamma^\mu, \gamma^\nu]$. A pair (ε, ψ) is a solution to the above equation if it obeys either the equation

$$\varepsilon = qA_0 + \sqrt{\varphi^{-1} D \varphi} \Rightarrow \varepsilon - qA_0 > 0 \tag{B.11}$$

or

$$\varepsilon = qA_0 - \sqrt{\varphi^{-1} D \varphi} \Rightarrow \varepsilon - qA_0 < 0. \tag{B.12}$$

Let us denote by $\varepsilon_{+,n}, \psi_{+,n}$ solutions for positive ε, i.e., for the upper branch of the energy spectrum, and by $\varepsilon_{-,\alpha}, \psi_{-,\alpha}$ solutions for negative ε, for the lower branch of the energy spectrum. Here n and α are some quantum numbers which account for the possible non-symmetry of the energy spectrum between both branches in the general case of an arbitrary potential A_0.

Solutions $\Psi_{+,n}(\mathbf{x}), \Psi_{-,\alpha}(\mathbf{x})$ of (B.10), constructed from $\psi_{+,n}$ and $\psi_{-,\alpha}$, obey the orthogonality and completeness relations

$$(\Psi_{+,n}, \Psi_{+,m})_D = \delta_{nm}, \quad (\Psi_{-,\alpha}, \Psi_{-,\beta})_D = \delta_{\alpha\beta}, \quad (\Psi_{+,n}, \Psi_{-,\alpha})_D = 0, \tag{B.13}$$

$$\sum_{n,\alpha} \left[\Psi_{+,n}(x) \Psi^{+}_{+,n}(y) + \Psi_{-,\alpha}(x) \Psi^{+}_{-,\alpha}(y) \right] = \delta(\mathbf{x} - \mathbf{y}), \quad x_0 = y_0, \tag{B.14}$$

where

$$\Psi_{+,n}(x) = e^{-i\varepsilon_{+,n} x^0} \Psi_{+,n}(\mathbf{x}), \quad \Psi_{-,\alpha}(x) = e^{-i\varepsilon_{-,\alpha} x^0} \Psi_{-,\alpha}(\mathbf{x}). \tag{B.15}$$

Let us define charge-conjugated states by the general rule (B.7),

$$\Psi^c_{+,\alpha} = \gamma^2 \Psi^*_{-,\alpha} = e^{i\varepsilon_{-,\alpha} x^0} \gamma^2 \Psi_{-,\alpha}(\mathbf{x}).$$

Thus defined states obey the eigenvalue equation

$$\hat{h}^c \Psi^c_{+,\alpha} = \varepsilon^c_{+,\alpha} \Psi^c_{+,\alpha}, \quad \hat{h}^c = \gamma^2 \hat{h}^* \gamma^2, \quad \varepsilon^c_{+,\alpha} = -\varepsilon_{-,\alpha}.$$

B.5 *Quantization*

Using the complete set (B.15), we may decompose the field operator $\hat{\Psi}(x)$ as

$$\hat{\Psi}(x) = \sum_n a_n \psi_{+,n} + \sum_\alpha b^+_\alpha \psi_{-,\alpha}. \tag{B.16}$$

From the commutation relations (B.9) and orthogonality conditions (B.13), we get

$$\left[a_n, a^+_m\right]_+ = \left[b_\alpha, b^+_\beta\right]_+ = \delta_{\alpha\beta}, \quad [a_n, a_m]_+ = [b_\alpha, b_\beta]_+ = 0,$$

which defines two sets of creation and annihilation operators, a_n, a^+_n and b_α, b^+_β.

Inserting the expansion (B.16) into the expression for the quantum field theory (QFT) Hamiltonian,

$$\hat{H}^{\mathrm{QFT}} = \int \hat{\Psi}^+ \hat{h} \hat{\Psi} d\mathbf{x},$$

obtained from (B.8), and taking into account the orthogonality conditions (B.13) and the algebra of creation and annihilation operators defined above, we write the QFT Hamiltonian as

$$\begin{aligned}
\hat{H}^{\mathrm{QFT}} &= \hat{H}^{\mathrm{QFT}}_R + E_0, \\
\hat{H}^{\mathrm{QFT}}_R &= \sum_n \varepsilon_{+,n} a^+_n a_n - \sum_\alpha \varepsilon_{-,\alpha} b^+_\alpha b_\alpha = \sum_n \varepsilon_{+,n} a^+_n a_n + \sum_\alpha \varepsilon^c_{+,\alpha} b^+_\alpha b_\alpha,
\end{aligned}$$

where $E_0 = \sum_\alpha \varepsilon_{-,\alpha} = -\sum_\alpha \varepsilon^c_{+,\alpha}$ is an infinite constant, and $\hat{H}^{\mathrm{QFT}}_R$ is a renormalized Hamiltonian, namely, the latter is selected to be the energy operator (in what follows, the subscript R will be omitted).

Using the conserved charge (B.4), we define the QFT charge operator as

$$\hat{Q}^{\mathrm{QFT}} = q \int d^2x \hat{\Psi}^+ \hat{\Psi} = \hat{Q}^{\mathrm{QFT}}_R + Q_0, \quad \hat{Q}^{\mathrm{QFT}}_R = q \sum_n a^+_n a_n - q \sum_\alpha b^+_\alpha b_\alpha, \tag{B.17}$$

where $Q_0 = q \sum_\alpha$ is the vacuum charge. We define the vacuum state $|0\rangle$ to be the zero vector of the annihilation operators: $a_n |0\rangle = b_\alpha |0\rangle = 0$.

B.6 *One-particle Sector*

Let us now construct the Hilbert space of one-particle states as the disjoint reunion $\mathcal{R} = \mathcal{R}_{10} \cup \mathcal{R}_{01}$, $\mathcal{R}_{10} \cap \mathcal{R}_{01} = \{0\}$, of the particle subspace $\mathcal{R}_{10}$ and the antiparticle subspace $\mathcal{R}_{01}$,

$$|\Psi\rangle = \left(\sum_n f_n a_n^+ |0\rangle, \sum_\alpha h_\alpha b_\alpha^+ |0\rangle\right) \in \mathcal{R}, \ \sum_n f_n a_n^+ |0\rangle \in \mathcal{R}_{10}, \ \sum_\alpha h_\alpha b_\alpha^+ |0\rangle \in \mathcal{R}_{01},$$

assuming $\sum_n |f_n|^2 < \infty$ and $\sum_\alpha |h_\alpha|^2 < \infty$. Therefore, physical states $|\Psi\rangle$ belong either to the particle subspace $\mathcal{R}_{10}$ or to the antiparticle subspace $\mathcal{R}_{01}$, in agreement with the superselection rule [8]. In other words, physical states $|\Psi\rangle$ are eigenstates of the charge operator (B.17),

$$\hat{Q}^{\mathrm{QFT}} |\Psi\rangle = \zeta q |\Psi\rangle, \quad \zeta = \pm 1. \tag{B.18}$$

We note that the spectrum of $\hat{H}^{\mathrm{QFT}}$ in the one-particle sector reproduces exactly that of particles and antiparticles without negative energy levels. States $|\Psi\rangle$ evolve in time according to the Schrödinger equation

$$i\partial_0 |\Psi(x_0)\rangle = \hat{H}^{\mathrm{QFT}} |\Psi(x_0)\rangle,$$

and remain in the one-particle sector due the fact that the Hamiltonian $\hat{H}^{\mathrm{QFT}}$ commutes with the particle number operator

$$\hat{N} = \sum_n a_n^+ a_n + \sum_\alpha b_\alpha^+ b_\alpha .$$

Consider the decompositions

$$\hat{\Psi} = \hat{\Psi}_{(-)} + \hat{\Psi}_{(+)}, \quad \hat{\Psi}^c = \hat{\Psi}^c_{(-)} + \hat{\Psi}^c_{(+)},$$

where $\hat{\Psi}^c$ is the charge-conjugate Heisenberg operator of the field $\hat{\Psi}$, defined by $\hat{\Psi}^c_{(\pm)} = \left(\hat{\Psi}^+_{(\mp)}\gamma^2\right)^T$, while the plus and minus terms are given by

$$\hat{\Psi}_{(-)} = \sum_n a_n \psi_{+,n}, \quad \hat{\Psi}_{(+)} = \sum_\alpha b_\alpha^+ \psi_{-,\alpha},$$
$$\hat{\Psi}^c_{(-)} = \sum_\alpha b_\alpha \psi^c_{+,\alpha}, \quad \hat{\Psi}^c_{(+)} = \sum_n a_n^+ \psi^c_{-,n}.$$

Let us consider a coordinate representation of the Fock space of time-dependent one-particle states $|\Psi(x_0)\rangle$ given by the representatives

$$\Psi(x) = \langle 0| \hat{\Psi}_{(-)} |\Psi(x_0)\rangle, \quad \Psi^c(x) = \langle 0| \hat{\Psi}^c_{(-)} |\Psi(x_0)\rangle.$$

Since $|\Psi(x_0)\rangle$ belongs either to the particle subspace $\mathcal{R}_{10}$ or to the antiparticle subspace $\mathcal{R}_{01}$, we can define its four-component coordinate representation $\boldsymbol{\Psi}(x)$ in one of the two forms

$$\boldsymbol{\Psi}(x) = \begin{pmatrix} \Psi(x) \\ 0 \end{pmatrix}, \quad \boldsymbol{\Psi}(x) = \begin{pmatrix} 0 \\ \Psi^c(x) \end{pmatrix}.$$

By means of the projection operator to the one-particle sector,

$$\int \left(\hat{\Psi}^+ |0\rangle \langle 0| \hat{\Psi} + \hat{\Psi}^{c+} |0\rangle \langle 0| \hat{\Psi}^c\right) d\mathbf{x} = 1,$$

we are able to write the QFT inner product $\left(\boldsymbol{\Psi}, \boldsymbol{\Psi}'\right)$ in terms of representatives as

$$\left(\boldsymbol{\Psi}, \boldsymbol{\Psi}'\right) = \left(\Psi, \Psi'\right)_D + \left(\Psi^c, \Psi^{c\prime}\right)_D,$$

where, as before, the Dirac scalar product is

$$\left(\Psi, \Psi'\right)_D = \int \Psi^+(x)\, \Psi(x)\, d\mathbf{x}.$$

It is not difficult to see that the equations

$$\hat{H}^{\mathrm{QFT}}\left|\Psi_n\right\rangle = \varepsilon_{+,n}\left|\Psi_n\right\rangle, \quad \hat{H}^{\mathrm{QFT}}\left|\Psi_\alpha^c\right\rangle = \varepsilon_{+,\alpha}^c\left|\Psi_\alpha^c\right\rangle,$$

where $\left|\Psi_n\right\rangle = a_n^+\left|0\right\rangle$, $\left|\Psi_\alpha^c\right\rangle = b_\alpha^+\left|0\right\rangle$, give in the coordinate representation

$$\hat{H}\boldsymbol{\Psi}_{+,n} = \varepsilon_{+,n}\boldsymbol{\Psi}_{+,n}, \quad \hat{H}\boldsymbol{\Psi}_{+,\alpha}^c = \varepsilon_{+,\alpha}^c\boldsymbol{\Psi}_{+,\alpha}^c,$$

so that the Hamiltonian $\hat{H}$ may be identified with

$$\hat{H} = \mathrm{bdiag}\left(\hat{h}, \hat{h}^c\right), \tag{B.19}$$

and

$$\boldsymbol{\Psi}_{+,n} = \begin{pmatrix} \Psi_{+,n} \\ 0 \end{pmatrix}, \quad \boldsymbol{\Psi}_{+,\alpha}^c = \begin{pmatrix} 0 \\ \Psi_{+,\alpha}^c \end{pmatrix}. \tag{B.20}$$

It is clear that in the coordinate representation the charge operator $\hat{Q}$ acts as

$$q\hat{\zeta}\boldsymbol{\Psi}(x) = q\boldsymbol{\Psi}(x), \quad q\hat{\zeta}\boldsymbol{\Psi}^c(x) = -q\boldsymbol{\Psi}^c(x), \tag{B.21}$$

where $\hat{\zeta} = \mathrm{bdiag}\,(I, -I)$, in accordance with the superselection rule (B.18).

Thus, states (B.20) form a basis in the coordinate representation of $\mathcal{R}$. They are eigenstates of the QFT charge operator and possess the inner product

$$\left(\boldsymbol{\Psi}_{+,n}, \boldsymbol{\Psi}_{+,m}\right) = \left(\Psi_{+,n}, \Psi_{+,m}\right)_D, \quad \left(\boldsymbol{\Psi}_{+,\alpha}^c, \boldsymbol{\Psi}_{+,\beta}^c\right) = \left(\psi_{+,\alpha}^c, \psi_{+,\beta}^c\right)_D, \quad \left(\boldsymbol{\Psi}_{+,n}, \boldsymbol{\Psi}_{+,\alpha}^c\right) = 0. \tag{B.22}$$

References

[1] L. Brink, S. Deser, B. Zumino, P. di Vecchia and P. Howe, Phys. Lett. **B64** (1976) 435; L. Brink, P. di Vecchia and P. Howe, Nucl. Phys. **B118** (1977) 76; R. Casalbuoni, Nuovo Cimento **A33** (1976) 115; 389; A. Barducci, R. Casalbuoni and L. Lusanna, Nuovo Cimento **A35** (1976) 377; A.P. Balachandran, P. Salomonson, B. Skagerstam and J. Winnberg, Phys. Rev. **D15** (1977) 2308; M. Henneaux, C. Teitelboim, Ann. Phys. **143** (1982) 127; D.M. Gitman and I.V. Tyutin, Pis'ma Zh. Eksp. Teor. Fiz. **51**, 3 (1990) 188; Class. Quantum Grav. **7** (1990) 2131.

[2] D.M. Gitman, A.E. Gonçalves and I.V. Tyutin, Phys. Atom. Nuc. **60** (1997) 665.

[3] S.P. Gavrilov and D.M. Gitman, Int. J. Mod. Phys. **A15** (2000) 4499.

[4] D.M. Gitman and I.V. Tyutin, Nucl. Phys. **B630** (2002) 509.

[5] D.M. Gitman and I.V. Tyutin, Int. Journ. Mod. Phys. **A12** (1997) 535.

[6] D.M. Gitman and I.V. Tyutin, *Quantization of Fields with Constraints* (Springer-Verlag, Berlin–Heidelberg, 1990).

[7] M.G. Alford, J. March-Russel and F. Wilczek, Nucl. Phys., **B328** (1989) 140.

[8] S. Weinberg, *The Quantum Theory of Fields*, vol. 1 (Cambridge University Press, New York, 1995).

UNIVERSIDADE DE SÃO PAULO
Instituto de Física
Cidade Universitária
Caixa Postal 66.318
05315-970 - São Paulo - Brasil