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**Quantization of Spinning Particle and Bifermionic
Constraint Problem**

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Abstract

This work is a natural continuation of our recent study in quantizing relativistic particles. There it was demonstrated that, by applying a consistent quantization scheme to a classical model of a spinless relativistic particle as well as to the Berezin-Marinov model of $3 + 1$ Dirac particle, it is possible to obtain a consistent relativistic quantum mechanics of such particles. In the present article we apply a similar approach to the problem of quantizing the massive $2 + 1$ Dirac particle. However, we stress that such a problem differs in a nontrivial way from the one in $3 + 1$ dimensions. The point is that in $2 + 1$ dimensions each spin polarization describes different fermion species. Technically this fact manifests itself through the presence of a bifermionic constant and of a bifermionic first-class constraint. In particular, this constraint does not admit a conjugate gauge condition at the classical level. The quantization problem in $2 + 1$ dimensions is also interesting from the physical viewpoint (e.g. anyons). In order to quantize the model, we first derive a classical formulation in an effective phase space, restricted by constraints and gauges. Then the condition of preservation of the classical symmetries allows us to realize the operator algebra in an unambiguous way and construct an appropriate Hilbert space. The physical sector of the constructed quantum mechanics contains spin- $1/2$ particles and antiparticles without an infinite number of negative-energy levels, and exactly reproduces the one-particle sector of the $2 + 1$ quantum theory of a spinor field.

1 Introduction

This work is a natural continuation of our recent study [1, 2, 3] which was devoted to the consistent quantization of classical and pseudoclassical models of relativistic particles. We recall that in the first article [1] it is demonstrated that, by applying the consistent quantization scheme (canonical quantization, combined with the analysis of constraints and symmetries, as well as of the physical sector) to the classical model of a spinless relativistic particle, it is possible to obtain a consistent relativistic quantum mechanics (QM) of such a particle. Remarkably, the problem of the infinite number of energy levels and of the indefinite metric is solved in the same manner as in the corresponding quantum field theory (QFT), i.e., by properly defining the physical sector of the Hilbert space. In external backgrounds that do not violate vacuum stability, the constructed QM turns out to be completely equivalent to the one-particle sector of the QFT. We stress that the Schrödinger equation of the QM is equivalent to a pair of relativistic wave equations (in this particular case to the pair of Klein-Gordon equations), namely, to an equation for a particle with charge q and to an equation for an antiparticle with charge $-q$.

As a logical extension of the approach [1], we consider the quantization of pseudoclassical models of spinning particles. In this relation one ought to recall that there does not exist a unique framework for the description of relativistic spinning particles which embraces all possible cases: particles with integer and half-integer spin, massive particles and massless particles, particles in even and in odd dimensions. In all these cases, the action's structure differs in an essential manner, and therefore each case requires a completely different treatment in the course of its quantization. In our works [2, 3], we started the quantization program of spinning particles considering the pseudoclassical Berezin-Marinov [4] action of the massive $3 + 1$ Dirac particle, the structure of which is typical for all massive relativistic particles with half-integer spin in even dimensions. The constraint structure of this model in the Hamiltonian formulation allows one to fix completely the gauge freedom at the classical level. In spite of the considerable technical difficulties involving the realization of the commutation relations and the Hamiltonian construction, one can carry out the canonical quantization scheme leading to the consistent (as in the spinless case) QM of the $3 + 1$ Dirac particle.

In the present article we consider the problem of quantizing the massive $2 + 1$ Dirac particle using the pseudoclassical model first proposed by Gitman, Gonçalves, and Tyutin (GGT) in [5]. From this particular model, one can devise the general quantization scheme for half-integer spinning particles in odd dimensions [6], since the action for the general case has the same essential structure as that of the GGT model. More remarks are in order regarding the choice of the model for the massive $2 + 1$ Dirac particle. Namely, we note that a number of alternative models have been proposed for the description of a spinning particle in $2 + 1$ dimensions (see, e.g., [5, 7]). We also note that in $2 + 1$ dimensions, a direct dimensional reduction of the Berezin-Marinov action does not reproduce the minimal quantum theory of spinning particles, which must provide only one spin projection value ($1/2$ or $-1/2$). In the papers [7, 8], two modifications were proposed. One of them is not minimal and is P - and T -invariant, so that an anomaly is present. The other does not possess the desirable gauge supersymmetries. The GGT action is gauge supersymmetric and reparametrization invariant. Furthermore, it is P - and T -noninvariant, in accordance with the expected properties of the minimal theory in $2 + 1$ dimensions.

We stress that the quantization of the GGT model differs in a nontrivial way from the one presented for

the Berezin-Marinov model. The point is that in $2 + 1$ dimensions (as well as in any odd-dimensional case) to each spin polarization corresponds a different particle, because distinct spin projections of a $2 + 1$ spinning particle belong to distinct irreducible representations, and thus describe different fermion species. Technically this fact manifests itself in the model through the presence of a bifermionic constant and of a bifermionic first-class constraint. This constraint does not admit a conjugate gauge condition at the classical level. However, since the corresponding operator has a compact spectrum, it can be consistently used to fix the remaining gauge freedom at the quantum level according to Dirac. Such problems do not appear in the Berezin-Marinov case. Our interest in the GGT model does not reside entirely upon these mentioned departures from the Berezin-Marinov model. The quantization problem of a particle in $2 + 1$ dimensions is a very interesting one from the physical viewpoint. There is a direct relation to field theory in $2 + 1$ dimensions [9, 10], which has recently attracted much attention, due to non-trivial topological properties, and especially due to the possible existence of particles with fractional spin and exotic statistics (anyons). There is also a strong relation of the $2 + 1$ quantum theory to the fractional Hall effect, high- T_c superconductivity, etc. [9]. Thus, we hope to have motivated the construction of a consistent relativistic QM of a spinning particle in $2 + 1$ dimensions.

The paper is organized as follows. In Section 2, we study the classical properties of the given pseudoclassical model and present its detailed Hamiltonian analysis. We focus on the selection of the physical degrees of freedom and on an adequate gauge-fixing. We obtain a Hamiltonian formulation of the model in an effective phase space, restricted by constraints and gauges. We gauge-fix two of the initial gauge symmetries, and retain an effective bifermionic first-class constraint, which does not admit gauge-fixing. In Section 3, we apply a quantization approach, being a combination of the canonical and the Dirac schemes, in which the bifermionic first-class constraint is imposed at the quantum level to select admissible state-vectors. We present a detailed construction of the Hilbert space. Then we reformulate the time-evolution in terms of the physical time, and verify that the constructed theory has the necessary symmetry properties. We select a physical sector which describes the consistent relativistic QM of particles in $2 + 1$ dimensions without an infinite number of negative-energy levels. In Section 4, we make a comparison of the constructed QM with the one-particle sector of the $2 + 1$ QFT. In Section 6, we summarize all the results obtained in our paper. In Appendix A, we justify the selected Hamiltonian realization, considering the semiclassical limit of the QM constructed. In Appendix B, we present some necessary constructions of the corresponding $2 + 1$ QFT, and we consider the problem of defining spin operators in $2 + 1$ dimensions. We actively use the results of the Appendix B throughout the article.

2 Pseudoclassical model and its constraint structure

2.1 Lagrangian and Hamiltonian formulations

In order to describe classically (that is, pseudoclassically) massive relativistic spin-1/2 charged particles in $2 + 1$ dimensions, we take the action first proposed in [5]. It has the form

$$S = \int_0^1 L d\tau, \quad L = -\frac{z^2}{2e} - e\frac{m^2}{2} - q\dot{x}^\mu A_\mu + ieqF_{\mu\nu}\xi^\mu\xi^\nu - im\xi^3\chi - \frac{1}{2}\theta m\kappa - i\xi_n\dot{\xi}^n, \\ z^\mu = \dot{x}^\mu - i\xi^\mu\chi + ie^{\mu\nu\lambda}\xi_\nu\xi_\lambda\kappa. \quad (1)$$

Here e , κ , and x^μ , $\mu = 0, 1, 2$, are even variables, while χ and ξ^n , $n = (\mu, 3)$, are odd variables; the Minkowski metric in $2 + 1$ dimensions reads $\eta_{\mu\nu} = \text{diag}(1, -1, -1)$, and in $3 + 1$ dimensions is $\eta_{mn} = \text{diag}(1, -1, -1, -1)$; θ is an even constant; $\varepsilon^{\lambda\mu\nu}$ is the Levi-Civita tensor in $2 + 1$ dimensions normalized as $\varepsilon^{012} = 1$, and summation over repeated indices is assumed. The particle interacts with an arbitrary external gauge field $A_\mu(x)$, which can be of Maxwell and/or Chern-Simons nature, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the strength tensor of this field, and q is the $U(1)$ -charge of the spinning particle. We assume that the coordinates x^μ and ξ^μ are $2 + 1$ Lorentz vectors; e , κ , ξ^3 , and χ are Lorentz scalars. All the variables depend on the parameter $\tau \in [0, 1]$, which plays here the role of time. Dots above the variables denote their derivatives with respect to τ . The action (1) is invariant under the restricted Lorentz transformations, but is P - and T -noninvariant, in accordance with the expected properties of the minimal theory in $2 + 1$ dimensions.

We recall that the action is invariant under the reparametrizations

$$\delta x^\mu = \dot{x}^\mu \varepsilon, \quad \delta e = \frac{d}{d\tau}(e\varepsilon), \quad \delta \xi^n = \dot{\xi}^n \varepsilon, \quad \delta \chi = \frac{d}{d\tau}(\chi\varepsilon), \quad \delta \kappa = \frac{d}{d\tau}(\kappa\varepsilon),$$

where $\varepsilon(\tau)$ is an even gauge parameter, and under two types of gauge supertransformations

$$\begin{aligned}\delta x^\mu &= i\xi^\mu \varepsilon, \quad \delta e = i\chi \varepsilon, \quad \delta \xi^\mu = \frac{z^\mu}{2e} \varepsilon, \quad \delta \xi^3 = \frac{m}{2} \varepsilon, \quad \delta \chi = \dot{\varepsilon}, \quad \delta \kappa = 0, \\ \delta x^\mu &= -i\varepsilon^{\mu\nu\lambda} \xi_\nu \xi_\lambda \vartheta, \quad \delta \xi^\mu = \frac{1}{e} \varepsilon^{\mu\nu\lambda} z_\nu \xi_\lambda \vartheta, \quad \delta \kappa = \dot{\vartheta}, \quad \delta e = \delta \xi^3 = \delta \chi = 0,\end{aligned}$$

where $\varepsilon(\tau)$ is an odd gauge parameter and $\vartheta(\tau)$ is an even gauge parameter.

We note that e , χ , and κ are degenerate coordinates, since their time derivatives are not present in the action. In what follows, we consider a reduced hamiltonization scheme for theories with degenerate coordinates [11], in which momenta conjugate to the degenerate coordinates are not introduced. To proceed with the hamiltonization, we introduce the velocities v^μ and α^μ and write the action (1) in the first-order formalism as

$$S^v = \int_0^1 \left[L^v + p_\mu (\dot{x}^\mu - v^\mu) + \pi_n (\dot{\xi}^n - \alpha^n) \right] d\tau = \int_0^1 \left(p_\mu \dot{x}^\mu + \pi_n \dot{\xi}^n - H^v \right) d\tau,$$

where

$$\begin{aligned}L^v &= -\frac{\bar{z}^2}{2e} - e \frac{m^2}{2} - qv^\mu A_\mu + iqe F_{\mu\nu} \xi^\mu \xi^\nu - im\xi^3 \chi - \frac{1}{2} \theta m \kappa - i\xi_n \alpha^n, \\ \bar{z}^\mu &= v^\mu - i\xi^\mu \chi + i\varepsilon^{\mu\nu\lambda} \xi_\nu \xi_\lambda \kappa, \quad H^v = p_\mu v^\mu + \pi_n \alpha^n - L^v.\end{aligned}$$

The variables p_μ and π_n should be treated as conjugate momenta to the coordinates x^μ and ξ^n , respectively. The ordering of variables in the Hamiltonian H^v complies with the usual convention for the choice of derivatives with respect to coordinates as right-hand ones and those with respect to momenta as left-hand ones.

The equations of motion with respect to the velocities and the degenerate coordinates read

$$\begin{aligned}\frac{\delta S^v}{\delta v^\mu} &= -p_\mu - \frac{1}{e} \bar{z}_\mu - qA_\mu = 0, \quad \frac{\delta S^v}{\delta e} = \frac{\bar{z}^2}{2e^2} - \frac{m^2}{2} + iqF_{\mu\nu} \xi^\mu \xi^\nu = 0, \\ \frac{\delta S^v}{\delta \kappa} &= -\frac{i}{e} \bar{z}^\mu \varepsilon_{\mu\nu\lambda} \xi^\nu \xi^\lambda - \frac{1}{2} \theta m = 0, \quad \frac{\delta S^v}{\delta \alpha^a} = \pi_n + i\xi_n = 0, \\ \frac{\delta S^v}{\delta \chi} &= \frac{1}{e} (v^\mu \xi_\mu + i\varepsilon^{\mu\nu\lambda} \xi_\mu \xi_\nu \xi_\lambda \kappa) - im\xi^3 = 0.\end{aligned}$$

The equations $\delta S^v / \delta \alpha^n = 0$ lead to the primary constraints

$$\varphi_n = \pi_n + i\xi_n, \quad (2)$$

and the equations $\delta S^v / \delta v^\mu = 0$ can be used to express the velocities v^μ , viz.,

$$v^\mu = -e(p^\mu + qA^\mu) + i\xi^\mu \chi - i\varepsilon^{\mu\nu\lambda} \xi_\nu \xi_\lambda \kappa. \quad (3)$$

Substituting this relation into the other equations, we obtain more primary constraints

$$\begin{aligned}\phi_1 &= (p + qA)^\mu \xi_\mu + m\xi^3, \\ \phi_2 &= (p + qA)^2 - m^2 + 2iqF_{\mu\nu} \xi^\mu \xi^\nu, \\ \phi_3 &= \varepsilon_{\mu\nu\lambda} (p + qA)^\mu \xi^\nu \xi^\lambda + \frac{i}{2} \theta m.\end{aligned} \quad (4)$$

Upon substituting (3) into the Hamiltonian H^v , we get the total Hamiltonian

$$H^{(1)} = \Lambda_1 \phi_1 + \Lambda_2 \phi_2 + \Lambda_3 \phi_3 + \lambda^n \varphi_n, \quad (5)$$

where $\Lambda_1 = -i\chi$, $\Lambda_2 = -e/2$, $\Lambda_3 = -i\kappa$, and $\lambda^n = -\alpha^n$ are henceforth Lagrange multipliers. One can see that the total Hamiltonian is proportional to the constraints. The resulting dynamically equivalent action is

$$S^{(1)} = \int_0^1 \left(p_\mu \dot{x}^\mu + \pi_n \dot{\xi}^n - H^{(1)} \right) d\tau.$$

In the following, we shall make use of the Dirac brackets with respect to a set φ of second-class constraints,

$$\{A, B\}_{D(\varphi)} = \{A, B\} - \{A, \varphi_a\} C^{ab} \{\varphi_b, B\}.$$

Here $C^{ab} \{\varphi_b, \varphi_c\} = \delta_c^a$ and the Poisson brackets of the functions F and G of definite Grassmann parities $\varepsilon(F)$ and $\varepsilon(G)$ are given by

$$\{F, G\} = \frac{\partial F}{\partial x^\mu} \frac{\partial G}{\partial p_\mu} + \frac{\partial F}{\partial \xi^n} \frac{\partial G}{\partial \pi_n} - (-1)^{\varepsilon(F)\varepsilon(G)} (F \leftrightarrow G). \quad (6)$$

One ought to say that in the classical theory the quantity θ is a bifermionic constant. In the quantum theory, though, it turns out to be a real number (see Sect. 5 Discussion).

2.2 Constraint reorganization and gauge-fixing

In analogy to [2, 3], we first reorganize the constraints (2) and (4) into an equivalent set (\mathbf{T}, φ) such that $\mathbf{T} = (T_1, T_2, T')$ is a set of first-class constraints, and φ is a set of second-class constraints,

$$\{\mathbf{T}, \mathbf{T}\}|_{\mathbf{T}, \varphi=0} = \{\mathbf{T}, \varphi\}|_{\mathbf{T}, \varphi=0} = 0.$$

The new constraints \mathbf{T} have the form

$$\begin{aligned} T_1 &= (p + qA)_\mu (\pi^\mu - i\xi^\mu) + m (\pi^3 - i\xi^3), \\ T_2 &= p_0 + qA_0 + \zeta r, \quad \zeta = -\text{sgn}[p_0 + qA_0], \\ T' &= \varepsilon_{\mu\nu\lambda} (p + qA)^\mu \xi^\nu \pi^\lambda + \frac{1}{2} \theta m, \end{aligned} \quad (7)$$

where $\zeta = \pm 1$, and $r = \sqrt{m^2 + (p_k + qA_k)^2 + 2qF_{\mu\nu}\xi^\mu\pi^\nu}$ is the principal value of the square root¹. One can see that T_2 and ϕ_2 are related as

$$\phi_2 = -2\zeta r T_2 - \frac{i}{2} \varphi_n \eta^{n\bar{n}} \{\varphi_{\bar{n}}, \phi_2\} + (T_2)^2.$$

In terms of the \mathbf{T} -constraints, the Hamiltonian (5) becomes

$$H^{(1)} = \Lambda_1 T_1 + \Lambda_2 T_2 + \Lambda' T', \quad (8)$$

with redefined Lagrange multipliers.

Our goal is to quantize this theory, so supplementary gauge conditions will be imposed upon the first-class constraints T_1 and T_2 , except the constraint T' , which is of bifermionic nature. The problem related to its gauge fixing is still open (see discussion in [6, 12, 13, 14]). In the end, there will remain a first-class constraint T' reduced on the constraint surface, while the second-class set of all the other constraints and gauge conditions will be used to construct Dirac brackets. The surviving first-class constraint will be enforced on state vectors of the quantized theory in order to fix the remaining gauge freedom according to Dirac.

We impose the following gauge-fixing conditions

$$\begin{aligned} \phi_1^G &= \pi^0 - i\xi^0 + \zeta (\pi^3 - i\xi^3), \\ \phi_2^G &= x_0 - \zeta \tau, \quad \zeta = \pm 1. \end{aligned} \quad (9)$$

The gauge (9), chosen to fix the gauge freedom related to T_1 , reduces the set of independent spin variables. The gauge (10), chosen to fix the gauge freedom related to T_2 , is the chronological gauge [15, 1, 2]. The resulting set of constraints $(T_1, T_2, \phi_1^G, \phi_2^G, \varphi)$ is second-class.

In order to simplify the Poisson brackets between constraints, and to write the dynamics in terms of independent variables, we pass to a set (Φ, \tilde{T}) which is equivalent to the set of constraints $(\mathbf{T}, \phi_1^G, \phi_2^G, \varphi)$. Here Φ are the second-class constraints

$$\begin{aligned} \Phi_1 &= p_0 + qA_0 + \zeta \tilde{\omega}, \quad \Phi_2 = \phi_2^G, \quad \Phi_3 = \varphi_1, \quad \Phi_4 = \varphi_2, \\ \Phi_5 &= -\frac{i}{2} T_1 + b T_2 + c \phi_2^G, \quad \Phi_6 = \phi_1^G, \quad \Phi_7 = \varphi_0, \quad \Phi_8 = \varphi_3, \end{aligned} \quad (11)$$

where

$$\begin{aligned} \tilde{\omega} &= \sqrt{\tilde{\omega}_0^2 + \frac{2\zeta q F_{k0}}{\tilde{\omega}_0 + m} (p + qA)_l (\xi^k \pi^l + \pi^k \xi^l)}, \\ \tilde{\omega}_0 &= \sqrt{m^2 + (p_k + qA_k)^2 + 2qF_{ik}\xi^i\pi^k}, \\ b &= \frac{i}{2} \frac{\{\phi_2^G, T_1\}}{\{\phi_2^G, T_2\}}, \quad c = -\frac{\{-\frac{i}{2} T_1 + b T_2, \Phi_1\}}{\{\phi_2^G, \Phi_1\}}, \end{aligned}$$

and \tilde{T} reads

$$\tilde{T} = T - \frac{\{T, \Phi_5\}}{\{\Phi_6, \Phi_5\}} \Phi_6, \quad T = T'|_{\Phi=0}.$$

¹We define the principal value of the square root of an expression containing Grassmann variables as the one which is positive when the generating elements of the Grassmann algebra are set to zero.

The constraint \tilde{T} is first-class, so it is orthogonal to all the second-class constraints, $\left\{ \tilde{T}, \Phi \right\} \Big|_{\Phi=0} = 0$.

However, for the further consideration, it is convenient to use the constraint T , which coincides with \tilde{T} and T' on the constraint surface,

$$T = -m\zeta (\xi^1 \pi^2 - \xi^2 \pi^1) + \frac{1}{2} \theta m. \quad (12)$$

The constraint T , which is responsible for parity violation with respect to reflection of one of the coordinate axes, does not depend on the space-time coordinates and momenta. The corresponding operator can be realized as a finite matrix. Therefore its spectrum is compact, and we thus do not expect standard difficulties with the Dirac quantization in such a case.

The nonzero Poisson brackets (taken on the constraint surface $\Phi = 0$) between second-class constraints are

$$\begin{aligned} \{\Phi_2, \Phi_1\} &= -\{\Phi_1, \Phi_2\} = 1, \quad \{\Phi_3, \Phi_3\} = \{\Phi_4, \Phi_4\} = -2i, \\ \{\Phi_5, \Phi_6\} &= \{\Phi_6, \Phi_5\} = \zeta (\tilde{\omega}_0 + m), \quad \{\Phi_7, \Phi_7\} = -\{\Phi_8, \Phi_8\} = 2i. \end{aligned}$$

In terms of the new constraints, the Hamiltonian (8) becomes

$$H^{(1)} = \Lambda_a \Phi_a + \Lambda \tilde{T}, \quad a = 1, 2, 5, 6.$$

with redefined Lagrange multipliers.

2.3 Effective dynamics in the reduced space

Evidently, not all variables are independent, due to the presence of constraints. In fact, it is possible to reduce the number of the variables using some of the second-class constraints. In doing so, we retain the following set of variables

$$\eta = (x^k, p_k, \xi^k, \pi_k, \zeta), \quad k = 1, 2. \quad (13)$$

We hereafter refer to these variables as the basic variables. All the initial phase-space variables can be expressed in terms of these basic variables. We remark that the basic variables are not independent, since there still are constraints among them.

Therefore, we seek to write the dynamics in the reduced space determined by the evolution of the basic variables (13) alone. Despite the fact that the corresponding constraints are time-dependent, it is still possible to write the evolution equation for the basic variables by means of Dirac brackets if we introduce a momentum ϵ canonically conjugate to the time-evolution parameter τ as was done in [16]. This equation reads

$$\dot{\eta} = \left\{ \eta, \Lambda \tilde{T} + \epsilon \right\}_{D(\Phi)}, \quad \Phi = 0, \quad \tilde{T} = 0, \quad (14)$$

where the Dirac brackets $\{, \}_{D(\Phi)}$ are constructed with respect to the constraints (11).

The new set of constraints (Φ, \tilde{T}) allows a number of simplifications in equation (14). In this connection, we divide the set Φ into second-class subsets U and V given by

$$U = \{\Phi_1, \Phi_2, \Phi_3, \Phi_4\}, \quad V = \{\Phi_5, \Phi_6, \Phi_7, \Phi_8\},$$

so that it is possible to apply the rule

$$\{A, B\}_{D(\Phi)} = \{A, B\}_{D(U)} - \{A, V_a\}_{D(U)} C^{ab} \{V_b, B\}_{D(U)}, \quad C^{ac} \{V_c, V_b\}_{D(U)} = \delta_b^a, \quad (15)$$

for any dynamical variables A and B . As a result, thanks to the vanishing of $\{V_b, \epsilon\}_{D(U)}$ on the constraint surface, equation (14) is simplified to

$$\dot{\eta} = \{ \eta, \Lambda T + \epsilon \}_{D(U)}, \quad U = 0, \quad T = 0. \quad (16)$$

In the same spirit, we further divide the subset U into two sets $u = (\Phi_3, \Phi_4)$ and $v = (\Phi_1, \Phi_2)$. This time, application of the rule (15) gives rise to a new simplification due to the vanishing of $\{ \eta, \epsilon \}_{D(u)}$. Thus,

$$\{ \eta, \epsilon \}_{D(U)} = -\{ \eta, v_a \}_{D(u)} c^{ab} \{ v_b, \epsilon \}_{D(u)} = \zeta \{ \eta, \Phi_1 \}_{D(u)}. \quad (17)$$

Since neither the basic variables nor the constraints u involve the coordinate x^0 , and p_0 is alone in the constraint Φ_1 as an additive factor, we can eliminate p_0 altogether from (17). Consequently, we are now able to take

$\Phi_2 \equiv 0$ identically, and thus substitute $x^0 = \zeta\tau$ into the same brackets. Finally, we express π_k in terms of ξ^k in $\tilde{\omega}$, using the constraints u , so that the equations of motion in the reduced phase space (13) become

$$\dot{\eta} = \{\eta, \mathcal{H}_{\text{eff}} + \Lambda T\}_{D(u)}, \quad u_k = \pi_k + i\xi_k = 0, \quad T = 0, \quad (18)$$

where \mathcal{H}_{eff} is an effective Hamiltonian given by

$$\begin{aligned} \mathcal{H}_{\text{eff}} &= [\zeta q A_0 + \omega]_{x^0 = \zeta\tau}, \quad \omega = \tilde{\omega}|_{\pi_k = -i\xi^k} = \sqrt{\omega_0^2 + \rho}, \\ \omega_0 &= \sqrt{m^2 + (p_k + qA_k)^2 - 2iqF_{kl}\xi^k\xi^l}, \quad \rho = \frac{-4i\zeta q F_{k0}}{\omega_0 + m} (p_l + qA_l)\xi^k\xi^l, \end{aligned} \quad (19)$$

and T (12) is an effective first-class constraint.

The nonzero Dirac brackets between the basic variables η are given by

$$\{x^k, p_l\}_{D(u)} = \delta_l^k, \quad \{\xi_k, \xi_l\}_{D(u)} = -\{\pi_k, \pi_l\}_{D(u)} = i\{\xi_k, \pi_l\}_{D(u)} = \frac{i}{2}\eta^{kl}. \quad (20)$$

3 Quantization

3.1 Operators of basic variables

The equal-time commutation relations for the operators \hat{X}^k , \hat{P}_k , $\hat{\Xi}^k$, and $\hat{\zeta}$, corresponding to the basic variables x^k , p_k , ξ^k , and ζ , are defined according to their Dirac brackets (20). The nonzero commutators $[\cdot, \cdot]$ (anticommutators $[\cdot, \cdot]_+$) are

$$[\hat{X}^k, \hat{P}_l] = i\hbar\delta_l^k, \quad [\hat{\Xi}^k, \hat{\Xi}^l]_+ = -\frac{\hbar}{2}\eta^{kl}. \quad (21)$$

We assume $\zeta^2 = 1$ and select a preliminary state space \mathcal{R} of \mathbf{x} -dependent 16-component columns $\underline{\Psi}(\mathbf{x})$,

$$\underline{\Psi}(\mathbf{x}) = \begin{pmatrix} \Psi_{+1}(\mathbf{x}) \\ \Psi_{-1}(\mathbf{x}) \end{pmatrix}, \quad (22)$$

where $\Psi_\zeta(\mathbf{x})$, $\zeta = \pm 1$, are 8-component columns. The inner product in \mathcal{R} is defined as follows²,

$$(\underline{\Psi}|\underline{\Psi}') = (\Psi_{+1}, \Psi'_{+1}) + (\Psi_{-1}, \Psi'_{-1}), \quad (\Psi, \Psi') = \int \Psi^\dagger(\mathbf{x})\Psi'(\mathbf{x})d\mathbf{x}. \quad (23)$$

Later on, we shall see this construction of the inner product is Lorentz-invariant.

We realize all the operators in the following block-diagonal form³,

$$\begin{aligned} \hat{X}^k &= x^k I_{16}, \quad \hat{P}_k = \hat{p}_k I_{16}, \quad \hat{p}_k = -i\hbar\partial_k, \\ \hat{\Xi}^k &= \text{bdiag}(\hat{\xi}^k, \hat{\xi}^k), \quad \hat{\zeta} = \text{bdiag}(I_8, -I_8). \end{aligned} \quad (24)$$

Here, I_{16} and I_8 are the 16×16 and 8×8 unit matrices, respectively, whereas $\hat{\xi}^k$ are 8×8 matrices which obey the equal-time commutation relations

$$[\hat{\xi}^k, \hat{\xi}^l]_+ = -\frac{\hbar}{2}\eta^{kl}.$$

²In what follows, we define the bilinear form (ψ, φ) as

$$(\psi, \varphi) = \int \psi^\dagger \varphi d\mathbf{x}$$

for vectors of any finite number of components.

³Here and in what follows we use the notation $\text{bdiag}(A, B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, where A and B are matrices.

3.2 Hamiltonian, first-class constraint and spin variables

Let us construct the operator which is a quantum version of the classical function \mathcal{H}_{eff} (19). We select it as follows

$$\mathcal{H}_{\text{eff}} \rightarrow \hat{H} = q\hat{\zeta}\hat{A}_0 + \hat{\Omega} = \text{bdiag}(\hat{H}_{+1}, \hat{H}_{-1}), \quad (25)$$

where $\hat{A}_0 = \text{bdiag}(A_0|_{x^0=\tau} I_8, A_0|_{x^0=-\tau} I_8)$, $\hat{\Omega} = \text{bdiag}(\hat{\Omega}_{+1}, \hat{\Omega}_{-1})$, $\hat{\Omega}_\zeta = \hat{\omega}_0|_{x^0=\zeta\tau}$, and

$$\hat{H}_\zeta = q\zeta A_0|_{x^0=\zeta\tau} I_8 + \hat{\Omega}_\zeta = (\zeta q A_0 I_8 + \hat{\omega}_0)|_{x^0=\zeta\tau}. \quad (26)$$

We realize the operator \hat{T} corresponding to the first-class constraint T (12) by the prescription $\hat{T} = T|_{\zeta=\hat{\zeta}, \xi=\hat{\xi}}$ and the classical bifermionic constant θ as a constant matrix $\hat{\Theta} = \text{bdiag}(\hat{\theta}, \hat{\theta})$, where $\hat{\theta}$ are 8×8 matrices. Thus, \hat{T} has a block-diagonal structure,

$$\hat{T} = m \left(2i\hat{\zeta}\hat{\xi}^1\hat{\xi}^2 + \frac{1}{2}\hat{\Theta} \right) = \text{bdiag}(\hat{t}_1, \hat{t}_{-1}), \quad \hat{t}_\zeta = m \left(2i\zeta\hat{\xi}^1\hat{\xi}^2 + \frac{1}{2}\hat{\theta} \right). \quad (27)$$

We are going to use this operator to fix the gauge at the quantum level according to Dirac, $\hat{\Psi} \in \mathcal{R}$, $\hat{T}\hat{\Psi} = 0$. One can see this condition implies

$$\hat{t}_\zeta \Psi_\zeta = 0. \quad (28)$$

The conservation of (28) in time is guaranteed by $[\hat{T}, \hat{H}] = 0$, which follows from the corresponding relation of the classical theory, $\{T, \mathcal{H}_{\text{eff}}\}_{D(u)} = 0$. It is equivalent to

$$[\hat{t}_\zeta, \hat{\Omega}_\zeta] = 0. \quad (29)$$

Now we postulate a manifest form for the operator $\hat{\Omega}_\zeta$, subject to the relation (29). This form ensures the hermiticity of the operator \hat{H} with respect to the inner product (23), provides the gauge invariance under $U(1)$ transformations, and Lorentz invariance of the inner product (23). Here, we note that $\hat{\omega}_0$ cannot be realized in analogy with the 3+1 dimensional case [2, 3], that is, in the form of the familiar one-particle Dirac Hamiltonian, and with $\hat{\xi}^k$ proportional to gamma-matrices. This is because \hat{t}_ζ contains the term $\hat{\xi}^1\hat{\xi}^2$, which would not commute with $\hat{\omega}_0$ in such a realization. In order to fulfill the condition (29), and simultaneously ensure that $\hat{\omega}_0^2$ corresponds to the classical ω_0^2 , we select

$$\hat{\omega}_0 = \begin{pmatrix} 0 & m - \gamma^k(\hat{p}_k + qA_k) \\ m + \gamma^k(\hat{p}_k + qA_k) & 0 \end{pmatrix}, \quad (30)$$

where γ^k , $k = 1, 2$, are any 4×4 matrices that obey the relation $[\gamma^k, \gamma^l]_+ = -2\delta_{kl}$. In fact, we can consider them as two matrices of a specific 4×4 realization of gamma-matrices in 2+1 dimensions, $[\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}$. Moreover, we can consider these matrices as a part of the complete set of gamma-matrices in 3+1 dimensions, for which it is convenient to select the following representation [17]

$$\gamma^0 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & i\sigma^1 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}, \quad (31)$$

where σ^k are the Pauli matrices, and I_2 is the 2×2 unit matrix.

Then, the expression $\hat{\Omega}^2 = \text{bdiag}(\hat{\omega}_0^2, \hat{\omega}_0^2)|_{x^0=\zeta\tau}$, where

$$\hat{\omega}_0^2 = \begin{pmatrix} m^2 + (\hat{p}_k + qA_k)^2 + \frac{i\hbar}{2} q F_{kl} \gamma^k \gamma^l & 0 \\ 0 & m^2 + (\hat{p}_k + qA_k)^2 + \frac{i\hbar}{2} q F_{kl} \gamma^k \gamma^l \end{pmatrix}, \quad (32)$$

is consistent with the semiclassical limit (see Appendix A).

The realization of the operators $\hat{\xi}^k$, $k = 1, 2$, and of the matrix $\hat{\theta}$, corresponding to the classical quantity θ , is constrained by the relation (29). The latter relation is a constraint on \hat{t}_ζ , and implies that $\hat{\xi}^1\hat{\xi}^2$ and $\hat{\theta}$ must commute with $\hat{\omega}_0$. Additionally, we require the condition $[\hat{\xi}^k, \hat{\theta}] = 0$ in accordance with the classical theory.

The above restraints are not sufficient to single out a representation, so we impose further restrictions to the form of $\hat{\xi}^1 \hat{\xi}^2$ and $\hat{\theta}$. The matrix of $\hat{\xi}^1 \hat{\xi}^2$ is chosen to be composed of blocks which are products of two 4×4 gamma-matrices, and the matrix $\hat{\theta}$ is chosen to be diagonal with eigenvalues $\pm \hbar$. The last restriction is consistent with the relation $\hat{\theta}^2 = \hbar^2$, valid in the subspace of states satisfying the condition (28), where $\hat{\theta}^2$ can be identified with $(4i\hat{\xi}^1 \hat{\xi}^2)^2 = \hbar^2$. Moreover, it is clear that $\hat{\xi}^1 \hat{\xi}^2$ cannot be the unit matrix, since this would lead to a contradiction with the commutation relations for $\hat{\xi}^k$. There is only one realization in the space of 8×8 matrices which fulfills all the aforementioned demands, viz.,

$$\hat{\theta} = \hbar \begin{pmatrix} \gamma^0 \Sigma^3 & 0 \\ 0 & \gamma^0 \Sigma^3 \end{pmatrix}, \quad \hat{\xi}^1 \hat{\xi}^2 = \frac{i\hbar}{4} \begin{pmatrix} 0 & \Sigma^3 \\ \Sigma^3 & 0 \end{pmatrix}, \quad (33)$$

where $\Sigma^3 = i\gamma^1 \gamma^2$. Then,

$$\hat{\xi}^1 = \frac{i}{2} \hbar^{1/2} \begin{pmatrix} 0 & \gamma^1 \\ \gamma^1 & 0 \end{pmatrix}, \quad \hat{\xi}^2 = \frac{i}{2} \hbar^{1/2} \begin{pmatrix} \gamma^2 & 0 \\ 0 & \gamma^2 \end{pmatrix}. \quad (34)$$

Taking into account the concrete realization of the operator \hat{t}_ζ , we can see that states Ψ_ζ that obey the condition (28) have the following form

$$\Psi_\zeta(\tau, \mathbf{x}) = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_\zeta(\tau, \mathbf{x}) \\ \zeta \gamma^0 \psi_\zeta(\tau, \mathbf{x}) \end{pmatrix}, \quad (35)$$

where the factor $1/\sqrt{2}$ has been inserted for convenience.

3.3 Schrödinger equation

The Schrödinger equation

$$i\hbar \partial_\tau \underline{\Psi} = (\underline{\hat{H}} + \Lambda \hat{T}) \underline{\Psi}, \quad (36)$$

with $\underline{\hat{H}}$ given by (25), for vectors $\underline{\Psi}$ subject to $\hat{T} \underline{\Psi} = 0$, has the form

$$i\hbar \partial_\tau \underline{\Psi} = \underline{\hat{H}} \underline{\Psi}. \quad (37)$$

Solutions of the above equation can be chosen as eigenstates of the matrix $\hat{\Theta} = \text{bdiag}(\hat{\theta}, \hat{\theta})$. Let us denote eigenstates of $\hat{\theta}$ by $\Psi_{\zeta, \theta}$, which are subject to $\hat{\theta} \Psi_{\zeta, \theta} = \theta \hbar \Psi_{\zeta, \theta}$, with the eigenvalues $\theta = \pm 1$. The latter implies that these solutions have the specific structure

$$\Psi_{\zeta, \theta}(\tau, \mathbf{x}) = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_{\zeta, \theta}(\tau, \mathbf{x}) \\ \zeta \gamma^0 \psi_{\zeta, \theta}(\tau, \mathbf{x}) \end{pmatrix}, \quad \psi_{\zeta, +1}(\tau, \mathbf{x}) = \begin{pmatrix} \psi_\zeta^{(+1)}(\tau, \mathbf{x}) \\ 0 \end{pmatrix}, \quad \psi_{\zeta, -1}(\tau, \mathbf{x}) = \begin{pmatrix} 0 \\ \sigma^1 \psi_\zeta^{(-1)}(\tau, \mathbf{x}) \end{pmatrix}, \quad (38)$$

where $\psi_\zeta^{(\theta)}(\tau, \mathbf{x})$ are 2-component columns. We can see that, due to the constraint (28), these states obey the eigenvalue equation $-2i\hat{\xi}^1 \hat{\xi}^2 \Psi_{\zeta, \theta} = \frac{\hbar}{2} \theta \zeta \Psi_{\zeta, \theta}$. Consequently, the eigenvalues θ label different particle species. Therefore, at the quantum level, we observe the noninvariance of the theory with respect to reflections of one of the coordinate axes.

In components, and in terms of the physical time $x^0 = \zeta \tau$, the Schrödinger equation (37) implies

$$[\gamma^\mu (i\hbar \partial_\mu - qA_\mu) - m] \psi_{\zeta, \theta}(\zeta x^0, \mathbf{x}) = 0. \quad (39)$$

By analogy with the $3+1$ case, one can regard $\hat{\zeta}$ as the charge-sign operator. Let us consider the states $\underline{\Psi}_\zeta$ with a definite charge $q\zeta$. These states satisfy the eigenvalue equation $\hat{\zeta} \underline{\Psi}_\zeta = \zeta \underline{\Psi}_\zeta$. Therefore, states with definite charge $\pm q$ are represented by (22) with $\Psi_{\mp 1} = 0$. The wave functions $\Psi_{\pm 1}$ are parameterized by the physical time $\tau = \pm x^0$.

It is clear that (39) for $\zeta = +1$ is the Dirac equation in the four-spinor representation (lacking the third spatial coordinate) for a particle with charge $+q$. The solutions of (39) with $\zeta = -1$ can be brought into correspondence with the Dirac equation in the four-spinor representation for a particle $\psi^c(x^0)$ with charge

$-q$, by the rule $\psi^c(x^0) = \gamma^2 \psi_{-1}^*(-x^0)$. In order to arrive at the two-spinor representation of the 2 + 1 Dirac equations for particles with charge $\pm q$, we shall use the decomposition (38) of the four-component column $\psi_{\zeta, \theta}$ into two-component columns $\psi_{\zeta}^{(\theta)}$. For $\zeta = +1$, the equation (39) decomposes into

$$[\Gamma_{\theta}^{\mu} (i\hbar\partial_{\mu} - qA_{\mu}) - m] \psi^{(\theta)}(x) = 0, \quad \psi^{(\theta)}(x) \equiv \psi_{+1}^{(\theta)}(x^0, \mathbf{x}), \quad (40)$$

where Γ_{θ}^{μ} are two inequivalent sets of gamma-matrices in 2 + 1 dimensions, given by (67). The analogous equation for $\zeta = -1$ has the form

$$[\Gamma_{\theta}^{\mu} (i\hbar\partial_{\mu} + qA_{\mu}) - m] \psi^{(\theta)c}(x) = 0, \quad \psi^{(\theta)c}(x) \equiv \Gamma_{\theta}^2 \psi_{-1}^{(\theta)*}(-x^0, \mathbf{x}), \quad (41)$$

which is the 2 + 1 Dirac equation in the two-spinor representation for a particle with charge $-q$.

Next, we define the x^0 -representation of states with definite θ -eigenvalue in terms of the physical time x^0 ,

$$\begin{aligned} \Psi_{\theta}(x) &= \begin{pmatrix} \Psi_{\theta}(x) \\ \Psi_{\theta}^c(x) \end{pmatrix}, \quad \Psi_{\theta}(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_{\theta}(x) \\ \gamma^0 \psi_{\theta}(x) \end{pmatrix}, \quad \Psi_{\theta}^c(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_{\theta}^c(x) \\ \gamma^0 \psi_{\theta}^c(x) \end{pmatrix}, \\ \psi_{+1}(x) &= \begin{pmatrix} \psi^{(+1)}(x) \\ 0 \end{pmatrix}, \quad \psi_{-1}(x) = \begin{pmatrix} 0 \\ \sigma^1 \psi^{(-1)}(x) \end{pmatrix}, \\ \psi_{+1}^c(x) &= \begin{pmatrix} \psi^{(+1)c}(x) \\ 0 \end{pmatrix}, \quad \psi_{-1}^c(x) = \begin{pmatrix} 0 \\ \sigma^1 \psi^{(-1)c}(x) \end{pmatrix}. \end{aligned} \quad (42)$$

The charge-conjugate components $\Psi_{\theta}^c(x)$ have been obtained by the rule

$$\Psi_{\theta}^c(x) = \begin{pmatrix} 0 & \gamma^0 \gamma^2 \\ \gamma^0 \gamma^2 & 0 \end{pmatrix} \Psi_{-1, \theta}^*(-x^0, \mathbf{x}),$$

and the inner product (23) in the x^0 -representation is reformulated as

$$(\Psi_{\theta}, \Psi'_{\theta}) = (\Psi_{\theta}, \Psi'_{\theta}) + (\Psi_{\theta}^c, \Psi'^c_{\theta}), \quad (43)$$

The states $\Psi_{\theta}(x)$ satisfy the evolution equation

$$i\hbar\partial_0 \Psi_{\theta}(x^0, \mathbf{x}) = \hat{\mathbf{H}} \Psi_{\theta}(x^0, \mathbf{x}), \quad \hat{\mathbf{H}} = \text{bdiag}(\hat{H}(x^0), \hat{H}^c(x^0)), \quad (44)$$

where

$$\begin{aligned} \hat{H}(x^0) &= qA_0 I_8 + \hat{\omega}_0, \quad \hat{H}^c(x^0) = \hat{H}(x^0) \Big|_{q \rightarrow -q} = -qA_0 I_8 + \hat{\omega}_0^c, \\ \hat{\omega}_0^c &= \hat{\omega}_0^c \Big|_{q \rightarrow -q} = \begin{pmatrix} 0 & \gamma^0 \gamma^2 \\ \gamma^0 \gamma^2 & 0 \end{pmatrix} \hat{\omega}_0^* \begin{pmatrix} 0 & \gamma^0 \gamma^2 \\ \gamma^0 \gamma^2 & 0 \end{pmatrix}. \end{aligned} \quad (45)$$

The operator $\hat{\Omega}$ in the x^0 -representation has the form $\hat{\Omega} = \text{bdiag}(\hat{\omega}_0, \hat{\omega}_0^c)$, while the operator \hat{T} in such a representation reads

$$\hat{T} = \text{bdiag}(\hat{t}, \hat{t}), \quad \hat{t} = 2i\xi^1 \xi^2 + \frac{1}{2} \hat{\theta}. \quad (46)$$

Then, states in the x^0 -representation satisfy the condition

$$\hat{T} \Psi_{\theta} = 0. \quad (47)$$

We can also see that the inner product (43) reduces to the standard inner product between 2 + 1 spinors

$$(\Psi_{\theta}, \Psi'_{\theta}) = (\psi^{(\theta)}, \psi^{(\theta)'}) + (\psi^{(\theta)c}, \psi^{(\theta)c'}). \quad (48)$$

Thus, in terms of $\psi^{(\theta)}$, treated as Dirac spinors, the inner product (43) is Lorentz-invariant.

We note that if one abandons condition (47), one gets a P -invariant QM, which can be obtained by dimensional reduction of the 3 + 1 QM given by [2]. In the event that (47) is no longer valid, states with distinct θ -values are allowed to interfere.

We have considered a realization in which both species of particles are described in the same Hilbert space. As we shall see in the Appendix B, this fact provides some advantages in studying questions related to spin polarization. For this reason, we have introduced in (22) the space \mathcal{R} of \mathbf{x} -dependent 16-component columns $\Psi(\mathbf{x})$. However, if one assumes a given value of the parameter θ fixed from the beginning, then one can obtain a physically equivalent realization of the Hilbert space of vectors $\Psi_{\zeta}(\mathbf{x})$ in (22) as 4-component columns. Accordingly, one should use the 2×2 gamma-matrices, instead of the 4×4 gamma-matrices, in the representations (30) and (34) for the operators $\hat{\omega}_0$ and $\hat{\xi}^k$. In such a realization, one immediately arrives at the above-described two-spinor representation of Dirac spinors, thus avoiding the intermediate description in terms of four-spinors.

3.4 Physical sector

The preliminary state space contains an infinite number of negative-energy states. We restrict it to a physical subspace \mathcal{R}_{ph} where these negative-energy states are absent. Namely, taking into account the correspondence principle, we demand that the operator $\hat{\Omega}$ be positive definite in \mathcal{R}_{ph} . The bases vectors of \mathcal{R}_{ph} , at a given instant of time, have the general structure (42), with the two-spinors being the solutions $\psi_{+,n}^{(\theta)}$ (80) and $\psi_{+,\alpha}^{(\theta)c}$ (81) of the stationary Dirac equation, corresponding to the energy eigenvalues $\epsilon_{+,n}^{(\theta)}$ and $\epsilon_{+,\alpha}^{(\theta)c}$. In addition, we require that the basis vectors be eigenvectors of the charge operator $q\hat{\zeta}$, exactly as is done in QFT when imposing the superselection rule (57). Thus, the basis vectors

$$\Psi_{\theta,+n} = \begin{pmatrix} \Psi_{\theta,+n} \\ 0 \end{pmatrix}, \quad \Psi_{\theta,+,\alpha}^c = \begin{pmatrix} 0 \\ \Psi_{\theta,+,\alpha}^c \end{pmatrix}, \quad (49)$$

are eigenvectors of the charge operator $q\hat{\zeta}$

$$q\hat{\zeta}\Psi_{\theta,+n} = q\Psi_{\theta,+n}, \quad q\hat{\zeta}\Psi_{\theta,+,\alpha}^c = -q\Psi_{\theta,+,\alpha}^c,$$

and of the Hamiltonian operator (44)

$$\hat{H}\Psi_{\theta,+n} = \epsilon_{+,n}^{(\theta)}\Psi_{\theta,+n}, \quad \hat{H}\Psi_{\theta,+,\alpha}^c = \epsilon_{+,\alpha}^{(\theta)c}\Psi_{\theta,+,\alpha}^c,$$

The positivity of $\hat{\Omega}$ can be easily established with respect to the basis vectors $\Psi_{\theta,+n}$ and $\Psi_{\theta,+,\alpha}^c$,

$$\begin{aligned} (\Psi_{\theta,+n}, \hat{\Omega}\Psi_{\theta,+n}) &= (\psi_{+,n}^{(\theta)}, (\epsilon_{+,n}^{(\theta)} - qA_0)\psi_{+,n}^{(\theta)}) > 0, \\ (\Psi_{\theta,+,\alpha}^c, \hat{\Omega}\Psi_{\theta,+,\alpha}^c) &= (\psi_{+,\alpha}^{(\theta)c}, (\epsilon_{+,\alpha}^{(\theta)c} + qA_0)\psi_{+,\alpha}^{(\theta)c}) > 0. \end{aligned} \quad (50)$$

Thus, the physical subspace \mathcal{R}_{ph} is formed by the vectors of the form $\Psi_{\theta,+}$ and $\Psi_{\theta,+}^c$, which are linear combinations of the states $\Psi_{\theta,+n}$ and $\Psi_{\theta,+,\alpha}^c$ respectively. The vectors from \mathcal{R}_{ph} satisfy the Schrödinger equation with the Hamiltonian (44). The inner product (43) between charged states from \mathcal{R}_{ph} of the same sign is given by

$$(\Psi_{\theta,+}, \Psi'_{\theta,+}) = (\Psi_{\theta,+}, \Psi'_{\theta,+}), \quad (\Psi_{\theta,+}^c, \Psi_{\theta,+}^c) = (\Psi_{\theta,+}^c, \Psi_{\theta,+}^c), \quad (51)$$

and it vanishes between charged states of different sign. And following (50), we see that the operator $\hat{\Omega}$ is positive definite,

$$(\Psi_{\theta,+}, \hat{\Omega}\Psi_{\theta,+}) > 0, \quad (\Psi_{\theta,+}^c, \hat{\Omega}\Psi_{\theta,+}^c) > 0, \quad (52)$$

Let us introduce the conserved spin polarization operator \hat{S} ,

$$\hat{S} = -2i\hat{\Xi}^1\hat{\Xi}^2\hat{\zeta}.$$

One can see (taking into account the relation (47)) that this is a conserved operator, whose eigenvectors are Ψ_{θ} and Ψ_{θ}^c ,

$$\hat{S}\Psi_{\theta} = \frac{\hbar}{2}\theta\Psi_{\theta}, \quad \hat{S}\Psi_{\theta}^c = -\frac{\hbar}{2}\theta\Psi_{\theta}^c, \quad (53)$$

The interpretation of \hat{S} as a spin operator can be also extracted from the QFT, see Appendix B. One can note that the operator $\hat{\zeta}\hat{S}$ acts on the states Ψ_{θ} and Ψ_{θ}^c as a particle specie operator,

$$\hat{\zeta}\hat{S}\Psi_{\theta} = \hbar\theta\Psi_{\theta}, \quad \hat{\zeta}\hat{S}\Psi_{\theta}^c = \hbar\theta\Psi_{\theta}^c.$$

In the present work, we do not exceed the limits of the one-particle consideration⁴ within the constructed QM. Thus, it is enough to study the overlaps of the type (51), as well as one-particle matrix elements for functions of physical operators,

$$\mathcal{F}(\hat{X}^k, \hat{P}_k, \hat{\Xi}^k, \hat{H}) = \text{bdiag} \left(f(x^k, \hat{p}_k, \hat{\xi}^k, \hat{H}(x^0)), f^c(x^k, \hat{p}_k, \hat{\xi}^k, \hat{H}^c(x^0)) \right),$$

$$(\Psi_{\theta,+}, \mathcal{F}\Psi'_{\theta,+}) = (\Psi_{\theta,+}, f\Psi'_{\theta,+}), \quad (\Psi_{\theta,+}^c, \mathcal{F}\Psi_{\theta,+}^c) = (\Psi_{\theta,+}^c, f^c\Psi_{\theta,+}^c).$$

⁴However, a generalization to the many-particle theory can be made on the basis of the constructed one-particle representation, and the existence of eigenvectors for the position operator \hat{X} of QFT can be demonstrated. This will be presented elsewhere.

Since all matrix elements of odd-component products of the operators $\hat{\Xi}^k$ are zero and the dynamics of the product of two operators $\hat{\Xi}^k$ is trivial due to (53), it is possible to reduce the physical subspace \mathcal{R}_{ph} of QM to an effective physical subspace required to calculate matrix elements of the functions of operators $\tilde{\mathcal{F}}(\hat{X}^k, \hat{P}_k, \hat{\mathbf{H}})$. Using the decomposition of the 16-component columns Ψ_θ and Ψ_θ^c in terms of two-component spinors $\psi^{(\theta)}$ and $\psi^{(\theta)c}$, we define the effective physical sector $\mathcal{R}_{\text{ph}}^{\text{eff}}$ as the space of states

$$\psi_+^{(\theta)} = \begin{pmatrix} \psi_+^{(\theta)} \\ 0 \end{pmatrix}, \quad \psi_+^{(\theta)c} = \begin{pmatrix} 0 \\ \psi_+^{(\theta)c} \end{pmatrix}, \quad (54)$$

where $\psi_+^{(\theta)}$ and $\psi_+^{(\theta)c}$ are the linear envelopes of the spinors $\psi_{+,n}^{(\theta)}$ and $\psi_{+,\alpha}^{(\theta)c}$, respectively. The dynamics in this representation is governed by the Hamiltonian

$$\hat{H}^{(\theta)} = \text{bdiag}(\hat{h}^{(\theta)}, \hat{h}^{(\theta)c}), \quad \hat{h}^{(\theta)} = qA_0 + \Gamma_\theta^0 [m + \Gamma_\theta^k (\hat{p}_k + qA_k)], \quad \hat{h}^{(\theta)c} = \hat{h}^{(\theta)} \Big|_{q \rightarrow -q}. \quad (55)$$

The operators $\hat{\zeta}$ and \hat{S} , which act in the space \mathcal{R}_{ph} , are reduced to the respective operators acting in the space $\mathcal{R}_{\text{ph}}^{\text{eff}}$

$$\hat{\zeta} = \text{bdiag}(I_2, -I_2), \quad \hat{S} = \theta \frac{\hbar}{2} \text{bdiag}(I_2, -I_2). \quad (56)$$

We can calculate the matrix elements of $\tilde{\mathcal{F}}(\hat{X}^k, \hat{P}_k, \hat{\mathbf{H}})$ using its representative

$$\text{bdiag}(\tilde{f}(x^k, \hat{p}_k, \hat{h}_\theta), \tilde{f}^c(x^k, \hat{p}_k, \hat{\xi}^k, \hat{h}_\theta^c))$$

in $\mathcal{R}_{\text{ph}}^{\text{eff}}$ as follows,

$$(\Psi_{\theta,+}, \tilde{\mathcal{F}}\Psi'_{\theta,+}) = (\psi_+^{(\theta)}, \tilde{f}\psi_+^{(\theta)}) , \quad (\Psi_{\theta,+}^c, \tilde{\mathcal{F}}\Psi_{\theta,+}^c) = (\psi_+^{(\theta)c}, \tilde{f}^c\psi_+^{(\theta)c}) .$$

4 Comparison with one-particle sector of QFT

We shall now give an interpretation of the constructed QM by making a comparison with the dynamics of the one-particle sector in the QFT of the Dirac field in 2+1 dimensions. To this end, we shall first demonstrate that the one-particle sector (in case it can be consistently defined) may be formulated as a consistent relativistic QM. Then, we shall demonstrate that this one-particle sector may be identified with the QM constructed in the previous section. Below, we make use of the notions, results and notation presented in the Appendix B.

To begin with, we recall that the one-particle sector of QFT (as well as any sector with a definite particle number) can be defined in a unique way for every moment of time only in the class of external backgrounds which do not create particles from the vacuum [18, 19, 20, 21], such as stationary magnetic fields and non-critical Coulomb fields. For this reason, we simplify the present discussion to this class of backgrounds. A generalization to arbitrary backgrounds which do not violate vacuum stability is possible.

Let us construct the Hilbert space of one-particle states as the disjoint union $\mathcal{R}_{\text{ph}}^{\text{QFT}} = \mathcal{R}_{10}^{\text{QFT}} \cup \mathcal{R}_{01}^{\text{QFT}}$, $\mathcal{R}_{10}^{\text{QFT}} \cap \mathcal{R}_{01}^{\text{QFT}} = \{0\}$, of the particle subspace $\mathcal{R}_{10}^{\text{QFT}}$ and the antiparticle subspace $\mathcal{R}_{01}^{\text{QFT}}$,

$$|\Psi\rangle = \left(\sum_n f_n a_n^+ |0\rangle, \sum_\alpha \lambda_\alpha b_\alpha^+ |0\rangle \right) \in \mathcal{R}_{\text{ph}}^1, \quad \sum_n f_n a_n^+ |0\rangle \in \mathcal{R}_{10}^{\text{QFT}}, \quad \sum_\alpha \lambda_\alpha b_\alpha^+ |0\rangle \in \mathcal{R}_{01}^{\text{QFT}},$$

where f_n and λ_n are arbitrary coefficients, and we assume $\sum_n |f_n|^2 < \infty$ and $\sum_\alpha |\lambda_\alpha|^2 < \infty$. Therefore, physical states $|\Psi\rangle$ belong either to the particle subspace $\mathcal{R}_{10}^{\text{QFT}}$ or to the antiparticle subspace $\mathcal{R}_{01}^{\text{QFT}}$, in agreement with the superselection rule [22]. In other words, physical states $|\Psi\rangle$ are eigenstates of the charge operator (85),

$$\hat{Q}^{\text{QFT}} |\Psi\rangle = \zeta q |\Psi\rangle, \quad \zeta = \pm 1. \quad (57)$$

We note that the spectrum of \hat{H}^{QFT} in the one-particle sector reproduces that of particles and antiparticles without an infinite number of negative-energy levels. A state vector of QFT in a given moment of time x^0 will be denoted as $|\Psi(x^0)\rangle$. This vector evolves in time according to the Schrödinger equation

$$i\partial_0 |\Psi(x_0)\rangle = \hat{H}^{\text{QFT}} |\Psi(x_0)\rangle,$$

and remains in the one-particle sector due the fact that the Hamiltonian \hat{H}^{QFT} commutes with the particle number operator (86). Consider the decompositions

$$\hat{\Psi} = \hat{\Psi}_{(-)} + \hat{\Psi}_{(+)}, \quad \hat{\Psi}^c = \hat{\Psi}_{(-)}^c + \hat{\Psi}_{(+)}^c,$$

where $\hat{\Psi}^c$ is the charge-conjugated Heisenberg operator of the field $\hat{\Psi}$, defined by $\hat{\Psi}_{(\pm)}^c = \left(\hat{\Psi}_{(\mp)}^{\dagger} \Gamma^2\right)^{\text{T}}$, while the plus and minus terms are given by

$$\hat{\Psi}_{(-)} = \sum_n a_n \psi_{+,n}, \quad \hat{\Psi}_{(+)} = \sum_{\alpha} b_{\alpha}^{\dagger} \psi_{-,\alpha}, \quad \hat{\Psi}_{(-)}^c = \sum_{\alpha} b_{\alpha} \psi_{+,\alpha}^c, \quad \hat{\Psi}_{(+)}^c = \sum_n a_n^{\dagger} \psi_{-,n}^c.$$

Let us consider a coordinate representation of the Fock space of time-dependent one-particle states $|\Psi(x_0)\rangle$, given by the representatives

$$\psi_+(x) = \langle 0 | \hat{\Psi}_{(-)} | \Psi(0) \rangle, \quad \psi_+^c(x) = \langle 0 | \hat{\Psi}_{(-)}^c | \Psi(0) \rangle.$$

Since $|\Psi(x_0)\rangle$ belongs either to the particle subspace $\mathcal{R}_{10}^{\text{QFT}}$ or to the antiparticle subspace $\mathcal{R}_{01}^{\text{QFT}}$, we can define its four-component coordinate representation $\psi(x)$ for each inequivalent representation of the gamma-matrices as

$$\psi_+(x) = \begin{pmatrix} \psi_+(x) \\ 0 \end{pmatrix}, \quad \psi_+^c(x) = \begin{pmatrix} 0 \\ \psi_+^c(x) \end{pmatrix}, \quad (58)$$

where $\psi(x) = \psi^{(\theta)}(x)$, and $\psi^c(x) = \psi^{(\theta)c}(x)$ in the Γ_{θ} representation. Using the projection operator to the one-particle sector,

$$\int \left(\hat{\Psi}^{\dagger} |0\rangle \langle 0| \hat{\Psi} + \hat{\Psi}^{c\dagger} |0\rangle \langle 0| \hat{\Psi}^c \right) dx = I_2,$$

we can present the QFT inner product $\langle \Psi | \Psi' \rangle$ in terms of representatives,

$$\langle \Psi | \Psi' \rangle = \begin{cases} \langle \psi_+, \psi'_+ \rangle, & \zeta = +1 \\ \langle \psi_+^c, \psi'^c_+ \rangle, & \zeta = -1 \end{cases}.$$

It is easy to see that the equations

$$\hat{H}^{\text{QFT}} |\Psi_n\rangle = \varepsilon_{+,n} |\Psi_n\rangle, \quad \hat{H}^{\text{QFT}} |\Psi_{\alpha}^c\rangle = \varepsilon_{+,\alpha}^c |\Psi_{\alpha}^c\rangle,$$

with $|\Psi_n\rangle = a_n^{\dagger} |0\rangle$ and $|\Psi_{\alpha}^c\rangle = b_{\alpha}^{\dagger} |0\rangle$, are written in the coordinate representation as follows

$$\hat{H} \psi_{+,n} = \varepsilon_{+,n} \psi_{+,n}, \quad \hat{H} \psi_{+,\alpha}^c = \varepsilon_{+,\alpha}^c \psi_{+,\alpha}^c,$$

so that for each inequivalent representation of the gamma-matrices the Hamiltonian \hat{H} can be identified with

$$\hat{H} = \text{bdiag} \left(\hat{h}, \hat{h}^c \right), \quad (59)$$

and

$$\psi_{+,n} = \begin{pmatrix} \psi_{+,n} \\ 0 \end{pmatrix}, \quad \psi_{+,\alpha}^c = \begin{pmatrix} 0 \\ \psi_{+,\alpha}^c \end{pmatrix}. \quad (60)$$

It is clear that in the coordinate representation the charge operator \hat{Q}^{QFT} acts as the charge operator in the space $\mathcal{R}_{\text{ph}}^{\text{eff}}$ of the QM,

$$q \hat{\zeta} \psi_+(x) = q \psi_+(x), \quad q \hat{\zeta} \psi_+^c(x) = -q \psi_+^c(x), \quad (61)$$

where $\hat{\zeta}$ is given by (56).

Thus, the states (60) form a basis in the coordinate representation of $\mathcal{R}_{\text{ph}}^{\text{QFT}}$. These states are eigenstates of the QFT charge operator, and

$$(\psi_{+,n}, \psi_{+,m}) = (\psi_{+,n}, \psi_{+,m}), \quad (\psi_{+,\alpha}^c, \psi_{+,\beta}^c) = (\psi_{+,\alpha}^c, \psi_{+,\beta}^c), \quad (\psi_{+,n}, \psi_{+,\alpha}^c) = 0. \quad (62)$$

We see that the Hamiltonian (59) in the coordinate representation of the one-particle sector of the 2 + 1 QFT is precisely the Hamiltonian (55) for a fixed representation $\theta = +1$ (or $\theta = -1$) of the gamma-matrices. Moreover, the physical vector space (60) of QFT in the one-particle sector coincides with (54). Thus, in backgrounds which do not violate vacuum stability, the QM with an appropriate definition of the Hilbert space can be identified with the one-particle sector of QFT.

5 Discussion

In this paper, we have quantized a P - and T -noninvariant pseudoclassical model of a massive relativistic spin-1/2 particle in $2 + 1$ dimensions, on the background of an arbitrary $U(1)$ gauge vector field. A peculiar feature of the model at the classical level is that it contains a bifermionic first-class constraint, which does not admit gauge-fixing. It is shown that this first-class constraint can be realized at the quantum level as a bounded operator, which is imposed as a condition on the state vectors (by analogy with the Dirac quantization method). This allows us to generalize the quantization scheme [1, 2, 3] in case there is a bifermionic first-class constraint.

In doing so, we encounter the phenomenon of quantizing classical constants, characteristic of pseudoclassical models. One ought to say that there are different viewpoints concerning the nature of classical constants in pseudoclassical models and their quantization [5, 6, 8, 13, 23, 24]. One of these viewpoints [8, 24] consists of replacing the classical constants by dynamical variables, prior to quantization, thus modifying the action. We think this approach is unnecessary, and refer to [25] for a thorough discussion. One may point out that the restraint on the range of values of a certain parameter is commonplace in quantum theory, and it is related to the details of the quantization and operator ordering. There is no reason one cannot treat classical constants in a similar manner, and restrict the range of their values at the classical level. In addition, one can also expect that the nature of these constants should change in the passage to quantum theory, which is the case for dynamical variables. In our model, on the classical level, θ is a bifermionic constant. In course of quantization it passes into a constant matrix, whose possible eigenvalues are defined by the quantum dynamics. In any case, the role of classical constants, in particular the parameter θ , may be better clarified in the context of the semiclassical limit of the QM, the discussion of which is presented in the Appendix A. Another viewpoint regarding the status of classical constants stems from the understanding that the sole purpose of pseudoclassical models is to provide a quantum theory. Therefore, one could fix the values of the parameters right at the beginning to be precisely those values which the QM dictates. In this sense, in the present model, one could interpret the parameter θ as a real number already in the pseudoclassical action.

We present a detailed construction of the Hilbert space and verify that the constructed QM possesses the necessary symmetry properties. We show that the condition of preservation of the classical symmetries under the restricted Lorentz transformations and the $U(1)$ transformations allows one to realize the operator algebra in an unambiguous way. Within the constructed relativistic QM, we select a physical subspace which describes the one-particle sector. The obtained realization of the operator algebra proves to be sufficiently different as compared to the quantization of a P -invariant pseudoclassical model of a massive relativistic spin-1/2 particle in $2 + 1$ dimensions. The physical sector of the QM contains both particles and antiparticles with positive-energy $\hat{\Omega}$ levels, and exactly reproduces the one-particle sector of the quantum theory of the $2 + 1$ spinor field.

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6 Appendix A. Semiclassical limit

Let us prove that the quantum Hamiltonian \hat{H} (25) provides a consistent realization of its classical analogue \mathcal{H}_{eff} . To this end, it is sufficient to show that the operator $\hat{\Omega}$ has a correct semiclassical limit. For simplicity, we assume that only a magnetic field is present. The contribution of an electric field to the semiclassical limit of the operator $\hat{\Omega}$ can be analyzed in complete analogy to the $3 + 1$ case, studied in [2] (see the discussion of the contributions $\hat{\rho}_1$ and $\hat{\rho}_2$).

From the standard viewpoint accepted in quantum mechanics, the semiclassical behavior of a wave packet corresponding to a particle takes place when the packet is sufficiently well-localized in the phase space of coordinates and momenta. Accordingly, it can be characterized by the coordinates of the average position \bar{x}^k and the average momenta \bar{p}_k . At the same time, the mean square deviations Δx^k should be small in comparison with the characteristic scale L of the system in question, $\Delta x^k \ll L$, while the mean square deviations Δp_k should be small as compared to $|p_k|$, $\Delta p_k \ll |p_k|$. In accordance with these conditions, the external field should be sufficiently homogeneous, and change with time sufficiently slowly, so that it should not vary considerably within distances commensurate with the size of a semiclassical wave packet (SWP), while the SWP should not disperse within the time interval of the observation. Thus, it is sufficient to restrict the analysis to the case of a time-independent field.

To prove that $\widehat{\Omega} = \text{bdiag}(\widehat{\Omega}_{+1}, \widehat{\Omega}_{-1})$ is a consistent quantum realization of ω ($\omega = \omega_0$ in the absence of an electric field), it suffices to show that the squared operator $\widehat{\Omega}^2 = \text{bdiag}(\widehat{\omega}_0^2, \widehat{\omega}_0^2)|_{x^0=\zeta\tau}$, given by (32), has a correct semiclassical limit, i.e., it has the same expectation value on semiclassical states $\underline{\Psi}_\theta^{cl}$ as the operator $\widehat{\Omega}^2$, obtained by direct quantization of ω_0^2 ,

$$\widehat{\Omega}^2 = \text{bdiag}(\widehat{\omega}_0^2, \widehat{\omega}_0^2), \quad \widehat{\omega}_0^2 = m^2 + (\hat{p}_k + qA_k)^2 - 2iqF_{kl}\hat{\xi}^k\hat{\xi}^l. \quad (63)$$

Thus, we need to show that

$$\left(\underline{\Psi}_\theta^{cl} | \widehat{\Omega}^2 \underline{\Psi}_\theta^{cl}\right) = \left(\underline{\Psi}_\theta^{cl} | \widehat{\Omega}^2 \underline{\Psi}_\theta^{cl}\right), \quad (64)$$

where the inner product is defined by (23).

In what follows, it is convenient to use the x^0 -representation. Thus, we pass from the vectors $\underline{\Psi}_\theta^{cl}$ to the vectors Ψ_θ^{cl} . The SWP is a superposition of state vectors from the physical subspace \mathcal{R}_{ph} , i.e., we have either a particle SWP, or an antiparticle SWP. Let us consider the case of a particle SWP (for charge-conjugated particles the consideration is analogous)

$$\Psi_\theta^{cl}(x) = \sum_n c_{+,n}^{(\theta)} \Psi_{\theta,+,n}(x),$$

where $c_{+,n}^{(\theta)}$ are the corresponding constant coefficients of the decomposition of the SWP in eigenvectors. The coefficients are defined at the initial time instant $x_{in}^0 = 0$ by the given mean values of the coordinates \bar{x}_{in} and momenta $\bar{\mathbf{p}}_{in}$ characterizing the SWP. The component structure of semiclassical states Ψ_θ^{cl} is that of the states Ψ_θ from the physical subspace \mathcal{R}_{ph} (49),

$$\begin{aligned} \Psi_\theta^{cl} &= \begin{pmatrix} \Psi_\theta^{cl} \\ 0 \end{pmatrix}, \quad \Psi_\theta^{cl} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_\theta^{cl} \\ \gamma^0 \psi_\theta^{cl} \end{pmatrix}, \\ \psi_{+1}^{cl} &= \begin{pmatrix} \psi^{(+1)cl} \\ 0 \end{pmatrix}, \quad \psi_{-1}^{cl} = \begin{pmatrix} 0 \\ \sigma^1 \psi^{(-1)cl} \end{pmatrix}, \end{aligned}$$

where

$$\psi^{(\theta)cl}(x) = \sum_n c_{+,n}^{(\theta)} \psi_{+,n}^{(\theta)}(x).$$

As shown in Section 3, we can calculate the mean values of the function $\tilde{\mathcal{F}}(\hat{X}^k, \hat{P}_k, \hat{H}_{x^0})$ by using its representative $\text{bdiag}(\tilde{f}(x^k, \hat{p}_k, \hat{h}_\theta), \tilde{f}^c(x^k, \hat{p}_k, \hat{h}_\theta^c))$ in the effective physical subspace $\mathcal{R}_{\text{ph}}^{\text{eff}}$. Thus, for example, the mean values of \mathbf{x} and $\hat{\mathbf{p}}$ at the time instant x^0 can be expressed as follows,

$$\begin{aligned} \bar{\mathbf{x}} &= \left(\Psi_\theta^{cl}(x^0), \hat{\mathbf{X}} \Psi_\theta^{cl}(x^0)\right) = \left(\psi^{(\theta)cl}(x^0), \mathbf{x} \psi^{(\theta)cl}(x^0)\right), \\ \bar{\mathbf{p}} &= \left(\Psi_\theta^{cl}(x^0), \hat{\mathbf{P}} \Psi_\theta^{cl}(x^0)\right) = \left(\psi^{(\theta)cl}(x^0), \hat{\mathbf{p}} \psi^{(\theta)cl}(x^0)\right). \end{aligned}$$

These mean values depend on the parameter x^0 , as well as on the initial values of \bar{x}_{in} and $\bar{\mathbf{p}}_{in}$: $\bar{\mathbf{x}} = \bar{\mathbf{x}}(x^0, \bar{x}_{in}, \bar{\mathbf{p}}_{in})$ and $\bar{\mathbf{p}} = \bar{\mathbf{p}}(x^0, \bar{x}_{in}, \bar{\mathbf{p}}_{in})$. In accordance with the above definition, we obtain the semiclassical behavior of a wave packet in case the equations of motion for the mean values to the leading order in the expansion obey the classical equations of motion

$$\begin{aligned} \frac{d\bar{x}^k}{dx^0} &= \{\bar{x}^k, \epsilon_\theta(\bar{\mathbf{x}}, \bar{\mathbf{p}})\}, \\ \frac{d(\bar{p}_k + qA_k(\bar{\mathbf{x}}))}{dx^0} &= \{\bar{p}_k + qA_k(\bar{\mathbf{x}}), \epsilon_\theta(\bar{\mathbf{x}}, \bar{\mathbf{p}})\}, \end{aligned}$$

where $\epsilon_\theta(\bar{\mathbf{x}}, \bar{\mathbf{p}})$ is the mean value of the Hamiltonian (in a stationary background, it does not depend on the time x^0)

$$\epsilon_\theta(\bar{\mathbf{x}}, \bar{\mathbf{p}}) = \left(\Psi_\theta^{cl}(x^0), \hat{\mathbf{H}} \Psi_\theta^{cl}(x^0)\right) = \left(\psi^{(\theta)cl}(x^0), \hat{h}^{(\theta)} \psi^{(\theta)cl}(x^0)\right).$$

Then we can see that thus determined SWP satisfies (with semiclassical accuracy) the evolution equation

$$i\hbar\partial_0\psi^{(\theta)cl}(x) = \hat{\epsilon}_\theta\psi^{(\theta)cl}(x),$$

$$\hat{\epsilon}_\theta = \epsilon_\theta(\bar{\mathbf{x}}, \bar{\mathbf{p}}) + \frac{\partial\epsilon_\theta(\bar{\mathbf{x}}, \bar{\mathbf{p}})}{\partial\bar{p}_k}(\bar{p}_k - \bar{p}_k) + \frac{\partial\epsilon_\theta(\bar{\mathbf{x}}, \bar{\mathbf{p}})}{\partial\bar{x}^k}(x^k - \bar{x}^k).$$

This spinor (with semiclassical accuracy) can be represented in the form

$$\psi^{(\theta)cl}(x) = \left[\Gamma_\theta^0 \hat{\epsilon}_\theta + \Gamma_\theta^j (i\hbar\partial_j - qA_j(\mathbf{x})) + m \right] \varphi^{(\theta)cl}(x),$$

where the function

$$\varphi^{(\theta)cl}(x) = \exp\left\{-\frac{i}{\hbar}\hat{\epsilon}_\theta x^0\right\} \varphi^{(\theta)cl}(\mathbf{x}),$$

obeys the squared Dirac equation (with semiclassical accuracy)

$$\left[(\hat{\epsilon}_\theta)^2 - D \right] \varphi^{(\theta)cl}(\mathbf{x}) = 0, \quad D = m^2 + (i\hbar\partial_k - qA_k(\mathbf{x}))^2 - \theta\hbar\sigma^3 qB(\mathbf{x}), \quad F_{21} = B. \quad (65)$$

Therefore, the explicit form of the mean energy reads

$$\epsilon_\theta(\bar{\mathbf{x}}, \bar{\mathbf{p}}) = \sqrt{m^2 + (\bar{p}_j + qA_j(\bar{\mathbf{x}}))^2 - \theta\hbar\sigma^3 qB(\bar{\mathbf{x}})},$$

where $\langle\sigma^3\rangle$ is the corresponding mean value of the matrix σ^3 , which will be specified below. We can see that a state with a given value of spin projection, and providing the necessary semiclassical limit, is obtained by choosing the column $\varphi^{(\theta)cl}(\mathbf{x})$ in the form

$$\varphi^{(\theta)cl}(\mathbf{x}) = \begin{pmatrix} f^{(\theta)cl}(\mathbf{x}) \\ 0 \end{pmatrix}.$$

Then we finally have

$$\epsilon_\theta(\bar{\mathbf{x}}, \bar{\mathbf{p}}) = \sqrt{m^2 + (\bar{p}_j + qA_j(\bar{\mathbf{x}}))^2 - \theta\hbar qB(\bar{\mathbf{x}})}.$$

It is exactly the function $\varphi^{(\theta)cl}(x)$ which defines (to the leading order in the approximation) the form of the SWP as a function of coordinates and momenta. This function can be represented as a well-localized wave packet of solutions of the squared Dirac equation,

$$\varphi^{(\theta)cl}(x) = \sum_n c_{+,n}^{(\theta)} \varphi_{+,n}^{(\theta)}(x),$$

with the same coefficients $c_{+,n}^{(\theta)}$ as those for $\psi^{(\theta)cl}(x)$.

Using the SWP of a particle in a magnetic field, we have

$$\left(\Psi_\theta^{cl}, \hat{\Omega} \Psi_\theta^{cl} \right) = \epsilon_\theta(\bar{\mathbf{x}}, \bar{\mathbf{p}}),$$

and, correspondingly,

$$\left(\Psi_\theta^{cl}, \hat{\Omega}^2 \Psi_\theta^{cl} \right) = (\epsilon_\theta(\bar{\mathbf{x}}, \bar{\mathbf{p}}))^2.$$

On the other hand, using the SWP we obtain the same result for the mean value of the operator $\widehat{\Omega}^2$ defined by (63). This completes the proof of the fact that $\hat{\Omega}$ has a correct semiclassical limit⁵.

To finish the semiclassical analysis, we find it useful to make a remark concerning the physical meaning of a semiclassical spinning particle. It should be noted that the term $\hbar\sigma^3 qB(\mathbf{x})$, referred to as a quantum correction, usually is not included into the expression for the classical energy. The reason for doing so is the fact that in the homogeneous field (for a SWP) we have the term $(\bar{p}_j + qA_j(\bar{\mathbf{x}}))^2 \sim 2|qB|\hbar n$, with Landau level number $n \gg 1$. Therefore, the term $\hbar qB(\mathbf{x})$, giving a contribution commensurate with the difference

⁵It is easy to see that in the general case, when we have an electric field satisfying the quasiclassical conditions (of being weak and sufficiently homogeneous within the QWP), we obtain a similar result,

$$\left(\Psi_\theta^{cl}, \hat{\Omega}^2 \Psi_\theta^{cl} \right) = (\epsilon_\theta(\bar{\mathbf{x}}, \bar{\mathbf{p}}) - qA_0(\bar{\mathbf{x}}))^2 = m^2 + (\bar{p}_j + qA_j(\bar{\mathbf{x}}))^2 - \theta qB(\bar{\mathbf{x}}).$$

Thus, the proof is also valid in the general case.

of the energy levels in the first summand, is extremely small in the expression for the semiclassical energy $\epsilon_\theta(\bar{\mathbf{x}}, \bar{\mathbf{p}})$. However, in a non-homogeneous field, without changing the conditions of semiclassics (i.e., within the conditions that permit one to characterize motion by classical coordinates and momenta), one can see the influence of spin magnetic moment on the classical trajectory. This can be observed from the equation

$$\frac{d(\bar{p}_k + qA_k(\bar{\mathbf{x}}))}{dx^0} = -qF_{kl}(\bar{\mathbf{x}}) \frac{d\bar{x}^l}{dx^0} + \theta q \frac{\partial B(\bar{\mathbf{x}})}{\partial \bar{x}^k} \frac{\hbar}{2\epsilon_\theta(\bar{\mathbf{x}}, \bar{\mathbf{p}})}.$$

Here, the term containing the field gradient $\partial B(\bar{\mathbf{x}})/\partial \bar{x}^k$ is much smaller than the preceding one. However, the first term causes acceleration, which is always perpendicular to the velocity of the particle. At the same time, the fluctuations of momenta or coordinates do not change anything in this respect, since the acceleration is simply a consequence of the structure of the minimal interaction (current-potential). The acceleration caused by the second term, although small, is directed alongside the field gradient, i.e., its direction is not related to the motion of the particle, and is not affected by the fluctuations of coordinates and momenta, because this is a consequence of the structure of the interaction between the field and the magnetic momenta. In principle, this permits one to separate the types of motion caused by different interactions. In the given case, the presence of the second term affects the particle in the same way as a spinless particle is affected by a weak electric field directed alongside the gradient of a magnetic field. The possibility of observing the effect of the interaction with the spin magnetic moment on a classical trajectory implies that one deals with a classical theory of a spinning particle in the usual sense: one can describe motion using the concept of a trajectory.

7 Appendix B. Spinor field in 2 + 1 dimensions

7.1 General

Consider here some required notions of the theory of the spinor field $\psi(x) = (\psi_\alpha(x), \alpha = 1, 2)$ in 2 + 1 dimensions, interacting with an external electromagnetic background (for convenience, we assume below $\hbar = c = 1$). We note that $\psi(x)$ are regarded as the generating elements of an infinite-dimensional Grassmann algebra. The action of the model may be written as

$$S_{\text{FT}} = \int dx \mathcal{L}, \quad \mathcal{L} = \bar{\psi} (\Gamma^\mu P_\mu - m) \psi, \quad P_\mu = (i\partial_\mu - qA_\mu), \quad \mu = 0, 1, 2, \quad (66)$$

where q is the algebraic charge, and Γ^μ are the gamma-matrices in 2 + 1 dimensions, $[\Gamma^\mu, \Gamma^\nu]_+ = 2\eta^{\mu\nu}$, $\mu, \nu = 0, 1, 2$, and $\bar{\psi} = \psi^\dagger \Gamma^0$.

Some remarks are in order. In 2 + 1 dimensions there exist two inequivalent representations for the gamma-matrices,

$$\Gamma_{+1}^0 = \Gamma_{-1}^0 = \sigma^3, \quad \Gamma_{+1}^1 = \Gamma_{-1}^1 = i\sigma^2, \quad \Gamma_{+1}^2 = -\Gamma_{-1}^2 = -i\sigma^1, \quad (67)$$

where σ^i are the Pauli matrices. Unless otherwise specified, we shall always assume $\Gamma = \Gamma_\theta$, $\psi = \psi^{(\theta)}$, $\bar{\psi} = \bar{\psi}^{(\theta)}$, and so on.

Returning to the action (66), we note that the corresponding Euler-Lagrange equation implies the Dirac equation for the spinor field,

$$\frac{\delta S_{\text{FT}}}{\delta \bar{\psi}} = [(i\partial_\mu - qA_\mu) \Gamma^\mu - m] \psi = 0 \iff i\partial_0 \psi = \hat{h} \psi, \quad (68)$$

where

$$\hat{h} = -\Gamma^0 \Gamma^k P_k + \Gamma^0 m + qA_0. \quad (69)$$

With the choice of the Lagrangian density (66), the field $\bar{\psi}$ does not carry a time derivative. In this connection, in order to quantize the model (66) it is convenient to apply the reduced Hamiltonization procedure [11]. We introduce the velocity $v = \dot{\psi}$ and regard $\bar{\psi}$ as a velocity. The first-order action reads

$$S^v = \int d^3x \left[\mathcal{L}^v + \pi (\dot{\psi} - v) \right] = \int d^3x \left(\pi \dot{\psi} - \mathcal{H}^v \right), \quad \mathcal{L}^v = \mathcal{L}|_{\dot{\psi}=v}, \quad \mathcal{H}^v = \pi v - \mathcal{L}^v. \quad (70)$$

It implies the following equations of motion

$$\begin{aligned} \frac{\delta S^v}{\delta v} &= \frac{\partial \mathcal{L}^v}{\partial v} - \pi = i\bar{\psi} \Gamma^0 - \pi = 0 \Rightarrow \bar{\psi} = -i\pi \Gamma^0, \\ \frac{\delta S^v}{\delta \bar{\psi}} &= \frac{\partial \mathcal{L}^v}{\partial \bar{\psi}} = -i\Gamma^0 v + (q\Gamma^0 A_0 - \Gamma^k P_k + m) \psi = 0 \Rightarrow v = -i\hat{h}\psi. \end{aligned}$$

Thus, the velocities are expressed in terms of the coordinates and momenta. Substituting them into the function \mathcal{H}^ν , we obtain the Hamiltonian density $\mathcal{H} = -i\pi\dot{\psi}$. Then the action (70) becomes the canonical Hamiltonian action $S^H = \int dx (\pi\dot{\psi} - \mathcal{H})$ for the phase space variables (ψ, π) . It implies the canonical Hamilton equations

$$\dot{\psi} = \{\psi, H^{\text{FT}}\}, \quad \dot{\pi} = \{\pi, H^{\text{FT}}\}, \quad H^{\text{FT}} = \int dx \mathcal{H} = \int \psi^\dagger \hat{h}(x^0) \psi dx. \quad (71)$$

The conserved charge reads

$$Q^{\text{FT}} = q \int \psi^\dagger \psi dx, \quad (72)$$

7.2 Dirac equation

In time-independent backgrounds the equation (68) is reduced to its stationary form

$$\hat{h}\psi(\mathbf{x}) = \varepsilon\psi(\mathbf{x}), \quad \psi(\mathbf{x}) = \exp(-i\varepsilon x^0)\psi(\mathbf{x}). \quad (73)$$

We square this equation with the ansatz

$$\psi(\mathbf{x}) = [\Gamma^0(\varepsilon - qA_0) + \Gamma^k(i\partial_k - qA_k) + m]\varphi(\mathbf{x}). \quad (74)$$

Then $\varphi(\mathbf{x})$ satisfies the equation

$$[(\varepsilon - qA_0)^2 - D]\varphi(\mathbf{x}) = 0, \quad (75)$$

where $D = m^2 - P_k P^k + \frac{i}{2} q F_{\mu\nu} [\Gamma^\mu, \Gamma^\nu]$. A pair (ε, φ) is a solution of the above equation if it obeys either

$$\varepsilon = qA_0 + \sqrt{\varphi^{-1} D \varphi} \Rightarrow \varepsilon - qA_0 > 0, \quad (76)$$

or

$$\varepsilon = qA_0 - \sqrt{\varphi^{-1} D \varphi} \Rightarrow \varepsilon - qA_0 < 0. \quad (77)$$

Let us denote by $(\varepsilon_{+,n}, \varphi_{+,n})$ solutions for positive values of $\varepsilon - qA_0$, i.e., those for the upper branch of the energy spectrum, and by $(\varepsilon_{-,\alpha}, \varphi_{-,\alpha})$ solutions for negative values of $\varepsilon - qA_0$, i.e., those for the lower branch of the energy spectrum. Here, n and α are quantum numbers which account for a possible asymmetry between both branches of the energy spectrum.

Solutions $\psi_{+,n}$ and $\psi_{-,\alpha}$ of (73), constructed from $\varphi_{+,n}$ and $\varphi_{-,\alpha}$, obey the orthogonality and completeness relations

$$(\psi_{+,n}, \psi_{+,m}) = \delta_{nm}, \quad (\psi_{-,\alpha}, \psi_{-,\beta}) = \delta_{\alpha\beta}, \quad (\psi_{+,n}, \psi_{-,\alpha}) = 0, \quad (78)$$

$$\sum_{n,\alpha} [\psi_{+,n}(x) \psi_{+,n}^\dagger(y) + \psi_{-,\alpha}(x) \psi_{-,\alpha}^\dagger(y)] = \delta(\mathbf{x} - \mathbf{y}), \quad x_0 = y_0, \quad (79)$$

where

$$\psi_{+,n}(x) = e^{-i\varepsilon_{+,n}x^0} \psi_{+,n}(\mathbf{x}), \quad \psi_{-,\alpha}(x) = e^{-i\varepsilon_{-,\alpha}x^0} \psi_{-,\alpha}(\mathbf{x}). \quad (80)$$

A solution of the eigenvalue problem $\hat{h}^c \psi^c(\mathbf{x}) = \varepsilon^c \psi^c(\mathbf{x})$ for the charge-conjugated Hamiltonian, $\hat{h}^c = \hat{h}(q)|_{q \rightarrow -q}$, can be analyzed in a similar manner, and

$$\psi_{+,\alpha}^c = \Gamma^2 \psi_{-,\alpha}^*, \quad \psi_{-,\alpha}^c = \Gamma^2 \psi_{+,\alpha}^*, \quad \varepsilon_{+,\alpha}^c = -\varepsilon_{-,\alpha}, \quad \varepsilon_{-,\alpha}^c = -\varepsilon_{+,\alpha}. \quad (81)$$

7.3 Quantization

In the passage to the quantum theory, the fields ψ and π (ψ and $\bar{\psi}$) become Heisenberg operators $\hat{\Psi}$ and $\hat{\pi}$ ($\hat{\Psi}$ and $\hat{\Psi}^\dagger$) with equal-time anticommutation relations given by

$$[\hat{\Psi}_\alpha(x), \hat{\pi}_\beta(y)]_+ \Big|_{x_0=y_0} = i [\hat{\Psi}_\alpha(x), \hat{\Psi}_\beta^\dagger(y)]_+ \Big|_{x_0=y_0} = i\delta_{\alpha\beta} \delta(\mathbf{x} - \mathbf{y}). \quad (82)$$

Using the complete set (80), we can decompose the field operator $\hat{\Psi}(x)$ as

$$\hat{\Psi}(x) = \sum_n a_n \psi_{+,n}(x) + \sum_\alpha b_\alpha^\dagger \psi_{-,\alpha}(x). \quad (83)$$

From the commutation relations (82) and orthogonality conditions (78), we obtain

$$[a_n, a_m^\dagger]_+ = [b_\alpha, b_\beta^\dagger]_+ = \delta_{\alpha\beta}, \quad [a_n, a_m]_+ = [b_\alpha, b_\beta]_+ = 0,$$

which defines two sets of creation and annihilation operators, a_n, a_n^\dagger and $b_\alpha, b_\alpha^\dagger$.

With the help of the expressions for the field operators (83) and for the classical Hamiltonian H^{FT} (71), we construct the quantum field theory (QFT) Hamiltonian

$$\hat{H}^{\text{QFT}} = \int : \hat{\Psi}^\dagger \hat{h} \hat{\Psi} : dx = \sum_n \varepsilon_{+,n} a_n^\dagger a_n + \sum_\alpha \varepsilon_{+,\alpha}^\dagger b_\alpha^\dagger b_\alpha. \quad (84)$$

The QFT charge operator and the particle number operator have the following form

$$\hat{Q}^{\text{QFT}} = \frac{q}{2} \int [\hat{\Psi}^\dagger, \hat{\Psi}] dx = q \left(\sum_n a_n^\dagger a_n - \sum_\alpha b_\alpha^\dagger b_\alpha \right), \quad (85)$$

$$\hat{N} = \sum_n a_n^\dagger a_n + \sum_\alpha b_\alpha^\dagger b_\alpha. \quad (86)$$

We define the vacuum state $|0\rangle$ to be the zero vector of the annihilation operators: $a_n |0\rangle = b_\alpha |0\rangle = 0$ for every n and α . Then we construct the Hilbert space \mathcal{R}^{QFT} of QFT as a Fock space.

7.4 Spin operators in 2 + 1 dimensions

Let us consider the Lorentz vector J_μ^{FT} ,

$$J_\mu^{\text{FT}} = \frac{1}{2} \varepsilon_{\mu\nu\lambda} \int \bar{\psi} \Gamma^0 J^{\nu\lambda} \psi dx,$$

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}, \quad L^{\mu\nu} = -x^\mu \hat{p}^\nu + x^\nu \hat{p}^\mu, \quad S^{\mu\nu} = \frac{i}{4} [\Gamma^\mu, \Gamma^\nu], \quad \hat{p}^\mu = -i\partial^\mu,$$

whose components in the absence of external fields correspond to the conserved Noether charges related to the infinitesimal Lorentz transformations. Following the interpretation of spin projection in 2 + 1 dimensions [26] as the projection to the x^0 -axis, we may introduce the FT spin operator S_0^{FT} as the spin part of J_0^{FT} . In the rest-frame, $p^1 = p^2 = 0$, only the term S^{12} of the angular momentum component J^{12} is non-vanishing. Therefore,

$$S_0^{\text{FT}} = \int \psi^{(\theta)\dagger} \left(\frac{i}{4} [\Gamma_\theta^1, \Gamma_\theta^{21}] \right) \psi^{(\theta)} dx = \frac{\theta}{2} \int \psi^{(\theta)\dagger} \sigma^3 \psi^{(\theta)} dx. \quad (87)$$

The corresponding QFT spin operator \hat{S}_0^{QFT} in the non-relativistic limit is proportional to the particle number operator,

$$\hat{S}_0^{\text{QFT}} = \frac{\theta}{2} \int : \hat{\psi}^{(\theta)\dagger} \sigma^3 \hat{\psi}^{(\theta)} : dx = \frac{\theta}{2} \hat{N}, \quad (88)$$

when

$$\psi_+^{(\theta)} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \psi_-^{(\theta)} \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The operator S_0^{FT} is only conserved in the non-relativistic limit. That is why we need to find a conserved spin operator to classify relativistic states.

The spin polarization of a relativistic particle can be ‘‘revealed’’ in a magnetic field, due to the fact that the interaction of the spin magnetic moment with the magnetic field gives a contribution to the energy of the particle. In 2 + 1 dimensions the energy spectrum in a magnetic field is determined by the squared Dirac equation (75), where the corresponding spin contribution to the energy square operator reads $\frac{i}{4} q F_{kl} [\Gamma^k, \Gamma^l]$. In conformity with the non-relativistic and semiclassical limits (see Appendix A), the solutions $\varphi_{+,n}^{(\theta)}$ and $\varphi_{-,\alpha}^{(\theta)}$ of (75) may be chosen as

$$\varphi_{+,n}^{(\theta)} = \begin{pmatrix} f_{+,n}^{(\theta)} \\ 0 \end{pmatrix}, \quad \varphi_{-,\alpha}^{(\theta)} = \begin{pmatrix} 0 \\ f_{-,\alpha}^{(\theta)} \end{pmatrix}.$$

Then one can see that these solutions are eigenvectors of $\frac{i}{4} q F_{kl} [\Gamma^k, \Gamma^l]$,

$$\frac{i}{4} q F_{kl} [\Gamma_\theta^k, \Gamma_\theta^l] \varphi_{+,n}^{(\theta)} = -\theta q F_{21} \varphi_{+,n}^{(\theta)}, \quad \frac{i}{4} q F_{kl} [\Gamma_\theta^k, \Gamma_\theta^l] \varphi_{-,\alpha}^{(\theta)} = \theta q F_{21} \varphi_{-,\alpha}^{(\theta)}.$$

Therefore, one can expect that a conserved operator of spin polarization can be introduced.

It is natural to expect that in 2 + 1 dimensions a spin polarization operator is, in some sense, trivial, since, in contrast to the 3 + 1 theory, it must have only one eigenvalue. Let us first examine the 3 + 1 case, where the meaning of the corresponding operator is more transparent. Here, to characterize spin states, one can use different spin polarization operators which are covariant and are integrals of motion [27]. The pseudoclassical model under consideration can be derived by a dimensional reduction from the pseudoclassical model [12] describing the Weyl particle in 3 + 1 dimensions. Consequently, it is natural to obtain the conserved spin polarization operator in 2 + 1 dimensions by a dimensional reduction of the (3 + 1)-dimensional theory of the Weyl field. The 3 + 1 left/right massless charged particle is described by a four-spinor, $\psi_{(s)}^{\text{Weyl}}(x)$, which obeys the Dirac equation

$$\gamma^n P_n \psi_{(s)}^{\text{Weyl}}(x) = 0, \quad n = 0, \dots, 3, \quad (89)$$

and the Weyl condition

$$(\gamma^5 - s) \psi_{(s)}^{\text{Weyl}}(x) = 0, \quad s = +1 (-1), \quad (90)$$

with $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$. Here, we also assume that $P_3 = i\partial_3$ ($A_3(x) = 0$) and $\partial_3 A_\mu(x) = 0$.

Dimensional reduction from 3+1 to 2+1 is a result of breaking the symmetry with respect to z -translations; therefore, the corresponding momentum is constant, $P_3 = m$. Accordingly, we reduce the group of Lorentz transformations eliminating the rotation generators which contain the z -component, in particular, the matrix γ^3 . Consider the following unitary transformation of the gamma-matrices

$$V^{-1}\gamma^\mu V = -\gamma^3\gamma^\mu, \quad V^{-1}\gamma^3 V = \gamma^3, \quad V = \frac{1}{\sqrt{2}}(1 + \gamma^3).$$

Using this transformation, we can represent equations (89) and (90) in the form

$$(\gamma^\mu P_\mu - m) \psi_{(s)}(x) = 0, \quad (\gamma^5\gamma^3 - s) \psi_{(s)}(x) = 0, \quad (91)$$

where $\psi_{(s)}(x) = V^{-1}\psi_{(s)}^{\text{Weyl}}(x)$. Note that the matrix $\gamma^5\gamma^3 = \gamma^0\Sigma^3$ in (91) is the matrix used in the realization of the operator $\hat{\theta}$ in (33). Now, we can present the 2 + 1 QFT spin polarization operator

$$\hat{S}^{\text{QFT}} = \frac{1}{2} \int : \hat{\psi}^\dagger \gamma^0 \Sigma^3 \hat{\psi} : d^2x, \quad (92)$$

where $\hat{\psi}$ and $\hat{\psi}^\dagger$ are the linear superpositions of the quantized four-component Dirac fields $\hat{\psi}_{(\pm)}$ and $\hat{\psi}_{(\pm)}^\dagger$, respectively. The operator (92) is a scalar under the 2 + 1 Lorentz transformations.

Thus, we arrive at QFT with both $\theta = +1$ and $\theta = -1$ species of fermions. The operator (92) is conserved in any external field. In the representation (31) of the gamma-matrices, we have $\gamma^0\Sigma^3 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$. Consequently, in order to be eigenvectors of this matrix, the fields $\hat{\psi}_{(s)}$ can be selected as follows,

$$\hat{\psi}_{(+1)} = \begin{pmatrix} \hat{\Psi}^{(+1)} \\ 0 \end{pmatrix}, \quad \hat{\psi}_{(-1)} = \begin{pmatrix} 0 \\ \sigma^1 \hat{\Psi}^{(-1)} \end{pmatrix}, \quad (93)$$

where $\hat{\Psi}^{(\theta)}$ is the two-component operator defined by (83) in the θ -representation of the gamma-matrices (67). Thus, we obtain

$$\hat{S}^{\text{QFT}} = \frac{1}{2q} (\hat{Q}_{+1}^{\text{QFT}} - \hat{Q}_{-1}^{\text{QFT}}), \quad (94)$$

where $\hat{Q}_\theta^{\text{QFT}}$ is the 2 + 1 charge operator (85) in the corresponding representation of the gamma-matrices. We can see that this construction is trivial only if either particle species $\theta = +1$ or $\theta = -1$ is present. If this is the case, then the conserved 2 + 1 QFT spin polarization operator is simply proportional to another conserved QFT scalar, the charge.

One-particle states $|\Psi\rangle = |\Psi^{(\theta)}\rangle$ are eigenstates of the spin polarization operator (94),

$$\hat{S}^{\text{QFT}} |\Psi^{(\theta)}\rangle = \theta \frac{\zeta}{2} |\Psi^{(\theta)}\rangle, \quad \zeta = \pm 1. \quad (95)$$

Despite negatively-charged $\theta = -1$ antiparticles and positively-charged $\theta = +1$ particles being characterized by the same value of spin polarization, they cannot be identified (in contrast to neutral particles), since

they carry different charges. Thus, depending on the representation of the gamma-matrices (or, equivalently, depending on the sign of mass in the Dirac equation), the one-particle state is either a $\theta = +1$ particle or a $\theta = -1$ particle. Since $\theta = +1$ and $\theta = -1$ particles cannot interact, in the general multiparticle case of QFT the particle numbers of these different types are not correlated. In the multiparticle case, since the spin polarization of particles of a given type is proportional to the total charge, the total spin is conserved in any external field, i.e., particles of a given species are created and annihilated in pairs whose total spin is zero. The species of a one-particle state is uniquely determined by the operator \hat{S}^{QFT} times the charge of the state.

Thus, in the spaces of four- and two-spinors we find respectively two representations, (92) and (94), for the conserved spin polarization operator. We can see that this operator is non-trivial in the $2 + 1$ extended (four-spinor) representation (92), since $\gamma^0 \Sigma^3$ is not a unit matrix. The relation between the $2 + 1$ extended representation and the two-spinor representation of the Dirac QFT is similar to the relation between the physical subspace \mathcal{R}_{ph} and the effective physical subspace $\mathcal{R}_{\text{ph}}^{\text{eff}}$, discussed in Section 3. One can say that the space of two-spinors is an effective space of the $2 + 1$ Dirac theory.

In order to compare the operators of spin polarization in QM and in QFT, it is instructive to use the four-spinor representation of QFT. By analogy with the results of Section 4, a coordinate representation of the Fock space of time-dependent one-particle states $|\Psi^{(\theta)}(x_0)\rangle$ is given by the representatives

$$\begin{aligned}\Phi_{(\theta)+}(x) &= \begin{pmatrix} \psi_{(\theta)+}(x) \\ 0 \end{pmatrix}, & \Phi_{(\theta)+}^c(x) &= \begin{pmatrix} 0 \\ \psi_{(\theta)+}^c(x) \end{pmatrix}, \\ \psi_{(+1)+}(x) &= \begin{pmatrix} \psi_+^{(+1)}(x) \\ 0 \end{pmatrix}, & \psi_{(-1)+}(x) &= \begin{pmatrix} 0 \\ \sigma^1 \psi_+^{(-1)}(x) \end{pmatrix}, \\ \psi_{(+1)+}^c(x) &= \begin{pmatrix} \psi_+^{(+1)c}(x) \\ 0 \end{pmatrix}, & \psi_{(-1)+}^c(x) &= \begin{pmatrix} 0 \\ \sigma^1 \psi_+^{(-1)c}(x) \end{pmatrix},\end{aligned}$$

where $\psi_+(x)^{(\theta)}$ and $\psi_+(x)^{(\theta)c}$ are two-component representatives defined by equation (58) for the θ -representation of the gamma-matrices. In this coordinate representation, the charge operator \hat{Q}^{QFT} acts as follows

$$q\hat{C}\Phi_{(\theta)+}(x) = q\Phi_{(\theta)+}(x), \quad q\hat{C}\Phi_{(\theta)+}^c(x) = -q\Phi_{(\theta)+}^c(x), \quad (96)$$

where $\hat{C} = \text{bdiag}(I_4, -I_4)$.

In the coordinate representation, the equation (95) has the form

$$\hat{s}\Phi_{(\theta)+}(x) = \frac{\theta}{2}\Phi_{(\theta)+}(x), \quad \hat{s}\Phi_{(\theta)+}^c(x) = -\frac{\theta}{2}\Phi_{(\theta)+}^c(x), \quad (97)$$

where $\hat{s} = \frac{1}{2}\text{bdiag}(\gamma^0 \Sigma^3, -\gamma^0 \Sigma^3)$.

In Subsection 3.4, we have defined the effective physical sector $\mathcal{R}_{\text{ph}}^{\text{eff}}$ of the QM as the space of states (54). Equivalently, we can realize the effective physical sector of the QM in the extended (eight-component) representation $\tilde{\mathcal{R}}_{\text{ph}}^{\text{eff}}$ as the space of states $\Phi_{(\theta)+}$ and $\Phi_{(\theta)+}^c$. It is easy to see that the representation of the operator \hat{s} in (97) is a representative of the operator \hat{S} in \mathcal{R}_{ph} , namely,

$$\left(\Psi_{\theta,+}, \hat{S}\Psi'_{\theta,+}\right) = \left(\Phi_{(\theta)+}, \hat{s}\Phi'_{(\theta)+}\right), \quad \left(\Psi_{\theta,+}^c, \hat{S}\Psi_{\theta,+}^c\right) = \left(\Phi_{(\theta)+}^c, \hat{s}\Phi_{(\theta)+}^c\right).$$

Thus, we conclude that the physical subspace $\tilde{\mathcal{R}}_{\text{ph}}^{\text{eff}}$ of the QM is identical with the one-particle sector of the $2 + 1$ spinor QFT in the four-spinor representation.

It should be noted that in the $2 + 1$ QM the corresponding expression for the QFT spin polarization operator arises in a natural way, without having recourse to the $3 + 1$ theory and the subsequent reduction to the $2 + 1$ theory. From the beginning, the $2 + 1$ QM possesses the conserved spin polarization operator \hat{S} .

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